The Quantum Hall Effects

Integer and Fractional

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Outline

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   - Quantization of Conductance
   - Edge States

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Experimental Setup

- Temperature: $< 5 \text{ K}$ (He cryostat)
- Small $B$ (some Tesla)
- Measure $\rho_{xx}$ and $\rho_{xy}$ vs. $B$

**Figure:** Hall bar
Two Dimensional Electron Gas

- QHE based on almost perfect realization of 2DEG

Figure: Si MOSFET

Figure: Energy level diagram

Ming Qiu – Johns Hopkins University: pha.jhu.edu/
QHE – Experimental Results

- $\rho_H = \rho_{xy} = -\frac{1}{n} \frac{h}{e^2}$ with $n = 1, 2, 3, \ldots$ with precision $10^{-10}$
- At plateaus $\rho_{xx} \to 0$
- Independent of material parameters

Figure: $\rho_{xx}$ and $\rho_{xy}$
Electrons in a Strong Magnetic Field

- Magnetic length: $\ell_0 = \sqrt{\frac{\hbar c}{eB}}$
- Cyclotron frequency: $\omega_c = \frac{eB}{m}$
- Landau levels: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$ with $n = 0, 1, 2 \ldots$
- Degeneracy of Landau levels: $N_S = \frac{e}{hc} \Phi = \frac{\Phi}{\Phi_0}$
- Filling factor $n = 2\pi \ell_0^2 n_0 = \frac{n_0}{n_B}$

**Figure:** Landau levels
2D Density of States

- Due to impurities degeneracy of states with different \((X, Y)\)
lifted \(\Rightarrow\) Extended and localized states

\[\text{Figure: DOS without impurities}\]
\[\text{Figure: DOS with } U(r)\]

 Qui: John Hopkins University
Quantization of Conductance

Since $n \in \mathbb{N} \Rightarrow n_0 = nn_B$ with $n \in \mathbb{N}$

$$\Rightarrow \sigma_H = -ecn_0/B = ecnn_B/B = \boxed{-ne^2/h}$$

Hall conductance

$$\sigma = \begin{bmatrix} 0 & ne^2/h \\ -ne^2/h & 0 \end{bmatrix}$$
The Quantum Hall Effects

- Integer Quantum Hall Effect
- Edge States

**Currents at the Edge**

- Why exact quantization independent of material?

**Hall conductance**

**Figure:** Hall ribbon Laughlin, *PRB* 23, 5632 (1980)
Charge transport in the IQHE

Adding one flux quantum

\[ x_k^m \rightarrow x_k^{m-1} \]

Charge transport between edges

- States in Landau level shift over one by one
  \[ \rightarrow \text{Edge to edge transfer of one } e^- \text{ per Landau level} \]

\[ I = -\frac{ne^2v_H}{h} \]
Phenomenology of the FQHE

- $B \approx 15 \text{T}$:
- $T < 1 \text{ K}$ and pure samples required

**Fractional $n$**

Hall conductance exhibits plateaus at at $n = 4/3, 5/3, 7/3, 1/5$ and so on

**Figure:** Hall resistance $R_H$ and dissipative resistance $R$

The Quantum Hall Effects
- Fractional Quantum Hall Effect
- The Laughlin Liquid

Laughlin Wave Functions

- Choose symmetric gauge → rotational symmetry about origin preserved
- Angular momentum $m$ good quantum number
- Consider only lowest Landau level:

$$\Psi(x, y) = f(z)e^{-\frac{1}{4}|z|^2}$$

- Choose: $f(z) = \prod_{j=1}^{N}(z - Z_j)$
The Quantum Hall Effects

Fractional Quantum Hall Effect

The Laughlin Liquid

Laughlin Wave Function

- Effect of Coulomb interaction non-trivial
- Interacting electron model required
- Two-body problem for particles:
  \[
  \Psi_{mM}(z_1, z_2) = (z_1 - z_2)^m (z_1 + z_2)^M e^{-\frac{1}{4}(|z_1|^2 + |z_2|^2)}
  \]
- For fermions \( m \) must be odd
- Exact solution for general \( V(|z_1 - z_2|) \)
- Only one state with fixed \( m \) and \( M \) with energy eigenvalue:

