Lecture 1: Special relativity - Tensors

" 2: Fluids ; equivalence principle

3: Curved spacetime and tensors

4: Einstein eqs. ; IVP

5: Linearized GWs

□

Suggested refs:

MTW : "Gravitation"

BFS : "A first course in GR"

RdI : "Introducing Einstein's Relativity"

Special Relativity

It describes the law of physics in the absence of gravitational fields ("flat spacetime")

Postulates

1) principle of relativity (Galileo)
No experiment can measure absolute velocity of an observer (if latter is moving at constant speed)

2) the speed of light is the same for all unaccelerated observers
$c = 2.99 \times 10^{10}\ cm\, s^{-1}$

$$\bar{F} = m\bar{a} = \frac{dv}{dt}$$

$$v \rightarrow v' = v + v_0$$

$$\frac{dv}{dt} = \frac{dv'}{dt} = \frac{dv}{dt}$$

SR selects therefore a class of observers which are special: inertial observers.

In GR these observers do not exist (all observers are accelerated) but it is possible to define observers which are inertial at one specific time and position

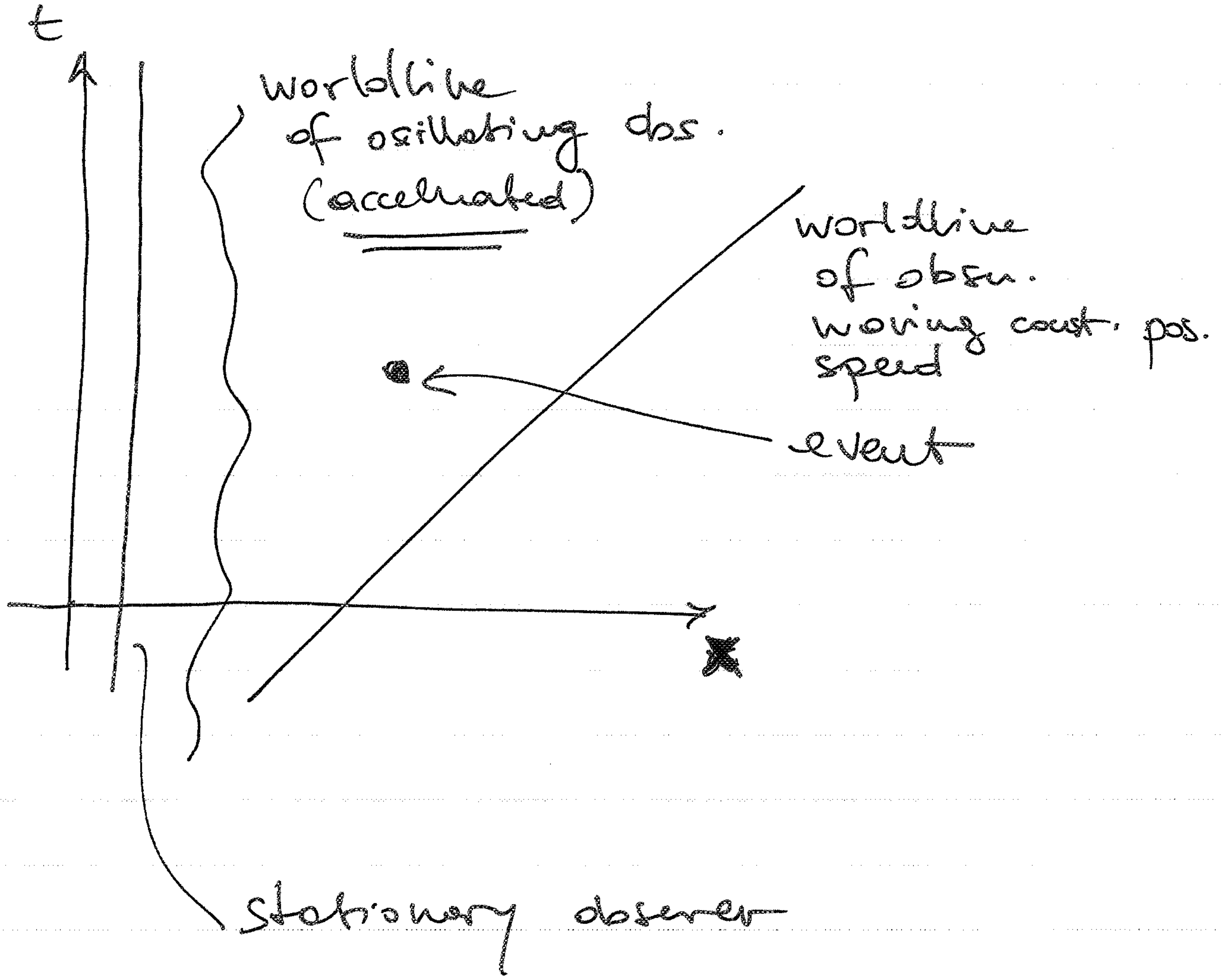
An inertial observer defines an inertial "frame" (reference system), ie a frame in which
- distance between spatial points does not change
- " " time points " "
- geometry is Euclidean.

