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Fachbereich Physik

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Frankfurt, 23.01.2020

Einführung in die Theoretische Festkörperphysik

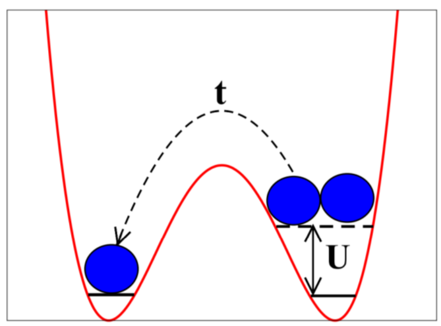
Winter term 2019/2020

Exercise 12

(Due date: 03.02.2020)

Problem 1 (**Double-well Potential**) (10 points)

Consider a quantum mechanical system of particles that are subject to a double-well potential (see figure).



a) First, we consider the case of two bosons (without spin) within the potential.
- Which are all possible Fock states?
- Give a symmetrized two-particle state that is built from the one-particle states where both bosons are in the left well.

b) Now, we consider the case of fermions with spin $s = 1/2$ in the same potential.
- What is the maximum number of fermions in this system?
- Which are all possible Fock states if we have two fermions in the potential?
- Which are all possible Fock states if both fermions have the same spin?
- Give an antisymmetric two-particle state that is built from the one-particle states where both fermions (with different spins) are in the left well.

Assume that the system can be described with the Hamilton operator $H = T + V$, where T describes the so-called hopping of particles between the wells and V the interaction energy (typically Coulomb repulsion) of two particles in the same well (see figure).

The bosons are described via creation operators $b^\dagger_\alpha$ and annihilation operators b_α where $\alpha = L, R$. The hopping and interaction contributions to the Hamilton operator read then ($t > 0$):

$$T = -t(b^\dagger_L b_R + b^\dagger_R b_L), \quad V = U \sum_\alpha \frac{n_\alpha(n_\alpha - 1)}{2} \qquad \text{mit} \quad n_\alpha = b^\dagger_\alpha b_\alpha.$$

In the case of fermions we have to take the spin into account, i.e. we use creation operators $c^\dagger_{\alpha,\sigma}$ and annihilation operators $c_{\alpha,\sigma}$. The two terms in the Hamilton operator are then ($t > 0$)

$$T = -t \sum_\sigma (c^\dagger_{L,\sigma} c_{R,\sigma} + c^\dagger_{R,\sigma} c_{L,\sigma}), \quad V = U \sum_\alpha n_{\alpha,\uparrow} n_{\alpha,\downarrow} \qquad \text{mit} \quad n_{\alpha,\sigma} = c^\dagger_{\alpha,\sigma} c_{\alpha,\sigma}.$$

c) Calculate the following commutators:

- For bosons: $[T, n_\alpha]$, $[V, n_\alpha]$
- For fermions: $[T, n_{\alpha,\sigma}]$, $[V, n_{\alpha,\sigma}]$

Show that the total number of particles N is conserved. It is

- $N = \sum_\alpha n_\alpha$ for bosons
- $N = \sum_{\alpha,\sigma} n_{\alpha,\sigma}$ for fermions.

d) Write the matrix elements of the Hamilton operator in the Fock basis for the case a) with 2 bosons. Calculate the energy eigenvalues and discuss the eigenvectors in both limits $\frac{U}{t} \to 0$ and $\frac{U}{t} \to \infty$.