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Theorie zu Magnetismus, Supraleitung und Elektronische Korrelation in Festkörpern

Sommersemester 2019

Blatt 12

(Abgabe: 09.07.2019)

Aufgabe 1 (**Wick's theorem and Feynman diagrams**) (10=6+4 Punkte)

Consider the zero temperature real time interacting Green's function

(1) $$G(p,t-t') = -i \langle| T c_p(t) c_p^\dagger(t') |\rangle ,$$

where p is the momentum quantum number, t, t' is the time, $c_p(t), c_p^\dagger(t)$ the fermionic annihilation/creation operators in the Heisenberg picture and $\langle|A|\rangle$ denotes the quantum mechanical expectation value. T is the time-ordering operator, which puts all operators to the right into descending time order, i.e.

(2) $$T a(t) b(t') = \begin{cases} a(t)b(t') & \text{for } t > t' \\ -b(t')a(t) & \text{for } t < t' \end{cases}$$

for two fermionic operators $a(t), b(t')$. The noninteracting Green's function is thus given as

(3) $$G_0(p,t-t') = -i \langle| T c_p(t) c_p^\dagger(t') |\rangle_0 ,$$

where $\langle|A|\rangle_0 = \langle \phi_0 | A | \phi_0 \rangle$ denotes the quantum mechanical expectation value for the ground state ϕ_0 of a noninteracting system.

Introducing the interaction $V(t)$, the full interacting Green's function can be obtained by a perturbation series expansion as

(4) $$G(k,t-t') = -i \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 ... \int_{-\infty}^{\infty} dt_n \langle| T c_p(t) c_p^\dagger(t') V(t_1) ... V(t_n) |\rangle_{0,conn.} ,$$

where only *connected* contributions are to be considered (explanation below). In our case we consider $V(t)$ to be the Coulomb interaction

(5) $$V(t) = \frac{1}{2} \sum_{kk'q} U_q c_{k+q}^\dagger(t) c_{k'-q}^\dagger(t) c_{k'}(t) c_k(t)$$

i) One way to evaluate Eq. (4) is to apply the Wick-Theorem which states that the expectation value of a product of creation and annihilation operators can be split into a sum of expectation values of time-ordered annihilation/creation operator pairs:
Wick's theorem: The time-ordered expectation value of a product of creation and annihilation operators is equal to the sum of the time-ordered expectation value of all possible permutation of annihilation/creation operator pairs. Each term has the sign $(-1)^N$, where N is the number of commuting neighbouring operators. Per convention the annihilation operators are put to the left and the creation operators are put to the right in the expectation value.

Example: Let us evaluate the following expression with Wick's theorem:

$$
\begin{aligned}
\langle| T\ c_a(t_1)\ c_b^\dagger(t_2)\ c_c(t_3)\ c_d^\dagger(t_4)|\rangle_0 &= \langle| T\ c_a(t_1)\ c_b^\dagger(t_2)|\rangle_0 \langle| T\ c_c(t_3)\ c_d^\dagger(t_4)|\rangle_0 \\
&\quad - \langle| T\ c_a(t_1)\ c_d^\dagger(t_4)|\rangle_0 \langle| T\ c_c(t_3)\ c_b^\dagger(t_2)|\rangle_0 \qquad (6)\\
&= \delta_{ab}\delta_{cd} \langle| T\ c_a(t_1)\ c_a^\dagger(t_2)|\rangle_0 \langle| T\ c_c(t_3)\ c_c^\dagger(t_4)|\rangle_0 \\
&\quad - \delta_{ad}\delta_{bc} \langle| T\ c_a(t_1)\ c_a^\dagger(t_4)|\rangle_0 \langle| T\ c_c(t_3)\ c_c^\dagger(t_2)|\rangle_0 \qquad (7)\\
&= (i)^2\delta_{ab}\delta_{cd} G_0(a, t_1 - t_2) G_0(c, t_3 - t_4) \\
&\quad - (i)^2\delta_{ad}\delta_{bc} G_0(a, t_1 - t_4) G_0(c, t_3 - t_2) \qquad (8)
\end{aligned}
$$

The last term has a negative sign because to obtain the ordering $c_a(t_1)\ c_d^\dagger(t_4)\ c_c(t_3)\ c_b^\dagger(t_2)$, we had to commute operators 3 times instead of 0 for the first term.

