Exercise 1: Natural units *(5 points)*

Natural units are defined by setting $c = \hbar = 1$.
Use natural units to express $1kg$ in GeV, as well as $1s$ in $1/GeV$. Use your results to express Newton's constant of gravity

$$G_N = 6,67 \times 10^{-11} \frac{m^3}{kgs^2} \tag{1}$$

in natural units.
Finally give the value of the Planck mass $M_{pl} = 1/\sqrt{G_N}$.

Exercise 2: Charged particle in a constant magnetic field *(5 points)*

We consider a particle with charge e and mass m. The Lagrangian of a charged particle in an electromagnetic field is given by

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 + e \frac{\dot{\mathbf{r}}}{c} \cdot \mathbf{A}(\mathbf{r}) - e\Phi(\mathbf{r}) \tag{2}$$

with the scalar potential $\Phi(\mathbf{r})$ and the vector potential $\mathbf{A}(\mathbf{r})$.
Calculate the Hamilton function $H(\mathbf{r}, \mathbf{p}, t)$ of the system, using a Legendre transformation.

Exercise 3: Continuity equation in quantum mechanics *(3 points)*

Consider a wave function $\psi(\mathbf{r}, t)$ satisfying the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})\right) \psi(\mathbf{r}, t) = \mathrm{i}\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t). \tag{3}$$

The probability density is defined as $\rho(\mathbf{r}, t) := \psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t)$. Show that it satisfies the continuity equation

$$\frac{\partial}{\partial t} \rho(\mathbf{r}, t) + \nabla \cdot \mathbf{j}(\mathbf{r}, t) = 0, \tag{4}$$

where $\mathbf{j}(\mathbf{r}, t)$ is the propability current

$$\mathbf{j}(\mathbf{r}, t) := \frac{\hbar}{2\,\mathrm{i}m} \left(\psi^* \left(\nabla\psi\right) - \left(\nabla\psi^*\right) \psi\right). \tag{5}$$

Exercise 4: Harmonic oscillator in quantum mechanics *(7 points)*

Recall the time-independent Schrödinger equation of the one-dimensional harmonic oscillator in quantum mechanics

$$\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)\psi(x) = E\psi(x), \tag{6}$$

with momentum operator $\hat{p}$ and the position operator $\hat{x}$.
It has been shown that the energy eigenvalues E can be calculated easily in an energy basis of the system $\{|n\rangle\}, n \in \mathbb{N}$, by making use of two operators defined as

$$\hat{a} := \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}\right), \qquad \hat{a}^\dagger := \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{\mathrm{i}}{m\omega}\hat{p}\right), \tag{7}$$

with the property to raise and lower ('create' and 'annihilate') the states

$$\hat{a}\,|n\rangle = c_n^-\,|n-1\rangle\,, \qquad \hat{a}^\dagger\,|n\rangle = c_n^+\,|n+1\rangle\,. \tag{8}$$

The occupation number operator acts as

$$\hat{N}\,|n\rangle = \hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle\,. \tag{9}$$

Calculate the commutator $[\hat{a}, \hat{a}^\dagger]$ and the normalization constants c_n^- and c_n^+. Use the operators $\hat{a}$ and $\hat{a}^\dagger$ to reformulate the Hamilton operator and give the energy eigenvalues E_n.