Lecture 1: Special relativity - Tensors

2: Fluids; equivalence principle

3: Curved spacetime and tensors

4: Einstein eqs.; IVP

5: Linearized GWs

Suggested refs:

MTW: "Gravitation"

BFS: "A first course in GR"

Riis: "Introducing Einstein's Relativity"
Special Relativity

It describes the law of physics in the absence of gravitational fields ("flat spacetime").

Postulates:

1) Principle of relativity (Galilean)
   No experiment can measure absolute velocity of an observer (if moving at constant speed).

2) The speed of light is the same for all unaccelerated observers.
   \( c = 2.99 \times 10^{10} \text{ cm/s} \)

\[
F = m \ddot{a} = \frac{dF}{dt}
\]

\[ s \rightarrow s' = s + vt\]

\[
\frac{ds}{dt} = \frac{ds'}{dt} \geq \frac{dt}{dt}
\]

SR selects therefore a class of observers which are special: *inertial* observers.
In GR these observers do not exist (all observers are accelerated) but it is possible to define observers which are inertial at one specific time and position.

An inertial observer defines an inertial "frame" (reference system), i.e. a frame in which spatial distance between points does not change and time points are Euclidean.

This seems obvious but is not longer true in GR.

In GR and G4 concepts are simpler if interpreted geometrically and therefore in a spacetime setting in which time and space are on equal footing and spacetime is a $1+1+1+1 = 4D$ object (manifold).

Our explanation is that of a four spectrum.
Worldline of oscillating body
(accelerated)

Worldline of observer moving past, pos.
speed event

Stationary observer
Consider observer 0 moving at speed $v$ wrt observer $\bar{0}$.

\[ \frac{dx}{dt} = \pm 1 \]

How do the two diagrams compare? Events are the important aspects of this picture of spacetime and coords are just relating to the observers and thus arbitrary.

An angle $\phi$ is present between the two.
and of course, the reverse is also true.

What is important is that the events "distances" measured in the two systems is the same even if the concept of simultaneity is lost.

Another fundamental quantity is the "distance" between two events.

Recall \( \frac{dx}{dt} = \pm 1 \) \( \Rightarrow \Delta x^2 - \Delta t^2 = 0 = \Delta \sigma^2 \)

ie the distance between two events connected by a lightbeam is zero.
This distance is the same in all coasts systems

\[ \Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 = 0 \]

It is therefore sensible to define

\[ \Delta s^2 : \text{distance between two events} \]

\[ \Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \]

\[ \Delta s^2_{AC} < 0 \quad \text{time-like interval} \]

\[ \Delta s^2_{AB} = 0 \quad \text{null} \]

\[ \Delta s^2_{AD} > 0 \quad \text{space-like interval} \]
We can now relate the coordiants in our system to the coordiants in the other one.

\[ -t^2 + x^2 = \text{const} \]

Events are equally distant out this hyp. A&B: simultaneous.

\[ \Delta t > \Delta \bar{t} \]: time dilatation!

\[ \Delta x < \Delta \bar{x} \]: Lorentz contraction!

In general, the transformation is called Lorentz transformation and is given by:

\[
\begin{cases}
\bar{t} = \gamma (t - vx) \\
\bar{x} = \gamma (x - vt) \\
\bar{y} = y \\
\bar{z} = z
\end{cases}
\]

where \( \gamma = \frac{1}{\sqrt{1 - v^2}} \)
to obtain
\[ x = f(z) \]
all is needed is
\[ \sqrt{\frac{W}{1 + W}} \]

Composition of velocities (Galileian)
\[ W = \frac{dx}{dt} = \frac{\bar{W} + \bar{v}}{1 + \bar{W} \bar{v}} \]
only \[ \bar{W} \ll 1 \]

In general the transformation (8) can be written as
\[ X^\alpha = \sum_{\beta=0}^{3} \Lambda^\alpha_\beta \times \beta = \Lambda^\alpha_\beta X^\beta \]
and similarly
\[ X^\alpha = \Lambda^\alpha_\beta \bar{v} \times \bar{\beta} \]

inverse transformation
\[ \Lambda^\alpha_\beta = \Lambda^\alpha_\beta (v) \]: Lorentz transf.
\[ \Lambda^\alpha_\beta = \Lambda^\alpha_\beta (v) \] : inverse of \[ \Lambda^\alpha_\beta \]
\[ = (\Lambda^\alpha_\beta)^{-1} \]
We have seen that what matters are the events and their separation which is independent of coordinates.

This is generally true: we want to write equations in a covariant (i.e., coordinate-independent) manner.

These equations will be valid in any system of coordinates.

To do this, we need to introduce the concept of tensors which make the derivation of covariant expressions very simple.

As a matter of fact, we already have introduced them, although not explicitly.
We can use this notation also to measure distances

\[ ds^2 = \sum_{\alpha, \beta} \gamma_{\alpha \beta} \, dx^\alpha \, dx^\beta \]

where \( \gamma_{\alpha \beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

is another \((4 \times 4)\) matrix

\[ \gamma_{00} = -1 ; \quad \delta_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]

Invariance implies

\[ ds^2 = d\bar{s}^2 = \gamma_{\alpha \bar{\beta}} \, d\bar{x}^\alpha \, d\bar{x}^\beta \]

\( \gamma \) is the \underline{metric} and is a matrix which allows to measure distances.

It's the most important quantity in GR/ER.
Because tensors are generalizations of vectors, let's recall some vector calculus you know well:

\[ \vec{V} = \{ V^x, V^y, V^z \} \]

Moreover, \( \vec{V} \) can be decomposed also in other coordinate systems, e.g., a spherical polar

\[ \vec{V} = \{ V_r, V^\theta, V^\phi \} \]

\[ V^r = |V| \cos \theta' \]
\[ V^\theta = |V| \sin \theta' \]

A bit of algebra shows that:

\[ \begin{align*}
V^x &= V^r \cos \theta + V^\theta \sin \theta \\
V^y &= V^r \sin \theta + V^\theta \cos \theta
\end{align*} \]
so that we can think of a transformation matrix \( \Lambda \)

\[
\mathbf{v}^I = \Lambda^e \mathbf{v}^e \tag{2}
\]

\[
\left( \begin{array}{c}
\mathbf{v}^x \\
\mathbf{v}^y
\end{array} \right) = |\mathbf{v}| \left( \begin{array}{cc}
\cos \theta & \sin \theta \\
\sin \theta & \cos \theta
\end{array} \right) \left( \begin{array}{c}
\mathbf{v}^x \\
\mathbf{v}^y
\end{array} \right)
\]

So for all is well established but for \(|\mathbf{v}|\). What is \(|\mathbf{v}|\)?

It's a measure of the length of \( \mathbf{v} \) and we have seen that \(|\mathbf{v}|\), the metric, does exactly this: measures the distance between two points, i.e., the "tip" and the "end" of the vector \( \mathbf{v} \). It's therefore clear that:

| Our considerations are fully generic and indeed we can extend expression (2) to a 4D spacetime |

\[
\mathbf{v}^a = \Lambda^e \mathbf{v}^e \quad : \text{transf. of vector in 4D}
\]
By analogy with $ds^2$

$$L V = V^a V^b \eta_{ab} = V^a V^b \eta_{ab}$$

frame independent

The length of $LV$ is the same for all coordinate systems.

The metric is therefore an operator (geometric object) that acts on a vector and returns its length:

$$\gamma = \gamma (V, V) = LV$$

If the two slots are occupied by two different vectors, then $\gamma$ returns the length of one in the direction of the other, i.e., the scalar product:

$$V \cdot W = V^a W^b \eta_{ab} = \text{scalar}$$

$$\gamma (V, W) = \text{scalar}$$
the metric is also defined as a \( \frac{\partial}{\partial x} \) tensor and thus as a function that takes as input two vectors and returns a scalar.

The metric is linear in its arguments
\[
y(\alpha U + \beta V, W) = \alpha y(U, W) + \beta y(V, W)
\]
\[
= \alpha y_{\mu \nu} U^\mu W^\nu + \beta y_{\mu \nu} V^\mu W^\nu
\]

To understand what the components \( y_{\mu \nu} \) are it is useful to consider the definition of a vector

\[
V = V^\mu e_\mu
\]
where \( e_\alpha \) : vector basis

bar goes here

\[
e_\alpha = \{e_0, e_1, e_2, e_3\}
\]
\[
e_0 = \{1, 0, 0, 0\}
\]
\[
e_1 = \{0, 1, 0, 0\}
\]
\[
e_2 = \ldots
\]
\[
e_3 = \{0, 0, 0, 1\}
\]
Let them have $\gamma$ act on $e_{\alpha}$

$$\gamma (e_{\alpha}, e_{\beta}) = e^\mu_{\alpha} e^\nu_{\beta} \eta_{\mu\nu}$$

$$= \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} \eta_{\mu\nu}$$

$$= \gamma_{\alpha\beta}$$

In other words: $\gamma_{\alpha\beta}$ are the components of the metric tensor in a coordinate vector basis.

Clearly, the metric tensor is symmetric in its arguments

$$\gamma (A, B) = \gamma (B, A)$$

In a similar way we can define a $(0, 1)$ form as a function (operator) that acting on a vector returns a number (scalar)

$$\beta (V) = p^a V^a$$
The graphical representation of a $(1)$-form or one-form is via surfaces.

So that \( \hat{p}(V) \) is the number of surfaces crossed by \( V \). The larger this number, the smaller the spacing among surfaces and the larger the magnitude of the one-form.

Just like vectors, one-forms have components and basis:

\[
\hat{p}(V) = p_a \cdot V^a : p_a \text{ are the components of } \hat{p}.
\]
\( \vec{p}(e_\alpha) = p_\alpha e^\alpha = p_\alpha \delta^\alpha_\alpha = p_\alpha \)

ie \( p_\alpha \) are the components of \( \vec{p} \) in the coordinate vector basis.

