

Exercise 1: Parity and charge conjugation(5+5=10 points)

Recall the following bilinear covariants from the lecture

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^5\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad \bar{\psi}\gamma^\mu\gamma^5\psi. \quad (1)$$

- i) In the lecture you discussed the parity transformation

$$P : (t, \mathbf{x}) \rightarrow (t, -\mathbf{x}), \quad (2)$$

and showed that the parity transformation of a Dirac spinor ψ is given as

$$\psi \rightarrow \psi^P = \gamma^0\psi. \quad (3)$$

Discuss the transformation properties of the bilinear covariants under parity transformation.

Hint: Split the vectorial bilinear covariants into $\bar{\psi}\gamma^0\psi$ and $\bar{\psi}\gamma^i\psi$ and discuss the parity transformation separately.

- ii) We consider a charged particle with mass m and charge q moving in an external electromagnetic field. In the lecture you discussed a relativistic description of such a system, given by the following Dirac equation

$$[i\gamma^\mu (\partial_\mu + iqA_\mu) - m] \psi(x) = 0. \quad (4)$$

Determine the corresponding Dirac equation for the charge conjugate spinor

$$C : \quad \psi \rightarrow \psi^C = i\gamma^2\gamma^0\bar{\psi}^T = i\gamma^2\psi^*, \quad (5)$$

as you have done it in the lecture in case of the free Dirac equation. Comment on the name „charge conjugation“.

Exercise 2: Solution of the Dirac equation(3+3+4=10 points)

In the lecture you showed that the general solution of the Dirac equation

$$[i\gamma^\mu\partial_\mu - m] \psi(x) = 0, \quad (6)$$

is given by the following Dirac spinor

$$\psi(x) = \int \sum_{r=1,2} (b_r(\mathbf{k})u_r(\mathbf{k})e^{-ikx} + d_r^*(\mathbf{k})v_r(\mathbf{k})e^{ikx}) \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}, \quad (7)$$

with $b_r(\mathbf{k}), d_r(\mathbf{k}) \in \mathbb{C}$ complex numbers. The $u_r(\mathbf{k}), v_r(\mathbf{k})$ are referred to as basis spinors.

- i) Show that the basis spinors are orthogonal

$$\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = -\bar{v}_r(\mathbf{k})v_s(\mathbf{k}) = 2m\delta_{rs}, \quad (8)$$

$$\bar{u}_r(\mathbf{k})v_s(\mathbf{k}) = \bar{v}_r(\mathbf{k})u_s(\mathbf{k}) = 0. \quad (9)$$

ii) Show the completeness relations

$$\sum_{s=1,2} u_{s,\alpha}(\mathbf{k}) \bar{u}_{s,\beta}(\mathbf{k}) = (\not{k} + m)_{\alpha\beta}, \quad (10)$$

$$\sum_{s=1,2} v_{s,\alpha}(\mathbf{k}) \bar{v}_{s,\beta}(\mathbf{k}) = (\not{k} - m)_{\alpha\beta}. \quad (11)$$

iii) We define the following two operators

$$\Lambda_{\alpha\beta}^+(\mathbf{k}) := \frac{1}{2m} \sum_{s=1,2} u_{s,\alpha}(\mathbf{k}) \bar{u}_{s,\beta}(\mathbf{k}), \quad \Lambda_{\alpha\beta}^-(\mathbf{k}) := -\frac{1}{2m} \sum_{s=1,2} v_{s,\alpha}(\mathbf{k}) \bar{v}_{s,\beta}(\mathbf{k}). \quad (12)$$

Show that they are orthogonal projection operators $(\Lambda^\pm)^2 = \Lambda^\pm$, $\Lambda^+ \Lambda^- = 0$ and let them act on the complete solution of the Dirac equation $\psi(x)$. What do they project out?