JORDAN – WIGNER, 1D – chains.

Particularly in CMP, we employ different operators and QP statistics in order to solve models... today, we will introduce a representation that is relevant to 1D spin models...

Consider 1D XYZ – model:

$$H = J \sum_{\langle ij \rangle} \left(S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z \right)$$

We can consider a representation in terms of <u>spinless</u> fermions similar to the Holstein–Primakoff bosonic representation from spin-wave theory:

<u>Consider</u> $\underset{n_i = 0}{\text{—}} \Rightarrow \downarrow \quad ; \quad \underset{n_i = 1}{\bullet} \Rightarrow \uparrow$

$$\hat{S}_i^z = a_i^\dagger a_i - \tfrac{1}{2} \qquad \hat{S}_i^+ = a_i^\dagger \qquad \hat{S}_i^- = a_i \qquad \text{X}$$

However, this representation <u>cannot</u> fulfill the spin commutation relations since

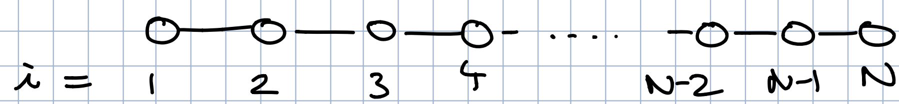
$$[S_i^+, S_j^+] = 0$$, but fermionic operators anticommute on different sites

We can solve this by introducing a phase factor:

$$\hat{S}_i^z = a_i^\dagger a_i - \tfrac{1}{2} \qquad S_i^+ = a_i^\dagger \exp\left\{ i\pi \sum_{j<i} a_j^\dagger a_j \right\}$$

$$S_i^- = \exp\left\{ -i\pi \sum_{j<i} a_j^\dagger a_j \right\} a_i$$

The new phase factor $\exp\{\ \}$ counts the number of fermions "to the left" of site i.



Let us substitute these operators in:

$$H = J \sum_{\langle ij \rangle} \left(S_i^x S_j^x + S_i^y S_j^y + \Delta\, S_i^z S_j^z \right)$$

$$= J \sum_i \left[\frac{1}{2}\left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + \Delta S_i^z S_{i+1}^z \right]$$

$$= J \sum_i \Bigg[\frac{1}{2}\Bigg(a_i^+ \exp\Big\{ i\pi \sum_{j<i} a_j^+ a_j \Big\} \exp\Big\{ -i\pi \sum_{\ell<i+1} a_\ell^+ a_\ell \Big\}\, a_{i+1}$$

$$+ \exp\Big\{ -i\pi \sum_{j<i} a_j^+ a_j \Big\}\, a_i\, a_{i+1}^+ \exp\Big\{ i\pi \sum_{\ell<i+1} a_\ell^+ a_\ell \Big\} \Bigg)$$

$$+ \Delta\left(a_i^+ a_i - \frac{1}{2} \right)\left(a_{i+1}^+ a_{i+1} - \frac{1}{2} \right) \Bigg]$$

Now, since $[a_j^+ a_j, a_i^+ a_i] = 0 \ \forall\, i,j$, then we can write:

$$H = J \sum_i \Bigg[\frac{1}{2}\Big(a_i^+ \exp\{ -i\pi a_i^+ a_i \}\, a_{i+1}$$

$$+ a_i \exp\{ +i\pi a_i^+ a_i \}\, a_{i+1}^+ \Big)$$

$$+ \Delta\left(a_i^+ a_i - \frac{1}{2} \right)\left(a_{i+1}^+ a_{i+1} - \frac{1}{2} \right) \Bigg]$$

$$= J \sum_i \Big[\tfrac{1}{2} \big(a_i^+ a_{i+1} \exp\{-i\pi a_i^+ a_i\} - a_{i+1}^+ a_i \exp\{i\pi a_i^+ a_i\} \big) + \Delta \big(a_i^+ a_i - \tfrac{1}{2}\big)\big(a_{i+1}^+ a_{i+1} - \tfrac{1}{2}\big) \Big]$$