I can also write \( \vec{p} = p_\alpha \tilde{\omega}^\alpha \)

and derive that

\[ p(V) = p_\alpha v^\alpha = p_\alpha \tilde{\omega}^\alpha(V) = p_\alpha \tilde{\omega}^\alpha(v^\beta e_\beta) \]
\[ = p_\alpha v^\beta \tilde{\omega}^\alpha(e_\beta) \]

\[ \Rightarrow \tilde{\omega}^\alpha(e_\beta) = \delta^\alpha_\beta \]

\( \tilde{\omega}^0 = \{1, 0, 0, 0\} \)
\( \tilde{\omega}^1 = \{0, 1, 0, 0\} \)
\( \tilde{\omega}^2 = \{0, 0, 1, 0\} \)
\( \tilde{\omega}^3 = \{0, 0, 0, 1\} \)

in other words \( \tilde{\omega}^\alpha \) are dual to \( e_\alpha \)

It's clear that \( \tilde{\omega}^\alpha \) are dual to \( e_\alpha \) vectors and just different indices of the same coin.

At this point that one-forms are dual to vectors and just different indices of the same coin.
Note that a one-form can be applied also to a function, in which case it represents the gradient of that function.

Consider a curve $\mathcal{C}: \{ x^\mu(t) \}$:

$x^\mu(t)$ could be the worldline of an observer and $t$ the proper time.

\[
\frac{dx^\mu}{dt} = u^\mu(t) : \text{tangent vector of } \mathcal{C} \text{ and frame velocity of the observer}
\]

Let $\phi = \phi(x^\nu)$ a scalar function along $\mathcal{C}$ and thus

\[
\phi(x^\nu) = \phi(x^\nu(t)) = \phi(t) \tag{2}
\]

\[
\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x^\mu} \frac{dx^\mu}{dt} = \partial_\mu \phi \frac{dx^\mu}{dt} = \partial_\mu \phi \ u^\mu
\]

Thus $\phi_\mu = \frac{\partial \phi}{\partial x^\mu} = \text{gradient of } \phi$ is a one-form.
\[ \frac{\partial}{\partial t} \phi = (2x \phi, 2y \phi, 2z \phi) \]

\[
\vec{V} \\
\vec{W}
\]

\[ \phi \text{ could be a pressure at a given altitude} \]

\[ \phi \text{ has little sense} \]

\[ \hat{T} \phi (V) = \partial \mu \phi \cdot V^\mu = V^\mu \partial \mu \phi \]

\[ = \text{gradient of } \phi \]

\[ \text{along } V \text{ is a meaningful number} \]

In the example above

\[ V^\mu \partial \mu \phi \gg W^\mu \partial \mu \phi \]

\[ \text{steep gradient in the North direction} \]
Recap

- $\gamma : (2)\text{-form, eg metric tensor}$
  \[ \gamma (v, w) = \gamma^{\alpha \beta} v_{\alpha} w_{\beta} \]

- $\hat{p} : (0)\text{-form, eg gradient}$
  \[ \hat{p}(v) = p_{\alpha} v^{\alpha} = v^{\alpha} \partial_{\alpha} \phi \]

- $\nabla : (1)\text{-form, ie vector and dual to one-form}$
  \[ \nabla (\hat{p}) = \nabla^{\alpha} p_{\alpha} = \hat{p}(v) \]

- $(M)\text{-form: operator combining}$
  \[ \text{M one-forms with N vectors} \]
  \[ R (\hat{p}, v, w) = R^{\alpha}_{\beta \delta \gamma} p_{\alpha} u^{\beta} v^{\delta} w^{\gamma} \]
Raising-lowering indices

We have seen that the metric tensor acts on vectors to yield a scalar but it can also act on one vector to yield a one-form

\[ \gamma (\mathbf{V}, \mathbf{W}) = \gamma_{\alpha\beta} V^\alpha W^\beta \]

\[ \Rightarrow \]

\[ \gamma (\mathbf{V}, \mathbf{W}) = \gamma_{\alpha\beta} V^\alpha = W_\beta \text{ : one-form} \]

If we act \( \mathbf{\nabla} \) on another vector, \( \mathbf{W} \), then we must end up with \( V_\alpha W^\alpha \)

\[ \nabla (\mathbf{V}, \mathbf{W}) = \nabla_{\alpha\beta} V^\alpha W^\beta = \gamma_{\alpha\beta} V^\alpha W^\beta \]

\[ \Rightarrow \]

\[ \gamma (\mathbf{V}, \mathbf{W}) = \gamma_{\alpha\beta} V^\alpha W^\beta = W_\beta \text{ : one-form} \]

\[ \boxed{\gamma_{\alpha\beta} V^\alpha = W_\beta} \]: the metric has mapped a vector into a one-form (lowered the index)
The formula is also true

\[ V^\beta = \eta^{\alpha \beta} V_\alpha \]

\[ \eta^{\alpha \beta} \text{ is inverse of } \eta_{\alpha \beta} \]

\[ \eta^{\alpha \beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

This can be done over and over again

\[ R^x_{\beta \gamma \delta \rho} p^\alpha U^\beta V^\gamma W^\delta = \]

\[ R^x_{\beta \gamma \delta \rho} p^\mu \eta^{\mu \alpha} U_\nu \eta^{\nu \beta} V^\rho W^\delta = \]

\[ \frac{1}{p^2} U^\beta \]

\[ \cdots R_{\mu \nu} = p^\mu U_\mu V^\nu W^\delta \]
A collection of particles whose properties can be defined in terms of averages performed on scales sufficiently large to contain a large (statistically significant) sample but as small as possible to allow for a continuum description.

\[ p(x), \quad f(x), \quad \sim \]

\[ l \text{ sufficiently large to contain a large number of particles} \]

\[ l \gg \ell : \text{mean free path and this implies that properties are constant in } \ell \]

\[ l \ll L : \text{scale of the problem} \]

These definitions are clearly weak and mathematically vague but for very large number of

\[ \hat{p}, \text{ mean velocity, density, etc...} \]
particles of ordinary fluids make them very unstable
Particles of ordinary fluids make them very natural. Note that I consider "fluids" as what is usually referred to as a gas. Given a single particle of the fluid, there are two important associated quantities:

1) $u^\alpha$: four velocity

$$u^\alpha = \frac{dx^\alpha}{dt}$$

2) proper time

$$ds^2 = -dt^2$$

Then the 4-velocity obeys the normalization condition:

$$u^\alpha u_\alpha = -1$$

Proof:

$$ds^2 = -dt^2 \Rightarrow \frac{ds^2}{dt^2} = -1$$

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \Rightarrow \frac{ds^2}{dt^2} = \eta_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}$$

$$= \eta_{\alpha\beta} u^\alpha u^\beta = u^\alpha u_\alpha = -1$$
Note

$u$ is called 4-velocity because the spatial components are those of a 3-velocity for small speeds.

In a momentarily comoving reference frame (MCRF)

$$u^\mu_{\text{MCRF}} = \{1, 0, 0, 0\} \quad u^\mu u_\mu = -1$$

$u = u^0 e^0$: along time direction

$$(u^0)^2 y_{00} = -1$$

In a frame which is moving at speed $v$ in the x-direction, then (i.e., a general frame at speed $v$)

$$u^\mu = \{x, y, z, 0\}$$

as deduced by Lorentz transform.

$$u^x = x v = \frac{v}{\sqrt{1 - v^2}} = \sqrt{1 + \frac{v^2}{2}} - \sqrt{\frac{2}{2}} \approx v$$
2) $p^\mu$: four-momentum

$$p^\mu = m_0 u^\mu$$

$$p^\mu_{\text{MCREF}} = m_0 u^\mu_{\text{MCREF}} = m_0 \left[ 1, 0, 0, 0 \right]$$

\(m_0\) is the "rest-mass"

(i.e. the mass in the frame in which it is at rest)

In general, for a particle moving in x-direction

$$p^\mu = m_0 \left[ 1, \gamma, \gamma \gamma^*, 0, 0 \right]$$

$$p^\mu = m_0 \gamma = m_0 \frac{\gamma}{\sqrt{1 - \gamma^2}} = m_0 + m_0 \gamma^2$$

because of this

$$p^\mu = E \text{ : energy of the particle}$$

$$p_0 = m_0 \gamma \gamma^* = -E$$
Note that $u$ cannot be defined for a photon $d\xi^2 = 0 = -dt^2$

$$u = \frac{dx \cdot dx}{dt dt} = \frac{0}{0}$$

This doesn't imply that there isn't a tangent to the trajectory of a photon, which is indeed given by $dx$.

Rather, it suggests that it is not for an observer to boost to a speed in which the photon is at rest. The speed of light is $c$ for all.
It's easy to show that the energy of the particle relative to our observer with four-velocity \( U \)
is \[ E = -p \cdot U = -p^\alpha u_\alpha. \]

(*) must be valid for any observer and so also for a MCRF one
\[
= -p^\alpha u^\beta_{\text{MCRF}} \gamma_{\alpha\beta}
= -p^0 u^0_{\text{MCRF}} \gamma_{00} = p^0 u^0_{\text{MCRF}}
= p^0 = E \quad \text{qed}
\]

If \( U \) and \( p \) are useful quantities for a single particle, there is one which is useful also in terms of a collection of particles: number flux density.

To simplify our analysis, let's consider a very special fluid: dust, i.e. a collection of identical particles which
... all at rest in one inertial frame (this is also referred to as a zero-pressure fluid; more later) let N be the number of particles and n its number density

\[ n = \frac{N}{V} \]

In another inertial frame moving at speed \( V \) the number density will be \( \bar{n} = \gamma n \)

\[ \bar{n} = \gamma n \]

ie the number density increases because of Lorentz contraction while \( N \) remain the same.

