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Fachbereich Physik

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Einführung in die Theoretische Festkörperphysik

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Exercise 5

(Due date: 15.11.2019)

Problem 1 (**Creation and Annihilation Operators**) (3 points)

The Hamiltonian for displacements in the harmonic approximation is given by:

$$H_{ph} = \frac{1}{2}\sum_{\mathbf{q}s}\left(\tilde{P}^{\dagger}_{\mathbf{q}s}\tilde{P}_{\mathbf{q}s} + \omega^2(\mathbf{q}s)\tilde{u}_{-\mathbf{q}s}\tilde{u}_{\mathbf{q}s}\right) \tag{1}$$

Show that, after the introduction of the creation and annihilation operators, by:

$$\tilde{u}_{\mathbf{q}s} = \sqrt{\frac{\hbar}{2\omega(\mathbf{q}s)}}(b_{\mathbf{q}s} + b^{\dagger}_{-\mathbf{q}s}) \tag{2}$$

$$\tilde{P}_{\mathbf{q}s} = \frac{1}{i}\sqrt{\frac{\hbar\omega(\mathbf{q}s)}{2}}(b_{-\mathbf{q}s} - b^{\dagger}_{\mathbf{q}s}) \tag{3}$$

the Hamiltonian can be put in the following form:

$$H_{Ph} = \sum_{\mathbf{q}s}\hbar\omega(\mathbf{q}s)\left(b^{\dagger}_{\mathbf{q}s}b_{\mathbf{q}s} + \frac{1}{2}\right) \tag{4}$$

Problem 2 (**Phonons in the Continuum Limit**) (3 points)

Consider a monoatomic chain of atoms with lattice constant a and coupling constant D (consider only nearest neighbour couplings).

a) Show that, for large wavelengths ($\lambda \gg a$), the equation of motion reduces to the continuum elastic wave equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2} \tag{5}$$

b) Compare the dispersion relation with the exact dispersion relation for this case:

$$\omega = 2\sqrt{\frac{D}{M}}\left|\sin\frac{qa}{2}\right| \tag{6}$$

Problem 3 (**Specific Heat of Europium Oxide**) (4 points)

The specific heat of EuO at low temperatures is proportional to $T^{3/2}$.

a) Determine the dispersion of the quasiparticles that are responsible for the observed behaviour. Suppose the dispersion is isotropic and has the form of a power law: $\omega \propto q^a$.
Note:
Prefactors are not relevant for this task; focus on the dependency on q. In order to determine the relationship between the specific heat c_V, and the dispersion $\omega(q)$, one should start with the density of states for three dimensions:

$$D(\omega) = \frac{V}{N} \sum_s \frac{1}{(2\pi)^3} \int_{S(\omega)} \frac{dS}{|d\omega/dq|} \tag{7}$$

where the integral is performed over a surface of constant ω. Then, using $\omega \propto q^\alpha$, show that:

$$D(\omega) \propto \omega^{\frac{3}{\alpha}-1} \tag{8}$$

Now, use the expression for the specific heat of bosons:

$$c_V = \frac{1}{V}\frac{\partial U}{\partial T} = \frac{\hbar}{V} \int d\omega \; D(\omega) \; \omega \; \frac{\partial}{\partial T} \frac{1}{e^{\hbar\omega/k_B T} - 1} \tag{9}$$

to determine the relationship between the temperature dependence of c_V and the dispersion $\omega \propto q^\alpha$. Assume that $c_V \propto T^\gamma$ at low temperatures, and determine how γ is related to α.

b) Discuss whether the observed specific heat can be attributed to phonons.