At this point, we can evaluate the phase factors... if we act

$$a_{i+1}^+ a_i |\psi\rangle$$

we must have either 0 or $|\psi\rangle$ must have 1 fermion at i. Therefore, we may write:

$$a_{i+1}^+ a_i \exp\{i\pi a_i^+ a_i\} |\psi\rangle$$

$$= a_{i+1}^+ a_i \exp\{i\pi (1)\} |\psi\rangle$$

$$= a_{i+1}^+ a_i |\psi\rangle$$

Similarily:

$$a_i^+ a_{i+1} \exp\{-i\pi a_i^+ a_i\} |\psi\rangle$$

$$= a_i^+ a_{i+1} \exp\{-i\pi (0)\} \; |\psi\rangle$$

$$= a_i^+ a_{i+1} |\psi\rangle$$

So, finally, we have:

$$H = \frac{N\Delta J}{4} + J \sum_i \Big[\tfrac{1}{2} \big(a_i^+ a_{i+1} + a_{i+1}^+ a_i - \Delta a_i^+ a_i - \Delta a_{i+1}^+ a_{i+1} \big) + \Delta a_i^+ a_i a_{i+1}^+ a_{i+1} \Big]$$

So, we have a model of fermions that interact via nearest neighbour interactions ...

We can diagonalize the single-particle part via Fourier transform

$$a_i = \frac{1}{\sqrt{N}} \sum_k e^{ik \cdot r_i} a_k$$

$$\begin{aligned} H = \frac{\Delta N J}{4} + \frac{J}{N} \sum_i \sum_{kk'} \frac{1}{2} \Big(& a_k^+ a_{k'} e^{-ik\cdot r_i + ik' \cdot (r_i + a)} \\ & + a_k^+ a_{k'} e^{-ik\cdot(r_i+a) + ik'\cdot r_i} \\ & - \Delta a_k^+ a_{k'} e^{-i(k-k')\cdot(r_i)} \\ & - \Delta a_k^+ a_{k'} e^{-i(k-k')\cdot(r_i+a)} \Big) \\ & + J\Delta \sum_i a_i^+ a_i a_{i+1}^+ a_{i+1} \end{aligned}$$

$$\begin{aligned} = \frac{\Delta N J}{4} + \frac{J}{2} \sum_k a_k^+ a_k \left(e^{ika} + e^{-ika} - 2\Delta \right) \\ + J\Delta \sum_i a_i^+ a_i a_{i+1}^+ a_{i+1} \end{aligned}$$

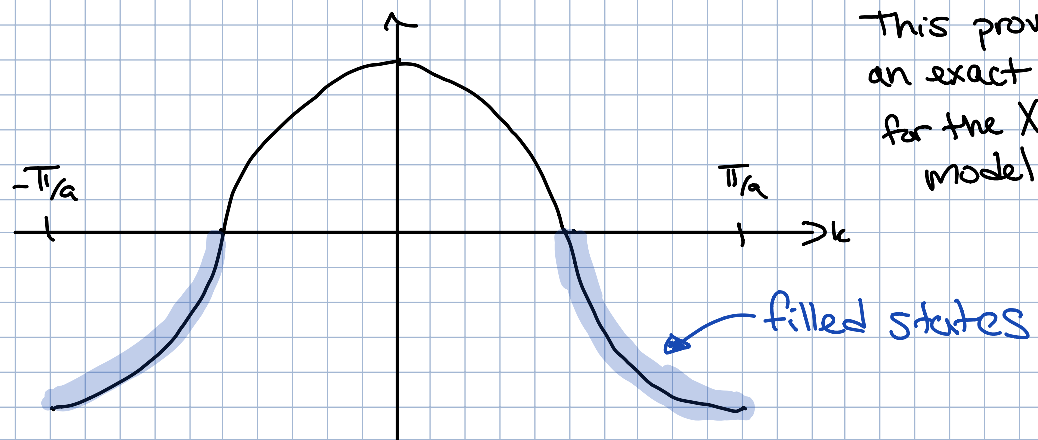
$$\boxed{\begin{aligned} H = \frac{\Delta N J}{4} + \sum_k J(\cos(ka) - \Delta)\, a_k^+ a_k \\ + J\Delta \sum_i a_i^+ a_i a_{i+1}^+ a_{i+1} \end{aligned}}$$

So, we find that we can map the spin model to a model with spinless fermions with nearest neighbor interactions.