Similarly we can measure the number flux

\[ (\text{flux}) \times = \frac{\text{number of particles}}{\text{unit area} \cdot \text{unit time}} \]
\[(\text{flux})^x = \frac{\gamma n \cdot \delta A}{\Delta t \cdot A}\]

I can therefore define the \( N \): number vector

\[N = n u\]

\[N^\mu = n \begin{pmatrix} 1 & \frac{v}{c} & \frac{v}{c} & 0 \end{pmatrix} \]

\[\uparrow \text{number flux across surface } x = \text{const}\]

\[\text{number density}\]

Note that \( N \cdot N = N^\mu N^\mu = -n^2 \)

\[n = (-N \cdot N)^{1/2}\]

\[\uparrow \text{number density in the MCRF}\]

\[\text{just like test mass is the one in the MCRF}\]
If a vector $\vec{N}$ is sufficient to measure the number of particles moving in a certain direction, it's clear that to measure the flux of momentum component in a specified direction we need a tensor of rank 2

$$T^{\alpha\beta} = e u^{\alpha} u^{\beta} : \text{stress-energy tensor for dust}$$

$$T^{00} = e u^{0} u^{0} = \gamma^{2} e = \frac{\beta}{\sqrt{1 - \beta^{2}}^{2}} : \text{energy density}$$

$$T^{0x} = e u^{0} u^{x} = \gamma^{2} e \sqrt{x} = \frac{\gamma^{2} e \sqrt{x}}{1 - \beta^{2}}: \text{energy flux}$$

$$T^{xx} = e u^{x} u^{x} = \gamma^{2} e (\gamma x)^{2} : \text{momentum flux}$$

Note that $\gamma$ and $\gamma x$ are distinct:

$\gamma$: velocity of observer

$\gamma x$: $\gamma x$ of dust particles
what is \( e \)?

\[
T_{\text{MCRF}} = e u^0 u^0 = e : \text{energy density}
\]

\[
= e (1 + e) \quad \text{specific internal energy}
\]

Note that the momentum flux is equivalent to pressure, indeed the pressure is the manifestation of a flux of momentum across a surface

\[
\text{pressure} = \frac{\text{force}}{\text{area}} = \frac{\text{momentum}}{\text{time} \cdot \text{area}} = \frac{\text{momentum}}{\text{time} \cdot \frac{\text{volume}}{\text{distance}}}
\]

\[
= \frac{\text{momentum} \cdot \text{velocity}}{\text{volume}}
\]

\[
= (\text{mom. dens}) \cdot \text{velocity}
\]

More generally

\[
T_{ij} : \text{energy flux in } x^i \text{ direction}
\]

\[
T_{ij} : (\text{mom. density flux})^j \text{ in the } x^i \text{ direction}
\]
The extension of the stress-energy tensor to a perfect fluid (i.e., a fluid with zero viscosity and heat losses) is straightforward. It contributes to energy density (same dimensions):

\[ T^{\alpha \beta} = (e+p) u^\alpha u^\beta + p g^{\alpha \beta} \]

\[ T^{00} = (e+p) u^0 u^0 + p g^{00} \]

\[ = \gamma^2 (e+p) - p \]

\[ \sim (1 + \gamma^2) (e+p) - p = e (1 + \gamma^2) + p \gamma^2 \]

\[ e \ll \rho \quad \text{and} \quad p \ll \rho \]

\[ \approx \frac{\rho}{(1 - \gamma^2) u^2} + \frac{p \gamma^2}{\rho} \]

\[ T^{0k} = (e+p) u^0 u^k = (e+p) \frac{\gamma^2 v^k}{\text{imortal mass-energy density}} \]

\[ T^{ij} = (e+p) \delta^{ij} v^j v^k + p \delta^{jk} \]

\[ j^{ik} \mid = (e+p) \frac{\gamma^2 (v^k)^2 + p}{\text{isochoric pressure contribution}} \]
Conservation laws

We want to derive expressions that quantify the conservation of baryon number, energy and momentum.

Let's consider a MCRF

\[ \frac{\partial}{\partial t} \left( n \frac{N}{2} \right) = (\text{flux-in}) - (\text{flux-out}) \]

\[ \frac{\partial}{\partial t} (n \nu_x A \Delta t)_{x=0} - (n \nu_x A \Delta t)_{x=L} + (l_{y=L}) - (l_{y=L}) + (l_{z=L}) - (l_{z=L}) \]

\[ = - \frac{1}{\text{Vol}} \frac{\partial}{\partial x} (n \nu_x) A \Delta t \text{L} - \frac{\partial}{\partial y} (n \nu_y) \text{L} - \frac{\partial}{\partial z} (n \nu_z) \text{L} \]

\[ \Rightarrow \frac{\partial}{\partial t} n \nu \approx - \frac{\partial}{\partial x} (n \nu_x) - \frac{\partial}{\partial y} (n \nu_y) - \frac{\partial}{\partial z} (n \nu_z) \]
\[ \frac{\partial (nu^\alpha)}{\partial x^\alpha} = 0 \]  

Rest mass conservation

Similarly, one can write for the energy and momentum

\[ \frac{\partial (T^\alpha{}\beta)}{\partial x^\alpha} = 0 \quad \beta = 0 : \text{energy} \]

\[ \beta = k : \text{momentum} \quad \text{(Enler eq.)} \]

These equations are those of special relativistic hydrodynamics and can also be written as

\[ \partial_\alpha (nu^\alpha) = 0 \iff (nu^\alpha), \alpha = 0 \]

\[ \partial_\alpha (T^\alpha{}\beta) = 0 \iff (T^\alpha{}\beta), \alpha = 0 \]

The same conservation laws can be defined for EM

\[ T^{\alpha\beta}_{\text{EM}} = \frac{1}{\mu_0} \left[ F^{\alpha\mu\nu} F_{\mu\nu} - \frac{1}{4} \eta^{\alpha\beta} F_{\mu
u} F^{\mu
u} \right] \]

where \( F_{\alpha\beta} = 2 u^\alpha (E_\beta) - \eta_{\alpha\beta} u_\gamma u^\gamma \quad \text{(Faraday tensor)} \)
Symmetric and antisymmetric tensor

\[ T_{\alpha\beta} = T_{\beta\alpha} : \text{sym.} \]

\[ T_{\alpha\beta} = -T_{\beta\alpha} : \text{anti-sym.} \]

\[ T[\alpha\beta] = \frac{1}{2} (T_{\alpha\beta} - T_{\beta\alpha}) : \text{anti-sym} \quad \text{by construction} \]

\[ T(c) = \frac{1}{2} (T_{\alpha\beta} + T_{\beta\alpha}) \]

\[ \text{Ex.} \]

\[ F_{\alpha\beta} = 2 u[\alpha E_\beta] + \eta_{\alpha\beta\gamma\delta} u^\gamma B^\delta \]

\[ = u^\alpha E_\beta + u_\beta E_\alpha + \eta_{\alpha\beta\gamma\delta} u^\gamma B^\delta \]
SR relied on the existence of inertial frames, i.e., set of coordinates such that the separation among points does not change in space and the time intervals are the same everywhere.

\[
\begin{align*}
\Delta x_1 &= \Delta x_2 \\
\Delta t_1 &= \Delta t_2
\end{align*}
\]

Different observers in relative motion will each have his own inertial frame, and a Lorentz transformation takes us from one to the other:

\[
\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu
\]

This mental construction is incompatible with the presence of a gravitational field. However, it is not totally useless...
GPS system is already a clear proof but to be convinced one can perform the "gravitational redshift" experiment first suggested by Einstein.

- take a particle of mass \( m \) and let it fall from height \( h \)
- on the ground it will have energy \( m + \frac{mV^2}{2} = m + mgh \)
- convert all the energy in a single photon of energy \( E = hv = m + mgh \) and send it back!
- convert the photon into a particle of mass \( m' = E' = hv' \)

Because we have not created energy \( m' = m \)

\[
\frac{E'}{E} = \frac{hv'}{hv} = \frac{m}{m + mgh} \Delta \mu - gh + o(V^4)
\]

\( E' < E \) or \( V' < V \) the photon at the top of the tower has been redshifted.
This result is incompatible with the existence of an inertial frame on Earth. In such a frame, in fact, the clocks at the bottom and top of the tower should tick at the same rate:

\[ \frac{\Delta t'}{\Delta t} = 1 = \frac{\nu^*}{\nu'} \]

but we have seen \( \nu' < \nu \Rightarrow \Delta t' > \Delta t \)

This shows that a global inertial frame is incompatible with a gravitational field but does not exclude the existence of a local inertial frame, i.e., the possibility of having an inertial frame in the neighborhood of any event.

This is indeed what expressed by the equivalence principle.
If gravity makes an inertial frame impossible, the way to produce an inertial frame is to remove gravity, i.e., to be in free fall.

The two frames are equivalent and inertial. The problem is that in both cases they are not global (i.e., they exist only in a portion of the spacetime).

This is the equivalence principle: the laws of physics in a free-falling frame are the same as in an inertial frame (i.e., in GR).

Note that there is the opposite (weak) equivalence principle.
Starting the equivalence between an accelerated frame and gravity.

The "height" in the two frames is the same.

How do we reveal the existence of gravitational fields?

Free particles in SR move on straight lines so that two particles on initially parallel lines do not ever intersect.

Loss of an inertial frame in a gravitational field will imply loss of parallelism and hence we can associate gravity with curvature.
The reason for this is that parallel lines intersect in a curved space.

The logic is therefore:

1. (Gravitational field) \leftrightarrow (Loss of global inertial frame)
2. (Loss of parallelism for free particles) \leftrightarrow (Presence of curvature)
3. (Parallel) \leftrightarrow (Intersect)
Before discussing the relation between gravity and curvature (i.e. Einstein equations), we need to define the mathematical tools necessary for its measurement.

