Exercise 1: Lorentz invariant integration measure (*5 points*)
Given the relativistic invariance of d^4k show that the integration measure

$$\frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \tag{1}$$

is Lorentz invariant, provided that $E(\mathbf{k}) = k_0 = \sqrt{m^2 + \mathbf{k}^2}$. Use this to argue that $2E(\mathbf{k})\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ is a Lorentz invariant distribution.
Hint: Start from the Lorentz invariant expression $\frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k_0)$ and use $\delta(x^2 - x_0^2) = \frac{1}{2|x_0|}(\delta(x + x_0) + \delta(x - x_0))$.

Exercise 2: Collider experiments at the LEP (*4 points*)
In the LEP storage ring at CERN, head-on collisions between (equally accelerated) electrons and positrons were produced, such that the total energy in the center of mass was equal to that of the Z boson ($m_z = 91$ GeV). What is the velocity of each particle before the collision? If an electron is accelerated toward a positron at rest, what velocity does it need in order to reach the same center-of-mass total energy?
Hint: The total momentum of two particles a, b in the center of mass frame is $\mathbf{p}_a + \mathbf{p}_b = 0$.

Exercise 3: Continuity equation in relativistic quantum mechanics (*3+2 points*)
In the lecture you derived the Klein-Gordon equation of relativistic quantum mechanics

$$-\frac{\partial^2}{\partial t^2}\phi(x) = \left[-\Delta + m^2\right]\phi(x). \tag{2}$$

i) Show that the current

$$j^\mu(x) = \mathrm{i}\left(\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi\right) \tag{3}$$

satisfies a continuity equation

$$\partial^\mu j_\mu = 0. \tag{4}$$

ii) Since the Klein-Gordon equation is a relativistic wave equation, it is solved by plane waves. Calculate the zero-component of the current $j^0 = \rho$ for a plane wave solution of the form $\phi \sim \exp(\mathrm{i}px)$ and interpret your result.

Exercise 4: Infinitesimal Lorentz transformation (*3+3 points*)
An infinitesimal Lorentz transformation and its inverse can be written as

$$x'^\alpha = (g^{\alpha\beta} + \epsilon^{\alpha\beta})x_\beta \qquad x^\alpha = (g^{\alpha\beta} + \epsilon'^{\alpha\beta})x'_\beta \tag{5}$$

where $(g^{\alpha\beta}) = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric and $\epsilon^{\alpha\beta}$ and $\epsilon'^{\alpha\beta}$ are infinitesimal.

i) Show that $\epsilon'^{\alpha\beta} = -\epsilon^{\alpha\beta}$ by making use of the definition of an inverse transformation.

ii) Show from the preservation of the norm that the infinitesimal shift $\epsilon^{\alpha\beta}$ is antisymmetric $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$.