Let us first consider the so-called XY-model with $J > 0$ (antiferromagnetic) and $\Delta = 0$.

$$\rightarrow H_{XY} = \sum_k \varepsilon_k a_k^+ a_k \quad ; \quad \varepsilon_k = J\cos(ka)$$

Since these are fermions, we fill up to the chemical potential, which is zero...



This provides an exact sol'n for the XY-model.

Now, let us consider different expectation values:

$$M_z = \langle \sum_i S_i^z \rangle = \langle \sum_i a_i^+ a_i - \tfrac{1}{2} \rangle$$

$$= \langle \sum_k a_k^+ a_k - \tfrac{1}{2} \rangle$$

This measures the number of fermions ... which is $\frac{N}{2}$ ...

$$M_z = \frac{N}{2} - \sum_k \frac{1}{2} \quad = 0$$

We can also compute correlation functions like

$$\langle S_i^x S_{i+r}^x \rangle$$

The calculation is quite "involved", however, we can find

$$\lim_{r\to\infty} \langle S_i^x S_{i+r}^x \rangle = 0$$

So, the XY-model does not have any long-range magnetic order. Order is suppressed by quantum fluctuations...

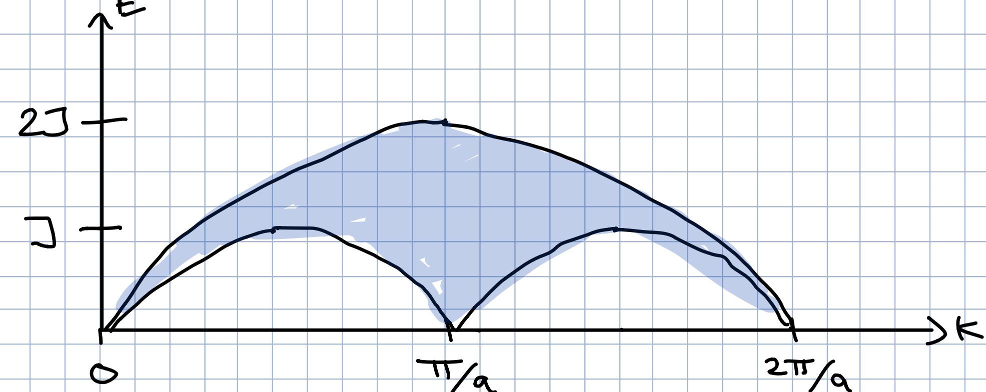
There are different types of excitations...

For example, we can consider excitations that preserve the magnetization... so the total # is constant.

These excitations consist of "particle-hole" excitations where we shift a fermion below the fermi surface to above the ε_F. These are gapless if the momentum transfer is $k = 0, \pi$.

However, there is a <u>continuum</u> of possible excited states at each momentum transfer...

Continuum of excitations



As we increase Δ to approach the Heisenberg model, we introduce interactions. The solution can then be obtained using the <u>Bethe Ansatz</u> which is beyond this course... however, the solution retains many features of the XY-Model, including:

- Since $M_z = 0$ for all Δ between 0 and 1, it follows that the interacting model is $\frac{1}{2}$-filled.
- The sol'n has no long-range order for $0 \leq \Delta \leq 1$. So, correlation functions have $\lim_{r\to\infty} \langle S_i^m S_{i+r}^m \rangle = 0$.
- The excitations form a continuum with similar momentum profile.

Kitaev Honeycomb Model.

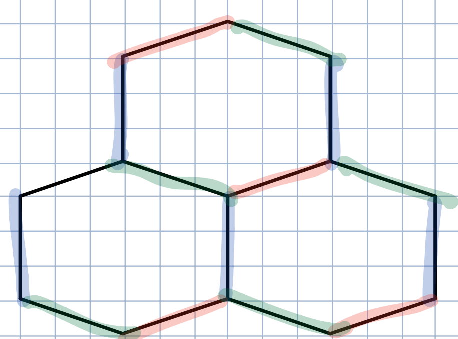
Now, we will consider another model that admits an exact solotion, but uses more "modern" technology:

A. Kitaev

"Anyons in an exactly solvable model and beyond" Annals of physics 321 (2006)

The model:

Consider the following Hamiltonian on the honeycomb (graphene) lattice:



$$H = \sum_{\langle ij \rangle} \sigma_i^\gamma \sigma_j^\gamma$$

where $\sigma_i^\gamma = 2 S_i^\gamma$

and $\gamma = x$ —

y —

z —

We consider an Ising-like Hamiltonian, but a different spin component is coupled on each bond...