The most important of these tools is the **covariant derivative**, i.e. a derivative that accounts also for the fact that the co-ords can change locally.

Note that we do not really need a curved space to define the covariant derivative.

Suppose we want to compute

\[
\frac{\partial}{\partial x^a} V^\beta = \frac{\partial}{\partial x^a} (V^\beta e_\beta)
\]

\[
= \frac{\partial V^\beta}{\partial x^a} e_\beta + V^\beta \frac{\partial e_\beta}{\partial x^a}
\]

- **Standard derivative**
  \[\frac{\partial V^\beta}{\partial x^a} e_\beta\]
- **New term** accounting for change of basis vectors
  \[V^\beta \frac{\partial e_\beta}{\partial x^a}\]
Let's restrict to 2D, cartesian coords $(x, y)$.

\[ ds^2 = g_{xx} \, dx^2 + g_{yy} \, dy^2 \]

\[ g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (g_{ij})^{-1} = g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ V^k \frac{\partial}{\partial x^k} \rightarrow V^j \frac{\partial}{\partial x^j} \]

If $i = x$:

\[ V^x \frac{\partial}{\partial x} + V^y \frac{\partial}{\partial y} = 0 \]

because both $e_x$ and $e_y$ do not vary in space (they're 1).

Let's consider in the same plane a polar coordinate system $(r, \theta)$.

\[ ds^2 = g_{rr} \, dr^2 + g_{\theta \theta} \, d\theta^2 = dr^2 + r^2 \, d\theta^2 \]

\[ g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \]
\[ \Rightarrow \quad \mathbf{e}_r \cdot \mathbf{e}_r = 1 \quad \mathbf{e}_\theta \cdot \mathbf{e}_\theta = r^2 \]

\[ \mathbf{e}_r \cdot \mathbf{e}_\theta = 0 \]

A bit of algebra shows that:

\[ \frac{\partial}{\partial r} \mathbf{e}_r = 0 \quad \frac{\partial}{\partial \theta} \mathbf{e}_r = \frac{1}{r} \mathbf{e}_\theta \]

\[ \frac{\partial}{\partial r} \mathbf{e}_\theta = \frac{1}{r} \mathbf{e}_\theta \quad \frac{\partial}{\partial \theta} \mathbf{e}_\theta = -r \mathbf{e}_r \]

What matters here is that \( \frac{\partial}{\partial x} \mathbf{e}_j = 0 \) so the derivative has a new term accounting for the change in the curvilinear coords.
Let's go back to eq.

\[ \frac{\partial e_\beta}{\partial x^\alpha} = \left( \Gamma^m_{\alpha \beta} \right) e_\mu \]

\[ \Rightarrow \]

\[ \frac{\partial \bar{V}}{\partial x^\alpha} = \frac{\partial V^\beta}{\partial x^\alpha} e_\beta + \nabla^m \Gamma^\beta_{\alpha \mu} e_\mu \]

\[ = \frac{\partial V^\beta}{\partial x^\alpha} e_\beta + \nabla^m \Gamma^\beta_{\alpha \mu} e_\mu \]

\[ = \left( \frac{\partial V^\beta}{\partial x^\alpha} + \nabla^m \Gamma^\beta_{\alpha \mu} \right) e_\beta \]

\[ \implies \quad V^\beta \nabla_\alpha = \nabla_\alpha V^\beta \]

ie

\[ V^\alpha \nabla_\beta = V^\alpha \beta + \Gamma^\alpha_{\beta \mu} V^\mu = \nabla_\beta V^\alpha \]
A corresponding derivative can be derived for a one-form

\[ \nabla_\beta V^\alpha = V^{\alpha,\beta} = V^\alpha,\beta - \Gamma^\alpha_{\beta\gamma} V^\gamma \]

or a rank-2 tensor

\[ \nabla_\alpha V^\mu_\nu = V^\mu_\alpha \gamma + \Gamma^\mu_{\alpha\beta} V^\beta_\nu + \Gamma^\mu_{\beta\nu} V^\beta_\alpha \]

\[ \Gamma^\alpha_{\beta\gamma} : \text{Christoffel symbol} \]

(affine connection)

\[ \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} : \text{sym. on lower indices} \]

Note that \( \Gamma^\alpha_{\beta\gamma} \) is not a tensor as it does not transform as a tensor.

Let's consider an example: 2D, \( \{r, \theta\} \)

\[ = V^r; r + \Gamma^r_{\beta\alpha} V^\beta \]

\[ = V^r; r + V^\theta; \theta + \Gamma^r_{\theta\alpha} V^r + \Gamma^\theta_{\theta\alpha} V^\theta \]
\[ \Gamma^\alpha_{r\alpha} = \Gamma^r_{rr} + \Gamma^\theta_{r\theta} = \frac{1}{r} \]
\[ \Gamma^\alpha_{\theta\alpha} = \Gamma^r_{\theta r} + \Gamma^\theta_{\theta\theta} = 0 \]

\[ = \nabla^r r + \nabla^\theta \theta + \frac{1}{r} \nabla^r r \]
\[ = \frac{1}{r} \partial_r (r \nabla^r r) + \partial_\theta \nabla^\theta \theta \]

: well-known expression of the divergence in polar co-ords.
A very important covariant derivative is that of the metric

\[ g_{\mu\nu;\alpha} = 0 \] (3)

This is easy to show and there are two different ways at least:

\[ = V_{\alpha;\beta} = \left( V^\mu \partial_\mu \right)_{;\beta} = V^\nu \partial_\nu g_{\mu\mu} + V^\mu \partial_\mu g_{\nu\nu} \]

\[ = (V^\nu;\beta) g_{\mu\nu} = V^\nu \partial_\nu g_{\mu\mu} \]

\[ = V^\mu \partial_\mu g_{\nu\nu} \]

\[ + V^\nu \partial_\nu g_{\mu\mu;\beta} \]

\[ V^\nu g_{\mu\mu;\beta} = 0 \implies g_{\mu\mu;\beta} = 0 \quad \text{q.e.d.} \]

The second one is that (3) is a tensor equation and thus valid in all frames.

If \( \mu \) go to a Cartesian coord system then \( g_{\mu\beta,\mu} = 0 \) \( \quad \Gamma^\nu_{\mu\beta} = 0 \implies g_{\mu\beta;\mu} = 0 \)

Another possibility is to see \( g_{\mu\nu;\alpha} \) as the components of a tensor \( \nabla g \) and if zero in one frame they are zero in all frames.
How do we calculate the $\Gamma$s?

Exploiting (3):

\[ g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^\rho_{\mu\lambda} g_{\nu\rho} - \Gamma^\rho_{\nu\lambda} g_{\mu\rho} = 0 \]  \hspace{1cm} (a)

\[ g_{\nu\lambda;\mu} = g_{\nu\lambda,\mu} - \Gamma^\rho_{\nu\mu} g_{\lambda\rho} - \Gamma^\rho_{\nu\lambda} g_{\mu\rho} = 0 \]  \hspace{1cm} (b)

\[ g_{\lambda\mu;\nu} = g_{\lambda\mu,\nu} - \Gamma^\rho_{\lambda\nu} g_{\mu\rho} - \Gamma^\rho_{\lambda\mu} g_{\nu\rho} = 0 \]  \hspace{1cm} (c)

\[(b) + (c) - (a) \Rightarrow \]

\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\delta} \left( \partial_\mu g_{\nu\delta} + \partial_\nu g_{\mu\delta} - \partial_\delta g_{\mu\nu} \right) \]

It looks complicated but it is just algebra. Just remember that $\Gamma \rightarrow$ partial derivatives of the metric
At this point we need a "curvature detector" i.e. an object that allows us to detect that there is curvature.

Any idea?

I can move a vector along a closed loop making sure to keep it always parallel to the same direction. This is a "parallel transport." In a flat spacetime the result of this operation yields the same vector but this is clearly not the case if the surface is curved.

The point is that one must always "stay" on the surface (be tangent) and have the vector drawn as parallel as possible to the previous one.
Initial and final vectors are clearly different.

Careful on your intuition of curved!

A cylinder is **intrinsically** flat: you can open and lay on a flat surface without creases.
We need now a definition of "parallel transport".

We want to move a vector along a curve without changing it. A curve is defined by its tangent vector

\[ U = \frac{dx^m}{dt} \]

\[ e^k(x) \]

(4) \[ \nabla_u V = 0 \] : the variation of \( V \) along the curve with \( V \) and vector are zero.

In coordinate form

\[ V^\alpha;_\beta U^\beta = 0 = V^\alpha,_{\beta} U^\beta + \Gamma^\alpha_{\beta \mu} U^\mu U^\beta \]

The concept of parallel transport allows also to define a "straight" curve.
A "straight" curve is a curve that parallel transports its tangent vector

\[ \nabla_{\mu} U^{\mu} = 0 \]

\[ U^{\alpha};_{\beta} U^{\beta} = U^{\alpha};_{\beta} U^{\beta} + \Gamma^{\alpha}_{\beta \mu} U^{\mu} U^{\alpha} \]

but \[ U^{\alpha} = \frac{dx^{\alpha}}{d\tau} \]
\[ U^{\alpha};_{\beta} = \frac{dx^{\alpha}}{d\tau} \frac{d}{d\tau} = \frac{d}{d\tau} \]

\[ = \frac{d}{d\tau} \left( \frac{dx^{\alpha}}{d\tau} \right) + \Gamma^{\alpha}_{\beta \mu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0 \quad (4) \]

This is a curve of "extremal" (smallest) length, i.e. a geodesic.

Note that a geodesic is "straight" only in flat spacetime and remain the curve of minimal distance in any spacetime.