  Haldane pseudopotential: \( \nu_m = \frac{\langle mM|V|mM\rangle}{\langle mM|mM\rangle} \)
Many-body states

- Without magnetic field only continuous spectrum

**Bound states**

Discrete spectrum

→ Bound states exist

**Figure:** $v_m$ vs. $m$ in units of $e^2/\epsilon_0 \ell$

Girvin *Séminaire Poincaré* 2 (2004) 53
Many-body states

- We need to find: \( \Psi (\{z\}) = f (\{z\}) e^{-\frac{1}{4} \sum_j |z_j|^2} \)
- \( f \) polynomial: Slater determinant with all states occupied
  \[
  f (\{z\}) = \begin{vmatrix}
  (z_1)^0 & (z_2)^0 \\
  (z_1)^1 & (z_2)^1 
  \end{vmatrix} = (z_1)^0(z_2)^1 - (z_2)^0(z_1)^1 \\
  = (z_2 - z_1)
  \]
- Single Slater determinant to fill first \( N \) \( m \)-states:
  \[
  f_N (\{z\}) = \prod_{i<j}^N (z_i - z_j)
  \]
- Thus full probability distribution:
  \[
  |\Psi (\{z\})|^2 = \prod_{i<j}^N |z_i - z_j|^2 e^{-\frac{1}{2} \sum_{j=1}^N |z_j|^2}
  \]
The Plasma Physics Analogy

Consider Boltzmann weight $|\Psi (\{z\})|^2 = \exp(-\beta U_{\text{class}})$

Potential energy of uniform electron gas

$$U_{\text{class}} = m^2 \sum_{i<j} (-\ln |z_i - z_j|) + \frac{m}{4} \sum_k |z_k|^2$$

interaction of particles with charge $m$

const. charge density

- Electrostatics: $\Phi(r) = Q \left(-\ln \frac{r}{r_0}\right)$
- Poisson equation
  $$\Delta \left(\frac{1}{4}|z|^2\right) = -(1/\ell^2) = 2\pi \rho_B \Rightarrow \rho_B = -1/(2\pi \ell^2)$$

Plasma neutrality condition

$$nm + \rho_B = 0 \Rightarrow n = \frac{1}{m} \frac{1}{2\pi \ell^2}$$
Adding One Flux Quantum

- Adding one flux quantum:

\[
\Psi_m^+(z_0; z_1, \cdots, z_N) = \prod_{j=1}^{N} (z_j - z_0) \Psi_m(z_1, \cdots, z_N)
\]

\[
= A_{z_0}^+ \Psi_m(z_1, \cdots, z_N)
\]

- Energy of many-body state with “quasi-hole”:

\[
\Phi_{pq}(z_0; z_1, \cdots, z_N) = \Phi(z_1, \cdots, z_N) - \frac{2}{m} \sum_{j=1}^{N} \ln |z_j - z_0|
\]

- Energy to destroy or create particle at \( B = 15 \, \text{T} \): \( 4 \, \text{K} \rightarrow \text{gap} \) to all excitations

- Excitation energy: Energy for adding quasi particle with \( Q = \pm e/m \)
Illustration of the Wave Function

Wave function for completely filled Landau level:

\[ \Psi_m(\{r\}) = \prod_{1 \leq j < k \leq N} (z_j - z_k)^{m-1} \Psi_1(\{r\}) \]

\[ \Psi_1(\{r\}) = \prod_{1 \leq j < k \leq N} (z_j - z_k)^{-1} \]

Figure: Representation of \( n = 1/3 \) state Eisenstein et al., Science 248, 1510 (1990)

Completely filled Landau level with each \( e^- \) having \( m - 1 \) flux quanta attached
Summary

- IQHE arises from boundary conditions and is carried by edge states
  - Disorder essential
  - Non-interacting electrons
- FQHE electrons in lowest Landau level condense into Laughlin liquid with fractional excitations having a gap
  - Pure samples needed
  - Interacting electrons
Advanced Solid State Physics
Westview Press

The Quantum Hall Effects – Fractional and Integral
Springer

Science 248, 1510 (1990)


Séminaire Poincaré 2 (2004) 53