In order to diagonalize the Hamiltonian, we will use a specific representation of the spin operators...

Introduce Majorana Operators ... $\{c_i, b_i^x, b_i^y, b_j^z\}$

$$\sigma_i^\gamma = i b_i^\gamma c_i$$

Where these operators satisfy:

(operators anticommute and square to one)

$$c_i c_j = \delta_{ij} + (\delta_{ij} - 1) c_j c_i$$

$$b_i^\alpha b_j^\beta = \delta_{ij}\delta_{\alpha\beta} + (\delta_{ij}\delta_{\alpha\beta} - 1)\, b_j^\beta b_i^\alpha$$

$$c_i b_j^\alpha = -b_j^\alpha c_i$$

We can conceptually think of these operators as being "half of a fermion"

$$c_A = f + f^+ \quad , \quad c_B = \frac{1}{i}(f - f^+)$$

this will turn out to be useful later.

e.g. $c_A c_A = (f + f^+)(f + f^+)$

$$= \underbrace{ff + f^+f^+}_{0} + \underbrace{ff^+ + f^+f}_{1}$$

Now, rewriting the Hamiltonian, :

$$H = \sum_{\langle ij \rangle} (i b_i^\gamma c_i)(i b_j^\gamma c_j)$$

$$= \sum_{\langle ij \rangle} b_i^\gamma b_j^\gamma c_i c_j$$

At first glance, it looks like we haven't done anything useful... we seem to have pure interactions!

However, we can notice that $[H, b_i^\gamma b_j^\gamma] = 0$ for i and j nearest neighbors along the γ-band...

Check $[H, b_i^z b_j^z]$

$$= \sum_{\langle lm \rangle} [b_l^\gamma b_m^\gamma c_l c_m, b_i^z b_j^z]$$

$$= \sum_{\langle lm \rangle} b_l^\gamma b_m^\gamma \underbrace{[c_l c_m, b_i^z b_j^z]}_{\to 0} + \underbrace{[b_l^\gamma b_m^\gamma, b_i^z b_j^z]}_{} c_l c_m$$

here, if $\gamma \neq z$, this is zero..
if $\gamma = z$, then we must have either $\begin{cases} l \neq i \text{ or } j \\ m \neq i \text{ or } j \end{cases} \Rightarrow 0$

or $\begin{cases} l = i \\ m = j \end{cases} \Rightarrow 0$

Since $[H, b_i^z b_j^z] = 0$, eigenstates of H must be eigenstates of $b_i^\gamma b_j^\gamma$ for all bonds.

so, motivated by this, let us introduce:

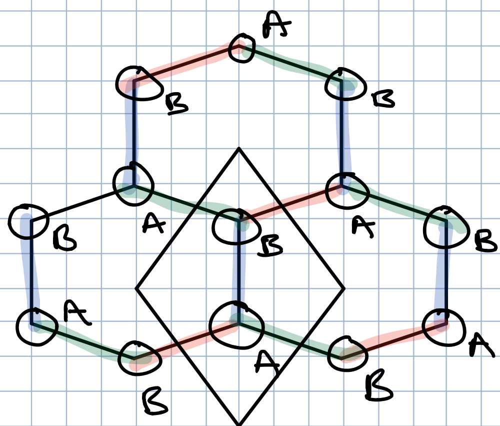
$$u_{ij} = -u_{ji} = i b_i^\gamma b_j^\gamma \quad ; \; (u_{ij})^2 = 1$$

this way $\langle u_{ij} \rangle = \pm 1$

$$H = -i \sum_{\langle ij \rangle}' u_{ij} c_i c_j$$

So, we have managed to write this as a hopping Hamiltonian for majorana c-fermions with a hopping strength determined by the u_{ij} bond variables, which can be ± 1.