If \( \Gamma^{\alpha}_{\mu \beta} = 0 \), \( (4) \Rightarrow (x^{\alpha})'' = 0 \Rightarrow (x^{\alpha})' = \text{const} \Rightarrow x^{\alpha} = k^2 \]
straight line.
We can finally define the "curvature" following the same logic: move along a closed loop parallelly transporting a vector.

\[ V_{\alpha \beta} V^M = V^M [\partial x \beta] + \frac{1}{2} (V^M, \alpha \beta - V^M, \beta \alpha) \]

\[ = \frac{1}{2} R^M_{\alpha \beta} \cdot V^\rho 
\]

\( R \) is the Riemann tensor.

Note that while \( V^M, \alpha \beta = V^M, \beta \alpha \), I cannot claim that \( V^M; \alpha \beta = V^M; \beta \alpha \).
$R$ is just a combination of $T$s, i.e. 1st and 2nd order terms of the metric

$$R^\mu_{\gamma\rho\sigma} = \Gamma^\mu_{\gamma\rho,\sigma} - \Gamma^\mu_{\gamma\rho,\sigma} + \Gamma^\mu_{\rho\sigma} \Gamma^{\rho\sigma} - \Gamma^\mu_{\rho\sigma} \Gamma^{\rho\sigma}$$

Note $R^a_{\beta\rho\sigma} = 0 \iff \nabla^a V^\rho \nabla^\rho V^\sigma = 0$

(zero curvature) $\iff$ (no difference in how sp. vector)

Properties of Riemann tensor

1) Anti-sym. on last pair of indices

$$R^\rho_{\beta\sigma\delta} = - R^\rho_{\sigma\delta\beta}$$

2) Sym. on exchange of 1st and 2nd pair of indices

$$R^\rho_{\beta\sigma\delta} = R^\rho_{\sigma\delta\beta}$$

3) Anti-sym. on last 3 indices

$$R^\rho_{\beta\sigma\delta} = 0$$
4) **Bianchi Identity**

\[ R^\gamma_{\beta [\rho \delta ; \mu ; \nu]} = 0 \]

\[ R^\gamma_{\beta \rho \delta ; \mu} + R^\gamma_{\beta \mu \delta ; \rho} + R^\gamma_{\beta \delta \rho ; \mu} = 0 \]

The Bianchi identities also define another tensor equation

\[ G^{\alpha \beta} ; \beta = 0 \quad \text{where} \]

\[ R^\alpha_\beta - \frac{1}{2} R g^{\alpha \beta} = G^{\alpha \beta} \quad \text{: Einstein tensor} \]

\[ R^{\alpha \beta} = R^\mu_{\alpha \mu \beta} = R_{\beta \alpha} \quad \text{: Ricci tensor} \]

\[ R = R^\nu_{\alpha \nu} = R^\mu_{\alpha \mu \nu} \quad \text{: Ricci scalar} \]

\( \sqrt{\text{single number containing all the information on curvature}}. \)
Let's consider a concrete example a 2-sphere

\[ ds^2 = R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

\[
\begin{pmatrix}
R^2 & 0 \\
0 & R^2 \sin^2 \theta
\end{pmatrix}; \quad g^{ij} = \begin{pmatrix}
R^2 & 0 \\
0 & (R^2 \sin^2 \theta)^{-1}
\end{pmatrix}
\]

Calculating the corresponding \( \Gamma^i_{\mu \nu} \) one finds that the only non-zero ones are

\[
\Gamma^i_{22} = -\sin \theta \cos \theta; \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot \theta \geq 0
\]

from which the only non-zero Ricci scalar

\[
R^i_{221} = -\sin \theta \cos \theta = -\frac{1}{R^2} \geq 2\end{pmatrix} \quad R_{i} = \frac{1}{R^2} \text{ and}
\]

\[
R^2_{121} = \frac{1}{R^2} \geq 1
\]

In general, for any 2-surface when \( R = \text{const} \)

\[
R^\alpha_{\mu \beta \nu} = \frac{1}{R^2} \left( \delta^\alpha_\mu \geq g_{\nu} - \delta^\alpha_\nu \geq g_{\mu} \right)
\]

Note that \( R^\alpha_{\mu \beta \nu} \propto 1/R^2 \)

to the inverse of the square of the radii.
Clearly: \( R \rightarrow \infty \) : flat spacetime

- One always finds a tangent space which is locally flat

**Important:** if all the components of the Riemann tensor vanish, the manifold is flat

**Note:** for this to happen you must have \( \Gamma, \Gamma_{\alpha} = 0 \). In other words, not only the first derivatives have to be zero, but also the second ones.
The Riemann tensor is a 4×4×4×4 matrix and hence it has 256 components! However, they are not all independent.

Indeed, the independent ones are only 20:

\[\mathcal{R} = \begin{array}{cccc}
\mathcal{R}^0_0 & \mathcal{R}^0_1 & \mathcal{R}^0_2 & \mathcal{R}^0_3 \\
\mathcal{R}^1_0 & \mathcal{R}^1_1 & \mathcal{R}^1_2 & \mathcal{R}^1_3 \\
\mathcal{R}^2_0 & \mathcal{R}^2_1 & \mathcal{R}^2_2 & \mathcal{R}^2_3 \\
\mathcal{R}^3_0 & \mathcal{R}^3_1 & \mathcal{R}^3_2 & \mathcal{R}^3_3
\end{array}\]

STRONG EP

We have discussed that there are no global inertial frames but that at least locally it is possible to find a reference frame which is inertial, i.e., freely falling.

In this frame, the laws of physics will be those of SR. Stated differently: any law which can be expressed in tensor notation in SR has exactly the same form in a locally inertial frame of a curved spacetime.
This is the "comma-to-semicolon" rule

\[ N^\alpha, \alpha = 0 \quad T^\alpha_\beta, \beta = 0 \quad (5) \]

In a local inertial frame this law is the same as in a curved spacetime

\[ N^\alpha; \alpha = 0 \quad T^\alpha_\beta; \beta = 0 \quad (6) \]

Note that (6) are valid in the whole spacetime. This is why we don't really trash SR and its extension to GR is very simple.

Another example is a geodesic. In a local inertial frame particles move on straight lines, i.e.

\[ U^\alpha, \mu \, U^\mu = 0 \quad \Rightarrow \quad \text{This is eq. to } x^\alpha = k \ell \]

\[ U^\alpha; \mu \, U^\mu = 0 \]

and hence freely falling particles move on time-like geodesics of the spacetime.
This is an important result because geodesics reflect curvature of spacetime and freely falling particles are obeying gravitational field's produced by matter.

The logic is therefore

\[
\text{(CURVATURE)} \leftrightarrow \text{( MATTER )}
\]

We know that the RHS is given by \( T^{\alpha \beta} \) and that \( T^{\alpha \beta} ; \beta = 0 \)

I need a tensor on the LHS that has these properties, e.g.

\[
R^{\alpha \beta} + c_1 g^{\alpha \beta} R + c_2 g^{\alpha \beta}
\]

This is indeed a generalization of Einstein's tensor so that the field eqs

\[
R^{\alpha \beta} - \frac{1}{2} g^{\alpha \beta} R + c_2 g^{\alpha \beta} = K T^{\alpha \beta}
\]

We will see that \( K = \frac{8\pi G}{c^4} \) and \( c_2 = \Lambda ; \) const.
We still need to compute the constant in the Einstein eqs. which hereafter we will consider in the case $\Lambda = 0$

\[
G^{\alpha \beta} = R^{\alpha \beta} - \frac{1}{2} g^{\alpha \beta} R = k T^{\alpha \beta}
\]

We need to obtain in the weak-field limit the standard Newtonian expression

\[
\nabla^2 \phi = 4\pi \rho G
\]

Let's contract with $g^{\alpha \beta} \Rightarrow$

\[
R^{\alpha \beta} g^{\alpha \beta} - \frac{1}{2} g^{\alpha \beta} g^{\alpha \beta} R = k T^{\alpha \beta} g^{\alpha \beta}
\]

\[
R^{\alpha \alpha} \frac{1}{2} R = k T^{\alpha \alpha} = k T
\]

\[R = -kT \Rightarrow\]

\[
R^{\alpha \beta} = k \left( T^{\alpha \beta} - \frac{1}{2} g^{\alpha \beta} T \right) \quad (6)
\]

In weak field $T^{00} \gg T^{0i} \gg T^{ij}$, using the covariant components, the $00$ component of (6) is

\[
R^{00} = k \left( T^{00} - \frac{1}{2} g^{00} T \right)
\]
Note

\[ \eta_{00} = -1; \eta^{00} = \frac{1}{\eta_{00}} = -1 \]

\[ h_{00} = h^{00} \neq -1 \]

In flat space, we have

\[ T_{00} \neq \eta_{00} T_{00} \]

\[ \eta^{0\alpha} \eta^{0\beta} T_{0\beta} = (\eta^{00})^2 T_{00} \]
\[ g_{\mu\nu} T = g_{\mu\nu} T^\mu \rho = g_{\mu\nu} \tilde{g}^\mu\nu T_{\mu\nu} \]
\[ = g_{\mu\nu} g_{\rho\sigma} T^{\rho\sigma} = T^{\mu\nu} \]
\[ = \frac{T_{\mu\nu}}{2} = \frac{\eta_{\rho\sigma} \eta_{\mu\nu} T^{\rho\sigma}}{2} \]
\[ = (\eta_{\rho\sigma})^2 \frac{T^{\mu\nu}}{2} = \frac{\rho c^2}{2} \]

\[ R_{\mu\nu} = \frac{\kappa}{2} \rho c^2 \]

The LHS can be computed easily to be

\[ R_{\mu\nu} = -\frac{1}{2} h_{\mu\nu}, \frac{\Omega^2}{2} h_{\mu\nu} \]

where we have introduced \( \eta_{\mu\nu} \) as a correction to the flat spacetime metric, i.e.