This directly extends to higher number of operators, i.e. the first terms can be

$$
\begin{aligned}
&\langle| T\ c_a(t_1)\ c_b^\dagger(t_2)\ c_c(t_3)\ c_d^\dagger(t_4)\ c_e(t_5)\ c_f^\dagger(t_6)|\rangle_0 \\
&= \langle| T\ c_a(t_1)\ c_b^\dagger(t_2)|\rangle_0 \langle| T\ c_c(t_3)\ c_d^\dagger(t_4)|\rangle_0 \langle|\ c_e(t_5)\ c_f^\dagger(t_6)|\rangle_0 \\
&- \langle| T\ c_a(t_1)\ c_b^\dagger(t_2)|\rangle_0 \langle| T\ c_c(t_3)\ c_f^\dagger(t_6)|\rangle_0 \langle|\ c_e(t_5)\ c_d^\dagger(t_4)|\rangle_0 \qquad (9)\\
&+ \ldots
\end{aligned}
$$

Therefore, a product of $2n$ operators (n creation-, n annihilation operators) is decomposed into a sum of $n!$ terms. For the special case of equal time operators we sort the creation operators to the left and the annihilation operators to the right which evaluates to

$$\langle| T\ c_p^\dagger(t)\ c_k(t)|\rangle_0 = \delta_{pk} n_F(\epsilon_k), \qquad (10)$$

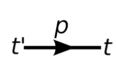
where n_F is the Fermi function and ϵ_k is the dispersion of the noninteracting system.

Use the Wick Theorem to evaluate the zeroth-order ($n = 0$) and first-order ($n = 1$) terms for the interacting Green's function (4) with the Coulomb interaction (5).

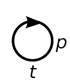
Remark: Considering only the connected contributions means that only pairs of operators where the operators from the Green's function are paired with other operator from the interaction are considered. I.e. in the sum of Eq. (4) all terms containing the pairs $\langle| T c_p(t) c_p^\dagger(t')|\rangle \langle| V(t_i) \cdots |\rangle_0 \cdots$ have to be excluded.

ii) Another possibility to evaluate Eq. (4) is the method of *Feynman Diagrams*. This is just a representation of the Wick theorem in terms of diagrams that represent the interacting Green's function as a combination of Green's function lines and interaction lines.

For each Green's function we draw a directed arrow, for each interaction we draw a wriggly line and for each number operator we draw a circle:

$G_0(p, t - t') =$ 

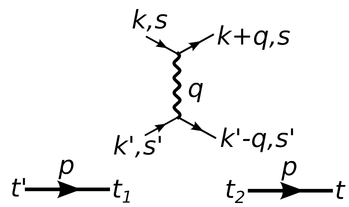
$U_q =$ 

$n_F(\epsilon_p) =$ 

To evaluate the terms in Eq. (4) one then simply starts with one incoming line and one outgoing line and sums up all possible graphs that can be obtained by connecting the Green's function and interacting lines. Then all internal variables (time, momentum) are integrated out. Each term gets assigned a prefactor of $(-1)^{2n}(-1)^{N_l}$, where n is the number of interaction lines and N_l is the number of closed loops contained in the diagram.

What are the ingredients for the zeroth-order diagram ($n = 0$, i.e. no interaction line)? Write the diagram down in terms of the noninteracting Green's function and compare to i).

Draw all possible connected diagrams for first-order, i.e. $n = 1$ resp. one interaction line. This means we have to draw all possible connections of the legs of interaction U_q with the "legs" of the in- and outgoing lines (t', p) and (t, p) and the remaining lines from the following figure, to obtain terms containing one interaction parameter and three Green's functions (some evaluate as number operators):



Write down the terms for the diagram and evaluate it by summing over all internal variables and compare to i).