To put this into a more familiar representation, let's "glue" the c-fermions in the same unit cell together...



$$H = \sum_{\langle ij\rangle} \sigma_i^\gamma \sigma_j^\gamma$$

where $\sigma_i^\gamma = 2S_i^\gamma$

and $\gamma = x$ ——

y ——

z ——

On A-sublattice: $c_i = f_n + f_n^+$

on B-sublattice: $c_i = \frac{1}{i}(f_n - f_n^+)$

(now, "n" labels unit cell)

Along z-bonds:

$$-i u_{ij}^z \overset{A}{c_i} \overset{B}{c_j}$$

$$= -\cancel{i} u_{ij}^z (f_n + f_n^+) \frac{1}{\cancel{i}} (f_n - f_n^+)$$

$$= -u_{ij}^z (\underbrace{f_n f_n - f_n^+ f_n^+}_{0} + f_n^+ f_n - f_n f_n^+)$$

$$= -u_{ij}^z (2 f_n^+ f_n - 1)$$

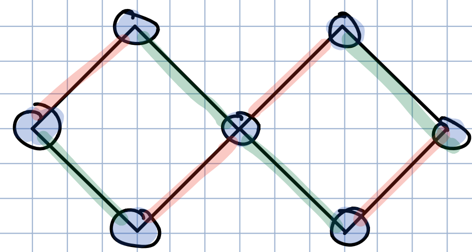
Along x-bonds and y-bonds.

$$-i u_{ij}^{x/y} \overset{A}{c_i} \overset{B}{c_j}$$

$$= -u_{ij}^{x/y} (f_n + f_n^+)(f_m - f_m^+)$$

$$= -u_{ij}^{x/y} (f_n f_m - f_n^+ f_m^+ + f_n^+ f_m - f_n f_m^+)$$

$$= -u_{ij}^{x/y} [f_n^+ f_m + f_m^+ f_n + f_n f_m - f_n^+ f_m^+]$$



$$H = -2\sum_{\substack{\text{unit cells}\\ n}} u_{ij}^{z}\left(f_n^{+}f_n - \tfrac{1}{2}\right) - \sum_{\langle nm\rangle} u_{ij}^{x/y}\left(f_n^{+}f_m + f_m^{+}f_n + f_nf_m - f_n^{+}f_m^{+}\right)$$

These are now spin-less fermions, which look a little like the J-W fermions from the 1D models. However, we also have terms like f^+f^+ and ff. You will learn in the next section of the course that these terms are associated with the fermion pairing in the BCS approach to superconductivity...

$\rightarrow$ sol'n to Kitaev model is like a spinon superconductor...

Great... are we done?

No!

Check whether they satisfy the commutation relations...

$$[\sigma_i^x, \sigma_i^y] = 2i\sigma_i^z$$

(remember $[S_x, S_y] = iS_z$)

$$\begin{aligned}
[\sigma_i^x, \sigma_i^y] &= (ib_i^x c_i)(ib_i^y c_i) - (ib_i^y c_i)(ib_i^x c_i) \\
&= -1\, b_i^x c_i b_i^y c_i + b_i^y c_i b_i^x c_i \\
&= b_i^x b_i^y c_i c_i - b_i^y b_i^x c_i c_i \\
&= 2 b_i^x b_i^y
\end{aligned}$$

We can see that the commutation relation is satisfied only if:

$$\begin{aligned}
i\sigma_i^z &= b_i^x b_i^y \\
i b_i^z c_i &= b_i^x b_i^y \\
(i b_i^z c_i)^2 &= i b_i^x b_i^y b_i^z c_i
\end{aligned}$$

$$\boxed{1 = i b_i^x b_i^y b_i^z c_i \equiv \hat{D}_i}$$

Why do we need this extra constraint?

So, our theory must be invariant under a local $\mathbb{Z}_2$ transformation $D_i^2 = 1$.

However, $[\hat{D}_i, u_{ij}^z]$

$$= [i b_i^x b_i^y b_i^z c_i, i b_i^z b_j^z]$$

$$= -\left(b_i^x b_i^y b_i^z c_i b_i^z b_j^z - b_i^z b_j^z b_i^x b_i^y b_i^z c_i \right]$$

$$= 2 b_i^x b_i^y c_i b_j^z$$

$$= 2 D_i u_{ij}^z \neq 0$$

Acting D_i on a wavefunction flips $\langle u_{ij} \rangle$ on all neighbouring bonds...