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

We need to relate \( h_{\mu\nu} \) to \( \Phi \) and we can do this by using the geodesic's equation which we know is the equation of a freely falling particle and the eq. of motion in the presence of gravity.
\[ \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \quad \text{vs} \quad \vec{\nabla} = -\vec{\nabla} \phi \]

Consider spectral components and neglect terms \( O(r^{-2}) \Rightarrow \Gamma^i_{jk} n^j n^k = \Gamma^i_{00} n^i n^0 \)

\[ \vec{\nabla} \phi = -\Gamma^i_{00} = -\frac{1}{2} \vec{\nabla} \cdot \mathbf{n} \]

Re-introduce a bit of algebra

\[ \vec{\nabla} \phi = -\frac{\mathbf{n}^2}{2} \vec{\nabla} \cdot \mathbf{n} = -\vec{\nabla} \phi \quad \text{Newtonian weak limit of geod. eq.} \]

\[ \Rightarrow \frac{\phi}{c^2} = -\frac{\mathbf{n}^2}{2} \quad \text{or} \quad \mathbf{n}^2 = -\frac{2\phi}{c^2} \]

Let's go back to the LHS of Einstein eqs

\[ \mathbf{R}^0 = -\frac{1}{2} \nabla^2 \mathbf{n} = \frac{1}{2} \nabla^2 \left( \frac{2\phi}{c^2} \right) = \frac{\nabla^2 \phi}{c^2} \]

Putting things together: Einstein eqs in weak field limit

\[ \frac{\nabla^2 \phi}{c^2} = \frac{G \rho c^2}{2} \quad \text{vs} \quad \nabla^2 \phi = 4\pi G \rho \]

\[ k = \frac{8\pi G}{c^4} \]
We have seen that geodesics are the trajectories of free particles and that they are parallel in an inertial frame and not in general.

Now we compute how the separation between two geodesics varies and relate this change to the curvature of spacetime. Before doing this, in GR let's do this calculation in Newtonian gravity.

\[ x^a = x^a(t) \text{ along } \gamma, \]
\[ z^a = z^a(t) \text{ and } \gamma_z = x^a(t) + y^a(t) \]

The equations of motion will be those of bodies in free fall.

\[ \ddot{x}^a = -(\partial^a \phi)_p \]

\[ \ddot{z}^a = -(\partial^a \phi)_p = \ddot{x}^a + y^b \partial_b (\phi)_p = - \left[ \partial_a + \frac{1}{2} \gamma^b \partial_b (\phi)_p \right] \]

\[ = -(\partial^a \phi)_p - \gamma^b \partial_a \partial_b (\phi)_p = -(\partial^a \phi)_p - k^a y^b \phi \]

(61)
where \( k^a b = \nabla^a \nabla_b \phi \) : 2nd deriv. of potential

Comparing (7) and (8)

\[
\dot{\gamma}^a = -k^a_b \gamma^b \tag{9}
\]

This is the eq. of motion of the vector \( \gamma \) separating the two trajectories.

If the field is uniform \( \dot{\gamma}^a \phi = \gamma^a \dot{\phi} = \gamma^a \ddot{\phi} \)

\[ \Rightarrow \dot{\gamma}^a = 0 \Rightarrow \gamma^a = \text{const} \]

Thus even in Newtonian gravity two freely falling particles will meet eventually.

\[
\Phi = -\frac{M}{r} \Rightarrow \dot{\gamma}^a \gamma_a \Phi = + \dot{\gamma}^r \left( -\frac{M}{r^2} \right) = -\frac{2M}{r^3}
\]

ie \( \dot{\gamma}^a = \dot{\gamma}^r = -\frac{2M}{r^3} \)

these are tidal fields and scale like \( r^{-3} \)
Back to GR

We have seen that deviations from II transport are measured by the curvature tensor

\[ 2 \nabla [\xi \nabla \beta] V^m = R^m_{\alpha \beta \gamma} V^\gamma \]

\[ \psi_1 (\tau, \nu) \]
\[ \psi_2 (\tau, \nu_2) \]

\( \mathbf{U} \) is tangent vector and thus

\[ \nabla_\mathbf{U} \mathbf{U} = 0 \]

for both curves since they are geodesics

Vectors connecting the two geodesics

\[ \xi^\alpha = \frac{dx^\alpha}{d\nu} \]

parameter used to distinguish the geodesics

It is now easy to show that the covariant derivative of \( \mathbf{U} \) along \( \xi \) is the same as the covariant derivative of \( \xi \) along \( \mathbf{U} \)
\[ \nabla_{\xi} \mathcal{U} = \nabla_{\mathcal{U}} \xi \]

Proof

\[ \nabla_{\xi} \mathcal{U}^{\mu} = \mathcal{U}^{\sigma}_{\beta} \xi^{\mu} = \mathcal{U}^{\sigma}_{\beta} \xi^{\mu} + \Gamma^{\mu}_{\beta \gamma} \mathcal{U}^{\gamma} \xi^{\beta} \]

\[ \nabla_{\mu} \xi^{\nu} = \xi^{\nu}_{\beta} \mathcal{U}^{\mu}_{\beta} = \xi^{\nu}_{\beta} \mathcal{U}^{\mu}_{\beta} + \Gamma^{\nu}_{\beta \mu} \mathcal{U}^{\beta} \xi^{\nu} \]

\begin{align*}
\underbrace{\mathcal{U}^{\sigma}_{\beta} \xi^{\mu} - \xi^{\nu}_{\beta} \mathcal{U}^{\mu}_{\beta} + \Gamma^{\nu}_{\beta \mu} (\xi^{\beta} \mathcal{U}^{\mu} - \mathcal{U}^{\beta} \xi^{\mu})}_{= 0} & = 0 \quad \text{because} \\
& \text{cov. hessian of sym. tensor} \\
& \text{and anti-sym. tensor} \\
T(\sigma) T^{\nu \beta} & = 0
\end{align*}

\[
= \frac{d x^{\mu}}{d \nu} \frac{d x^{\nu}}{d \xi^{\beta}} - \frac{d x^{\beta}}{d \xi^{\nu}} \frac{d x^{\nu}}{d \xi^{\beta}}
\]

I next want to take another covariant derivative of \( \xi \) along \( \mathcal{U} \).
$$\nabla_u \nabla_u \xi^x = \nabla_u \nabla_{\xi^x} u^x$$

$$= 2 \nabla \left[ u \nabla_{\xi^x} \right] u^x$$

(9)

**proof**

$$2 \nabla \left[ u \nabla_{\xi^x} \right] u^x = \nabla_u \nabla_{\xi^x} u^x - \nabla_{\xi^x} \nabla u / u^x$$

$$= \nabla_u \nabla u \xi^x$$

on the other hand it is possible to calculate \(\text{(good exercise)}\)

$$2 \nabla \left[ u \nabla_{\xi^x} \right] u^x = R^{x}_{\beta \mu \nu} u^\mu \xi^\nu u^\beta$$

(10)

(9), (10) \(\Rightarrow\)

$$\nabla_u \nabla_u \xi^x = R^{x}_{\beta \mu \nu} u^\mu \xi^\nu u^\beta$$

This is the eq. of geodesic deviation
(second cov. deriv. along \(\xi \) of \(\xi \))

Let's search the analogy with Newtonian

$$\nabla_u = \frac{D}{D\tau}$$

Then (10) \(\Leftrightarrow\)
\[
\frac{D^2 \xi^a}{dt^2} = R^a_{\beta \mu \nu} U^\beta U^\mu \xi^\nu = \kappa^a \xi^a
\]

where \( \kappa^a = R^a_{\beta \mu \nu} U^\beta U^\mu \)

This is to be compared with the corresponding Newtonian expression

\[
\ddot{\xi}^a = -K^a \xi^b
\]

As in Newtonian physics, if \( R^a_{\nu} = 0 \) (i.e. \( R^a_{\beta \mu \nu} = 0 \)) then

\[
\frac{D^2 \xi^a}{dt^2} = 0 : \text{geodesics that are initially parallel remain parallel.}\]

Alternative view: the convergence/divergence of geodesics initially parallel are telling us about the curvature of space-time.

I can monitor two freely falling particles and check if their distance changes. This is indeed a GW detector!
Let's consider a concept already mentioned: tidal fields in the exterior of a black hole.

It is possible to calculate the geodesic deviation for particles freely falling onto a black hole and obtain

\[
\frac{D^2 y^r}{D t^2} = \frac{2M}{r^3} y^r
\]

\[
y^\sigma = h^{\rho \sigma} \tilde{g}^\rho
\]

\[
\frac{D^2 y^\vartheta}{D t^2} = -\frac{M}{r^3} y^\vartheta
\]

\(h_{\rho \sigma} = g_{\rho \sigma} + u^\rho u_\sigma \) projector

\[
\frac{D^2 y^\phi}{D t^2} = -\frac{M}{r^3} y^\phi
\]

\[
\frac{D^2 y^r}{D t^2} > 0
\]

\[
\frac{D^2 y^\vartheta}{D t^2} < 0
\]

\[\text{XXX}XX\]
The starting point are EFES

\[ G_{\mu\nu} = 8\pi T_{\mu\nu} \]  \hspace{1cm} (1)

but we need to look for solutions in a linearized theory of gravity, i.e. in a limit in which the fields and hence the curvature is small.

