Exercise 1: Complex scalar field (10 points)

Consider the action of a complex scalar field

$$S[\phi, \phi^*, \partial^{\mu}\phi, \partial^{\mu}\phi^*] = \int d^4x (\partial_{\mu}\phi^*\partial^{\mu}\phi - m^2\phi^*\phi).$$
(1)

i) Treating ϕ and ϕ^* as dynamical variables, find their conjugate momenta π and π^* and show that the Hamiltonian has the following form

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi).$$
(2)

ii) Show hat the Lagrange density is invariant with respect to the following (global) transformation

$$\phi(x) \to \phi'(x) = e^{-i\alpha}\phi(x), \qquad \phi^*(x) \to \phi^{*\prime} = e^{i\alpha}\phi^*(x), \tag{3}$$

with α being a constant. Calculate the associated Noether current

$$j^{\mu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \frac{\delta\phi}{\delta\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^{*})} \frac{\delta\phi^{*}}{\delta\alpha}$$
(4)

and show that it is conserved

$$\partial_{\mu}j^{\mu} = 0. \tag{5}$$

iii) The energy momentum tensor of a theory with $n \in \mathbb{N}$ scalar fields ϕ_n , described by a Lagrange density $\mathcal{L} = \mathcal{L}(\phi_n, \partial_\mu \phi_n)$ is given as

$$\Theta^{\mu\nu} = \sum_{n} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_{n})} \partial^{\nu}\phi_{n} - g^{\mu\nu}\mathcal{L}.$$
 (6)

Treating ϕ and ϕ^* as dynamical variables calculate the energy momentum tensor for the theory of a complex scalar field and show that it is symmetric $\Theta^{\mu\nu} = \Theta^{\nu\mu}$.

Exercise 2: The energy momentum tensor in Electrodynamics (10 points) Consider the Lagrange density of classical electrodynamics without external currents or charges

$$\mathcal{L}(A^{\nu},\partial^{\mu}A^{\nu}) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu},\tag{7}$$

with the electromagnetic field-strength tensor

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}.$$
 (8)

i) Calculate the associated energy-momentum tensor

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}A_{\rho})} \partial^{\nu}A_{\rho} - g^{\mu\nu}\mathcal{L}.$$
(9)

ii) Show that it is neither symmetric $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$, nor gauge invariant under a gauge transformation

$$A_{\mu} \to A'_{\mu} = A_{\mu} + \partial_{\mu} f(x). \tag{10}$$

iii) Our aim is now to construct the symmetric and gauge-invariant energy momentum tensor of electrodynamics. For this purpose we add the following quantity

$$\tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \partial_{\beta} K^{\beta\mu\nu}, \qquad (11)$$

with $K^{\beta\mu\nu}$ being antisymmetric in β and μ . To prove that the additional term does not change energy momentum conservation, show that the energy momentum tensor is conserved

$$\partial_{\mu}\tilde{\Theta}^{\mu\nu} = 0. \tag{12}$$

Finally show that the choice

$$K^{\beta\mu\nu} = F^{\mu\beta}A^{\nu} \tag{13}$$

leads to a symmetric and gauge-invariant energy momentum tensor $\tilde{\Theta}^{\mu\nu}$.

iv) Show that the zero component of $\tilde{\Theta}^{00}$ reproduces indeed the energy density familiar from electrodynamics

$$\tilde{\Theta}^{00} = \epsilon = \frac{1}{2} \left(|\mathbf{E}|^2 + |\mathbf{B}|^2 \right), \tag{14}$$

and the $\tilde{\Theta}^{0i}$ components give the Poynting vector

$$\tilde{\Theta}^{0i} = \mathbf{S}^i = (\mathbf{E} \times \mathbf{B})^i.$$
(15)