Quantum Field Theory and the Standard Model of Particle Physics

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Exercise 1: Quantized real scalar field, number operator. (*4 points*)
Prove by induction that for the occupation number operator of the lecture

$$\int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\, a^\dagger(\mathbf{p})a(\mathbf{p})\, \left|\mathbf{k}^{(1)}, \ldots, \mathbf{k}^{(n)}\right\rangle = n\, \left|\mathbf{k}^{(1)}, \ldots, \mathbf{k}^{(n)}\right\rangle . \tag{1}$$

Exercise 2: Quantized complex scalar field. (*16 points*)
Consider a complex scalar field ϕ

$$S[\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*] = \int d^4x \left((\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi\right) . \tag{2}$$

The general solution of the classical Klein-Gordon equation of a complex scalar field is given as

$$\phi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left(a(\mathbf{k})e^{-\mathrm{i}kx} + b^*(\mathbf{k})e^{\mathrm{i}kx}\right) , \tag{3}$$

with a and b^* independent complex coefficients.

i) Calculate the conjugate fields π and π^* and give their explicit form as a function of a and b.

ii) Let us now quantize the theory, applying the procedure of canonical quantization, identifying

$$\begin{aligned} \phi(x) &\to \hat{\phi}(x), \qquad \pi(x) \to \hat{\pi}(x), \\ \phi^*(x) &\to \hat{\phi}^\dagger(x), \qquad \pi^*(x) \to \hat{\pi}^\dagger(x), \end{aligned} \tag{4}$$

and setting the following commutators unequal zero

$$\begin{aligned} [\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] &= i\delta^3(\mathbf{x} - \mathbf{y}), \\ [\hat{\phi}^\dagger(t, \mathbf{x}), \hat{\pi}^\dagger(t, \mathbf{y})] &= i\delta^3(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{5}$$

All other commutators are zero.
In analogy to the case of a real scalar field, show that the following commutators of the creation- and annihilation operators are non-zero

$$\begin{aligned} [\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{k})] &= 2E(\mathbf{p})(2\pi)^3\delta^3(\mathbf{p} - \mathbf{k}), \\ [\hat{b}(\mathbf{p}), \hat{b}^\dagger(\mathbf{k})] &= 2E(\mathbf{p})(2\pi)^3\delta^3(\mathbf{p} - \mathbf{k}). \end{aligned} \tag{6}$$

iii) On the last sheet you have calculated the Hamilton function of the complex scalar field given as

$$H(\pi, \pi^*, \phi, \phi^*) = \int d^3x \left(\pi^*\pi + (\nabla\phi^*)(\nabla\phi) + m^2\phi^*\phi\right) . \tag{7}$$

Apply the procedure of canonical quantization and use the explicit expressions of $\hat{\phi}$ and $\hat{\pi}$ to diagonalize it. Show that it describes two sets of particles with mass m.

iv) On the last sheet you showed that the Lagrangian of the complex scalar field theory is invariant under the following global transformation

$$\phi \to \phi' = e^{\mathrm{i}\alpha}\phi, \tag{8}$$

with associated, conserved Noether current

$$j^\mu = \mathrm{i}\left(\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi\right). \tag{9}$$

Calculate the charge operator from the zero component of the Noether current

$$\hat{Q} = \int d^3x \hat{j}^0, \tag{10}$$

then express it as a function of the creation and annihilation operators and calculate the charge of the two kinds of particles, by letting it act on states $\hat{Q}\left|a(\mathbf{q})\right\rangle = \hat{Q}\hat{a}^\dagger(\mathbf{q})\left|0\right\rangle$ and $\hat{Q}\left|b(\mathbf{q})\right\rangle = \hat{Q}\hat{b}^\dagger(\mathbf{q})\left|0\right\rangle$.

Comment: In this exercise you showed that the quantized complex scalar field describes two massive charged particles with spin 0.