Let's start from flat space-time and move away from there:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

where

\[ h_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

and

\[ \left| \frac{h_{\mu\nu}}{\eta_{\mu\nu}} \right| \ll 1 \]

Ex

\[ h_{\mu\nu} \sim \phi \sim M_\odot \sim 10^{-6} \]

\[ \frac{\phi}{R_0} \]
If we want to rewrite (1) we need to work out the affine connections

\[ \Gamma^\mu_{\alpha \beta} = \frac{1}{2} g^{\mu \rho} \left( g_{\nu \rho, \beta} + g_{\rho \nu, \alpha} - g_{\nu \alpha, \beta} \right) \]

\[ g_{\alpha \beta} = g_{\alpha \beta} + h_{\alpha \beta} \Rightarrow \]

\[ \Gamma^\mu_{\alpha \beta} = \frac{1}{2} g^{\mu \rho} \left( h_{\nu \rho, \alpha} + h_{\rho \nu, \alpha} - h_{\nu \alpha, \beta} \right) = \frac{1}{2} \left( h^{\mu \alpha} \cdot h_{\nu \rho} - h^{\mu \nu} \cdot h_{\rho \alpha} \right) \]

Note that the indices of \( h_{\mu \nu} \) are raised and lowered using \( g^{\mu \nu} \) and \( g_{\mu \nu} \) and \( \eta_{\mu \nu} \).

Similarly, the linearized Ricci tensor

\[ R_{\mu \nu} = \Gamma^\sigma_{\mu \nu, \alpha} - \Gamma^\sigma_{\mu \alpha, \nu} \]

\[ = \frac{1}{2} \left( h^{\sigma \mu} \cdot h_{\nu \alpha} + h^{\sigma \nu} \cdot h_{\mu \alpha} - h^{\sigma \alpha} \cdot h_{\mu \nu} \right) \]

where \( h_{\mu \nu} = h^{\mu \nu} \cdot h_{\mu \nu} = \text{trace of } h_{\mu \nu} \).
(*)

\[ \overline{h} = -h \]

**proof**

\[ \overline{h} = \overline{h}_\mu = h_\mu - \frac{1}{2} \eta_\mu^\nu h^\nu \]
\[ = h - \frac{1}{2} \eta_\mu^\nu \overline{h}^\nu \]
\[ \therefore \overline{h} = -h \]
\[ R = g^{\mu \nu} R_{\mu \nu} = \eta^{\mu \nu} R_{\mu \nu} \]

\[ \Rightarrow 2G_{\mu \nu} = 16\pi T_{\mu \nu} \quad \Rightarrow \]

\[ h_{\mu \nu} \omega^a + h_{\mu \nu} \omega^a = h_{\mu \nu} \omega^a - h_{\mu \nu} \omega^a - \eta_{\mu \nu} (\eta_{\alpha \beta} \omega^\alpha \omega^\beta) = 16\pi T_{\mu \nu} \]

(2)

Expression (2) can be made more compact if we use the trace-free part of \( h_{\mu \nu} \)

\[ \bar{h}_{\mu \nu} = h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} \bar{h} \]

where the bar can be applied to any symmetric tensor.

\[ G_{\mu \nu} = \bar{R}_{\mu \nu} \quad \text{trace of } \bar{h}_{\mu \nu} \]

Proof:

\[ \bar{h}_{\mu \nu} = \bar{h}_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} \bar{h} \]

\[ = \bar{h}_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} \eta_{\alpha \beta} \omega^\alpha \omega^\beta \]

\[ = \bar{h}_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} \eta^{\alpha \beta} (\eta_{\alpha \beta} - \frac{1}{2} \eta_{\alpha \beta} h) \]

\[ = \bar{h}_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h + \frac{1}{2} \eta_{\mu \nu} h = \bar{h}_{\mu \nu} \]
In this case the EFES take the form

\[-\bar{\eta}^{\mu\nu},_a - \gamma^{\mu\nu} \bar{\eta}^{\rho\sigma} \gamma_{\rho\sigma} + \bar{\eta}^{\mu\nu},_a, \mu = 16\pi \pi^{\mu\nu} \ (3)\]

\[-\Box \bar{\eta}^{\mu\nu} = - \left( \partial_t^2 + \partial_x^2 - \partial_y^2 - \partial_z^2 \right) \bar{\eta}^{\mu\nu} \]

We can now exploit the gauge freedom and specify \( \eta^{\mu\nu} \) so that

\[-\bar{\eta}^{\mu\nu},_a = 0 \quad (4)\]

(4) is the equivalent of the Lorentz gauge condition and as usual of that

**DIGRESSION**

We have already seen that if we make an infinitesimal coordinate transformation

\[x^\alpha' = x^\alpha + \xi^\alpha, \text{ then the new "perturbed" metric} \]

\[\bar{\eta}^{\mu\nu} + h^{\mu\nu} = g^{\mu\nu} + h^{\mu\nu} + \xi^{\mu\rho} - \xi^{\nu\rho} \]

i.e. \( h_{\mu\nu} = h_{\mu\nu} + \xi_{\mu\nu} - \xi_{\nu\mu} \)

where \( \xi^{\mu} \) are arbitrary but small functions.

For any of these all of the observables are unmodified.
Think of electromagnetism

\[ A_\mu \rightarrow A_\mu' = A_\mu + \gamma_\mu \]

then \[ F^{\text{new}}_{\mu'\nu'} = A_\mu' \nu' - A_\mu' \nu = A_\mu' \nu' + \gamma_{\mu' \nu} \]

\[ A_\mu' - \gamma_{\mu'} = F_{\mu' \nu} \]

Recap: the linearized theory of gravity

with

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \]

satisfies the field equations

\[ -\gamma_{\mu\nu,\kappa} = 16\pi T_{\mu\nu} \]  \hspace{1cm} (5)
Gauge transformations and coordinate transformations in linearized gravity

In their linearized form, the EFE are invariant to both global Lorentz (coord.) transformations and local ones.

1) Global Lorentz transf.

\[ x'^\mu \rightarrow x'^\mu' \]

\[ x'^\mu = \Lambda^\mu_\nu x^\nu \]

\[ g'^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\rho g^{\sigma\rho} \]

The metric then transforms, and the perturbations also transform like tensors:

\[ \epsilon^\mu_\nu = \Lambda^\mu_\sigma \Lambda^\nu_\rho \epsilon^{\sigma\rho} \] (1)

2) One can also consider small local coordinate transformations

\[ x'^\mu = x^\mu(P) + \epsilon^\mu \]

where

\[ \epsilon^\mu = x^\mu \]
one can always perform these transformations if they leave \( h_{\mu \nu} \ll 1 \)

As an example consider the temperature as a function of position \( T = T(x^\mu) \) and perform a small coordinate trans.

\[
T(x^\mu - q^\mu) = T(x^\mu + \delta x^\mu) = T(x^\mu - q^\mu - \delta x^\mu)
\]

temp. at new position = temp. at old position displaced by \( \delta x^\mu \)

\[
\approx T(x^\mu) - T_{\mu \nu} \delta x^\mu
\]

If \( T = \cos^2(x^0) \) and \( \delta x^0 = 0.001 \sin(x^2) \)

\[
T_{\mu \nu} \delta x^\mu = 2 \cos(x^0) \sin(x^0)
\]

\[
= \cos^2(x^0) + 0.002 \cos(x^0) \sin(x^0) \sin(x^2)
\]

these changes cannot be ignored in the metric which is already measuring deviations from flat and therefore gravity.
\[ g^\mu_\nu^\prime (x^\nu (x')) = g^\mu_\nu (x^\nu (x')) \frac{\partial x^\nu}{\partial x'^\nu} \]

so that

\[ h^\nu_\mu^\prime \text{ new} = h^\nu_\mu \text{ old} - \delta^\nu_\mu \frac{\partial}{\partial x'^\nu} \]

\[ = \text{ the result of the local coordinate transform} \]

\[ (**) \]

are reminiscent of what happens in EM, there too the eq. are invariant for transformations of the type

\[ A^\mu_\nu = A^\mu_\nu + \delta^\mu_\nu \]

so that

\[ F^\mu_\nu^\prime \text{ new} = A^\mu_\nu^\prime, \nu^\prime - A^\nu_\mu^\prime, \mu^\prime = A^\mu_\nu \delta + \delta^\mu_\nu \]

\[ \frac{\partial}{\partial x'^\mu} \]

\[ \frac{\partial}{\partial x'^\nu} \]

\[ = F^\mu_\nu \text{ old} \]

In a similar way one can show that

\[ R^\mu_\nu \text{ new} = R^\mu_\nu \text{ old} \]
because all equations are invariant under the transformation (28) one has the freedom to choose \( \xi^m \) in the most suitable way.

Note: fixing the GLOBAL gauge does not fix the LOCAL one.

If one chooses the Lorentz gauge

\[ \gamma^\mu, \alpha = 0 \]

the equations are still invariant for any choice of the \( \xi^m \) such that

\[ \xi^\mu, \chi = 0 \]

we will exploit this freedom later on.
Let's consider now the linearized eps of gravity in vacuum.

\[ \nabla^a \epsilon^{\mu \nu} = 0 \quad \Rightarrow \quad \square \nabla^a \epsilon^{\mu \nu} = 0 \quad (6) \]

\[ \text{Dolomieuian} \]

In the Lorentz gauge, the gravitational field propagates in spacetime as a wave.

Note: in linearized theory the spacetime is curved although not spherically.

Q: What's the simplest solution to (6)?

It's a plane wave

\[ \epsilon^{\mu \nu} = \text{Re} \left[ A^{\mu \nu} e^{-i k^a x^a} \right] \quad (7) \]

Where \( k^a \) is a null wave vector

\[ k^a k_a = 0 \] and therefore the plane wave described by (7) is propagating in the direction \( \frac{k}{k^0} = \left( k_x, k_y, k_z \right) \) with frequency \( \omega = k^0 = \left( k_0 \right)^{1/2} \).
Solutions (7) indicate there are \( 16 - 6 = 10 \) independent components in the amplitude tensor, but we know there are only 2 dynamical degrees of freedom in GR (spin-2 fields).