In the physical Hilbert space where the commutation relations are preserved, D_i must act as an identity operator, but in our approach, it changes u_{ij} configurations...

WHAT IS GOING ON?

For every unit cell, i.e. two sites...

<u>Physical Space</u>

$$+ \;\; 2 \text{ spins}$$

4 total states per 2N

$\rightarrow (4)^{N/2} = 2^N$ total states

<u>Majorana space</u>

$\{c_A, c_B\}$ majorana = 1 real fermion = 2 states

$$+ \;\; 3 \times u_{ij}^\gamma = 2 \times 2 \times 2$$

16 states per 2N

$\rightarrow (16)^{N/2} = 4^N$ total states

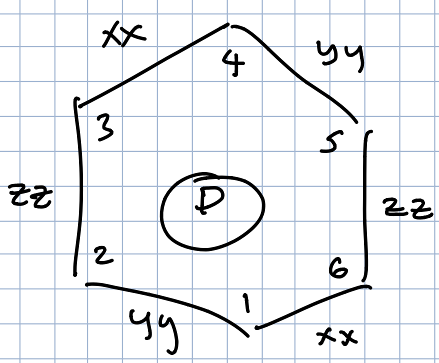
Our effective Hilbert Space is too large...

Some configurations in our expanded Hilbert space correspond to the same physical state!

$\longrightarrow$ we are discussing a $\mathbb{Z}_2$ gauge theory.

We need to have some way to "fix our gauge"

- Similar to E+M, the "$\mathbb{Z}_2$ flux" through any loop is gauge invarient. This is the product of u_{ij} around any closed loop.



For example:

$$W_p = u_{12}u_{23}u_{34}u_{45}u_{56}u_{61}$$
$$= \sigma_1^z \sigma_2^x \sigma_3^y \sigma_4^z \sigma_5^x \sigma_6^y$$

We have $[W_p, H] = 0$, $W_p^2 = \mathbb{1}$, so $\langle W_p \rangle = \pm 1$

eigenstates of H must be eigenstates of W_p

Features of the model:

1. It can be shown (numerically) that the ground state of the model has all $\langle W_p \rangle = +1$. With a particular choice of gauge, the ground state has all $u_{ij} = +1$.

2. There are two types of excitations ...

(a) flipping u_{ij} on one bond $\longrightarrow$ flipping $\langle W_p \rangle$ on two neighboring plaquettes.

(b) making particle-hole excitations in the fermion sector.

3. Since the ground state has $\langle W_p \rangle = +1$ ... we can show:

$$W_p |\Psi_{gs}\rangle = |\Psi_{gs}\rangle$$

$$W_p \sigma_i^\mu W_p = -\sigma_i^\mu \ldots \quad \text{if } i \in P$$

$$\langle \sigma_i^\mu \rangle = \langle \Psi_{gs} | \sigma_i^\mu | \Psi_{gs} \rangle = \langle \Psi_{gs} | W_p \sigma_i^\mu W_p | \Psi_{gs} \rangle$$
$$= -\langle \sigma_i^\mu \rangle$$

$$\longrightarrow \langle \sigma_i^\mu \rangle = 0$$

The ground state cannot have magnetic order.

We can use a similar method to show that $\langle \theta_i^\gamma \theta_j^\gamma \rangle = 0$ if i and j are further than nearest neighbours. So, spin-spin correlations are extremely short-ranged.

How big is our Majorana Hilbert Space?

We could "glue" the majoranas back together in a different way... for example...

At every site, introduce..

$$c_i = f_{i,1} + f_{i,1}^{\dagger} \qquad b_i^z = \frac{1}{i}\left(f_{i,1} - f_{i,1}^{\dagger}\right)$$

$$b_i^y = f_{i,2} + f_{i,2}^{\dagger} \qquad b_i^x = \frac{1}{i}\left(f_{i,2} - f_{i,2}^{\dagger}\right)$$

our Hilbert space is now equivalent to two real fermions per site...

Hilbert space is spanned by

$n_{i,1}$	$n_{i,2}$	
0	0	4 states per site.
0	1	
1	0	
1	1	

But, physical Hilbert space has 2 degrees of freedom, i.e. spin $\uparrow, \downarrow$.

So, our majorana Hilbert space is 2x times too large per site 2^N times