Exercise 1: Lorentz Transformation of Dirac Spinors(*5 points*) Given the transformation property of the Dirac spinor $\psi(x)$ under Lorentz transformations, show the the adjoint spinor transforms as

$$\bar{\psi}'(x') = \bar{\psi}(x) S^{-1}(\Lambda) \tag{1}$$

Exercise 2: Dirac and Klein-Gordon equation(*4 points*)
Let ψ be a Dirac spinor solving the Dirac equation

$$\left[\mathrm{i}\gamma^\mu \partial_\mu - m\right] \psi(x) = 0. \tag{2}$$

Show that it is also a solution of the Klein-Gordon equation

$$\left[\partial_\mu \partial^\mu + m^2\right] \psi(x) = 0. \tag{3}$$

Exercise 3: Bilinear covariants(*2+4=6 points*)

i) First show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{4}$$

and use your results to show that

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0. \tag{5}$$

ii) Using these transformations, show that the equation

$$S^{-1}(\Lambda) \gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu, \tag{6}$$

familiar from the lecture, is satisfied up to first order in $\delta\omega$.

iii) From Exercise 1 and what you were shown in the lecture, one can easily see that $\bar{\psi}\psi$ transforms as a Lorentz scalar and $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz vector. Furthermore an additional gamma matrix $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ was introduced with the transformation property

$$S^{-1}(\Lambda) \gamma^5 S(\Lambda) = \det(\Lambda) \gamma^5. \tag{7}$$

Prove the behaviour under Lorentz transformation of the following bilinear covariants

$$\bar{\psi}\gamma^5\psi \to \det(\Lambda) \bar{\psi}\gamma^5\psi \qquad \text{pseudoscalar} \tag{8}$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \to \det(\Lambda) \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\gamma^5\psi \qquad \text{axial vector} \tag{9}$$

$$\bar{\psi}\sigma^{\mu\nu}\psi \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \bar{\psi}\sigma^{\alpha\beta}\psi \qquad \text{antisymmetric tensor} \tag{10}$$

Exercise 4: Continuity equation in relativistic quantum mechanics II(*4 points*)

Let the Dirac spinor ψ be a solution of the Dirac equation

$$[\mathrm{i}\gamma^\mu \partial_\mu - m]\,\psi(x) = 0. \tag{11}$$

Show that the current

$$j^\mu(x) = \bar{\psi}\gamma^\mu\psi \tag{12}$$

satisfies a continuity equation

$$\partial_\mu j^\mu(x) = 0. \tag{13}$$

Discuss wether $j^0 = \rho$ is positive definite in the case of the Dirac equation.