The excess of components is due to the fact that we still have not yet constraining all the possible gauge conditions.

4) \( h^{\mu\nu} = 0 \) Lorentz gauge

\[ A_\mu \cdot k^\nu = 0 \quad (A \text{ is orthogonal to } k) \]  

5) Select a global Lorentz frame (as in SR)

with 4-vel \( u^\mu \) such that

\[ A_\mu \cdot u^\nu = 0 \quad (*) \]

6) Choose the remaining gauge freedom to set \( A_\mu^{\nu} = 0 \)

This condition is equivalent to fixing the infinitesimal gauge transformation, i.e., choosing the displacement vector

\[ \xi^\mu = -i C^\mu e^i k^a x^a \]

Note that \( A_\mu \cdot u^\nu = 0 \) are just 3 conditions because I also know that

\[ k^\mu A_\mu \cdot u^\nu = 0 \]

\[ A_0 u^0 + A_{01} u^1 + \ldots + A_{05} u^3 = 0 \]

\[ A_1 u^0 + \ldots + A_{50} u^0 = 0 \]
If we now choose a Lorentz frame in which $u^\alpha = (1, 0, 0, 0)$, i.e., $u^0 = 1; u^1 = 0$ the conditions expressed before for \( h_{\mu \nu} \) can be translated in conditions for \( h_{\mu \nu} \).

In particular

(B) \( A_{\mu \nu} k^\nu = 0 \iff h_{\mu 0} = 0 \): only the spatial components of \( h_{\mu 0} \) are nonzero.

(A) \( A_{\mu \nu} k^\nu = 0 \iff h_{ij}, j = 0 \): the spatial component is divergence free.

(c) \( A^\alpha = 0 \iff h = h^k k = 0 \): the spatial components are trace free.

Conditions (A), (B), (C) define the TT gauge.

Because \( h = 0 \) there is no difference between \( h_{\mu \nu} \) and \( T_{\mu \nu} \) in this gauge.

Q: One might wonder: how general is this treatment?

We know from EM that any EM wave can be decomposed as the superposition of plane waves.

The same is true in linearized gravity.
Further more because the gauge conditions (A)-(C) are all bi-linear in \( h_{\mu\nu} \) we can say that any gravitational wave can be written in a TT gauge.

Indeed there is a theorem:

Pick a specific global Lorentz frame (i.e. a specific 4-vel. vector such that \( u^\alpha = \delta^\alpha_0 \)). For any gravitational wave satisfying

\[ \square h_{\mu\nu} = 0 \]

one can find suitable gauge conditions such that

\[ \square h_{ij} = 0 \quad \& \quad h_{\mu\nu} = h_{\mu\nu}^T \iff h_{\mu\nu} \text{ satisfies (A)-(C)} \]

only 6 eqs

MAKING SENSE OF THE TT-GAUGE

The use of the TT gauge introduces important advantages and simplifications.

The most relevant one is the time-changing time-space components of the Riemann tensor are:

\[ R_{joko} = R_{ojok} = -R_{jojk} = -R_{joko} \]

where

\[ R_{joko} = -\frac{1}{2} h_{ijk,00} \]

(8)
(b) indicates that a travelling gw introduces a local oscillation of the curvature tensor

\( h_{ij} \propto e^{i\omega t} \)

This oscillatory behaviour appears as oscillations in the separation between two neighbouring particles A and B.

Consider a coordinate system \( \xi^a \) attached to A and comoving with \( T^a : \xi^a = \gamma \) \( x^a = 0 \) proper time.

All along A's worldline the line element is

\[ ds^2 = -c^2 dt^2 + \delta_{ij} dx^i dx^j + O \left( (x^0)^2 \right) dx^0 dx^3 \]

The presence of the gw will alter the geodesic motion of AB and be seen as this will manifest in the geodesic deviation equation.
\[ u^\mu u^\nu \nabla_\mu \nabla_\nu \pi_{\mu \nu} = -R^\rho_{\mu \nu \sigma} u^\mu u^\nu u^\sigma \]

\[ \frac{d^2 x^j}{dt^2} + \Gamma^j_{\rho \sigma} \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt} = R^\rho_{\mu \nu \sigma} u^\mu u^\nu u^\sigma \]

Define \( n^j_B = -x^j_A + x^j_B = x^j_B \)

\[ \frac{D^2 n^j_B}{Dt^2} = -R^3_{\mu \nu \kappa} \cdot n^\mu n^\nu n^\kappa \]

(9)

Because \( \Gamma^j_{\mu \nu} = 0 \) in the vicinity of \( A \), (9)

\( \Rightarrow \)

\[ \frac{d^2 x^j_B}{dt^2} = -R^3_{\mu \nu \kappa} \cdot x^\mu B \]

\( \Rightarrow t = 2 \)

After the time integration

\[ x^j_B(t) = x^j_B(0) \left[ 1 - \frac{1}{2} R^\mu_{\nu \kappa} x^\nu B \right] \]

(10)

Eq. (10) tells us that in the reference frame comoving with \( A \), the particle \( B \) is subject to an oscillation with amplitude \( R^\mu_{\nu \kappa} \).

If the particles are in the direction of the wave (ie \( \vec{n} \parallel \vec{E} \)), then no oscillation is recorded (these are transverse waves).
The wave is really transverse: not only in the mathematical description but also in the physical behavior (geod. deviation).

Let's specify to a concrete example: consider a wave propagating in the positive $z$-direction.

Then

\[ h_{xx} = - h_{yy} = \text{Re} \left\{ A^+ e^{-i\omega(t-z)} \right\} \]

\[ h_{xy} = h_{yx} = \text{Re} \left\{ A_x e^{-i\omega(t-z)} \right\} \]

$A^+$ and $A_x$ represent the two independent modes of polarization.

Just like in EM, one can decompose a GW in two linearly polarized plane waves or in two circularly polarized ones.
The polarization can be expressed in terms of the *linear polarization tensors*

\[
\begin{align*}
\bar{e}_+ &= \bar{e}_x \otimes \bar{e}_x - \bar{e}_y \otimes \bar{e}_y \\
\bar{e}_x &= \bar{e}_x \otimes \bar{e}_x + \bar{e}_y \otimes \bar{e}_y
\end{align*}
\]

Similarly, one can define the *circular polarization tensors*:

\[
\begin{align*}
\bar{e}_e &= (\bar{e}_x + i\bar{e}_y) / \sqrt{2} \\
\bar{e}_e &= (\bar{e}_x - i\bar{e}_y) / \sqrt{2}
\end{align*}
\]

Deformations of rings of particles through the action of giant with single polarization, e.g.

\[
h_{jk} = \text{Re} \left[ A + e (t) \right] (t_{jk})
\]
GENERATION OF GWs

We will estimate the generation of GWs using Newtonian physics. Modulo factors \( O(10) \), these estimates are reliable.

How do you produce EM waves? Oscillating EM fields, such as those produced by an oscillating electric dipole.

\[
\text{electric dipole} = \frac{\text{energy emitted}}{\text{Unit time}}
\]

\[
= \frac{2}{3} e^2 a^2 = \frac{2}{3} \frac{d^2}{dt^2} \quad \text{(second time derivative of electric dipole)}
\]

\[
d = e \times \rightarrow \quad d = e \times = e a
\]

Can we have dipolar GWs?

\[
d = \sum \frac{\mathbf{m}_a x_a}{A}
\]

\[
d = \sum \mathbf{m}_a \times x_a = \mathbf{p}
\]

\[
d = \dot{\mathbf{p}} = 0 \quad \text{(Euler eq. \ (mom. const.)})
\]

\[
\Rightarrow \text{linear dipole} = 0
\]

(this is equivalent to the impossibility of having non-unipolar EM radiation)
Next in EM one can calculate the energy loss due to varying electric current (radiating magnetic dipole)

\[ \mu = B \mathbf{i} \rightarrow \text{electric current} \]

\[ \mathbf{j} \times \mathbf{B} \rightarrow \text{second time derivative of the electric current} \]

The gravitational analogue is

\[ \mu = \sum m_a (r_a) \times (r_a) = \mathbf{j} \]

\[ \mathbf{j} \times \mathbf{j} = 0 \]

Next we need to consider changing quadrupoles

Electric quadrupole \( \frac{1}{20} Q^2 \) \( Q_{ij} Q_{ij} \)

where \( Q_{ij} = \sum \frac{2}{3} i (x_i x_j - \frac{1}{3} \delta_{ij} x_k x_k) \)

Similarly one can show that the GW luminosity is

First quadrupole \( \frac{1}{5} \langle \mathbf{E}^2 \rangle = \frac{1}{5} \langle \mathbf{E}_{ij} \mathbf{E}_{jk} \rangle \)
\[ I_{jk} = \frac{1}{2} \sum_k (x_j x_k + x_k x_j) - \frac{1}{3} \delta_{jk} (x_1^2 + x_2^2 + x_3^2) \]

\[ = \int dV \left( x_j x_k - \frac{1}{3} \delta_{jk} x^2 \right) \]

Consider two masses \( M_1 \) and \( M_2 \), coupled by a spring of length \( L \) and oscillating at \( \omega \)

\[ F_{ij} \propto \frac{M_1 M_2}{r^2} \]

\[ F_{ij} \sim \frac{1}{12} \frac{M_1 L^2 \omega^3}{L} \sim \frac{M_1 L}{10^8} \left( \frac{L}{20m} \right)^2 \left( \frac{\text{GHz}}{4\text{Hz}} \right)^3 10^{-23} \text{ ergs}^{-1} \]

only if one considers very large masses, and relativistic velocities will

\[ T_{\text{aw}} \sim 10^{30} - 10^{30} \text{ ergs}^{-1} \]