This seems obvious[⊙] but is not longer true in GR

□

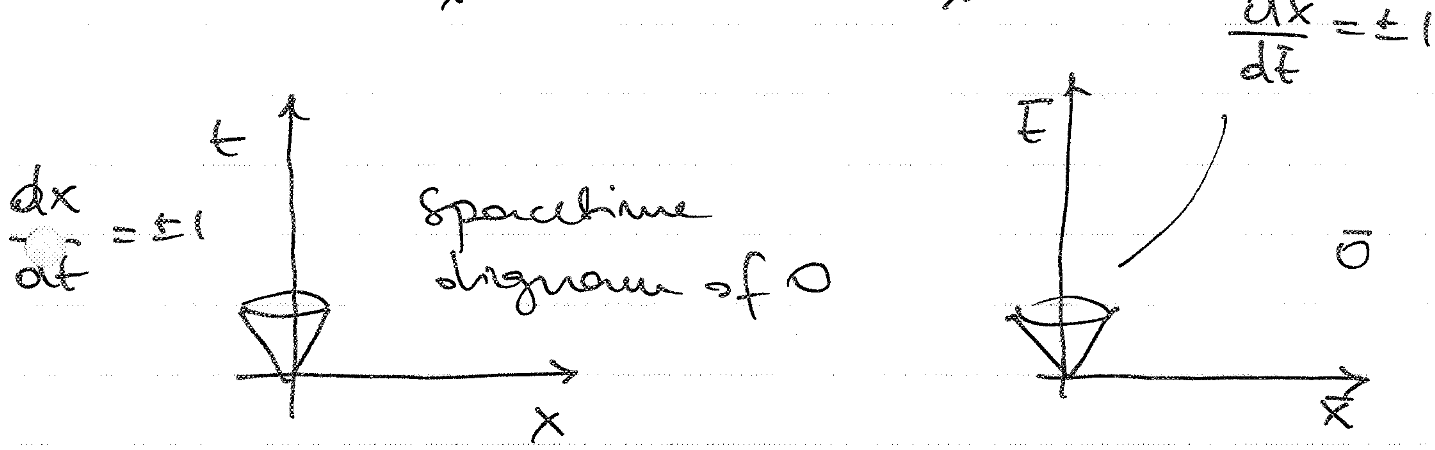
In SR and GR concepts are simpler if interpreted geometrically and therefore in a spacetime setting in which time and space are on equal footing and spacetime is a 1+1+1+1 = 4D object (manifold)

⊙ our experience is that of a flat spacetime

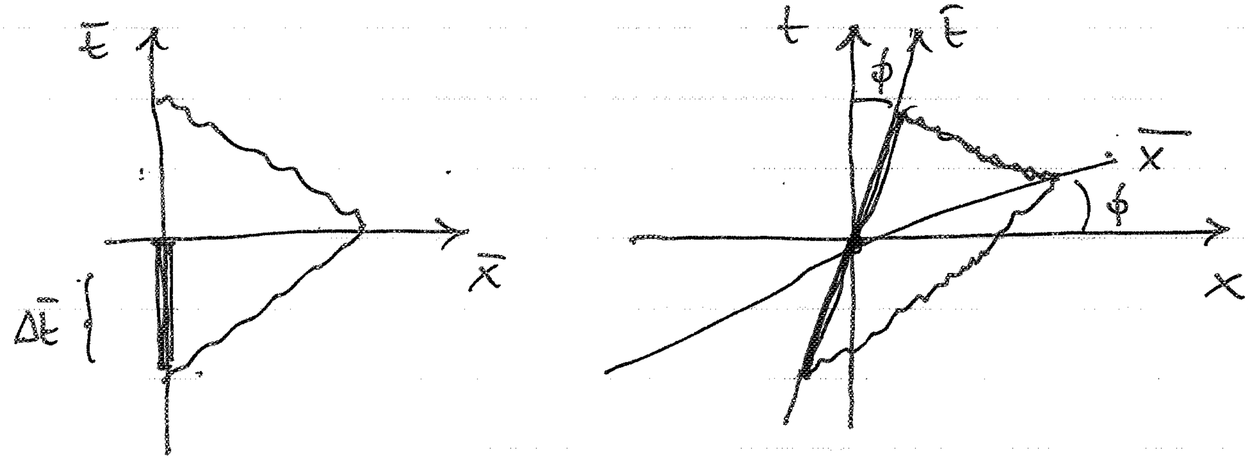


t
worldline of osillating obs. (accelerated)
worldline of obsv. moving const. pos. speed
event
stationary observer

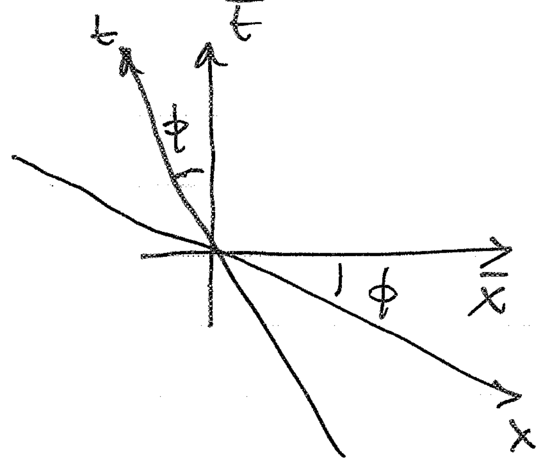
Consider observer $\bar{O}$ moving at speed v wrt observer O.



How does the two diagrams compare? Events are the important aspects of this picture of spacetime and coords are just relations to the observers and thus arbitrary

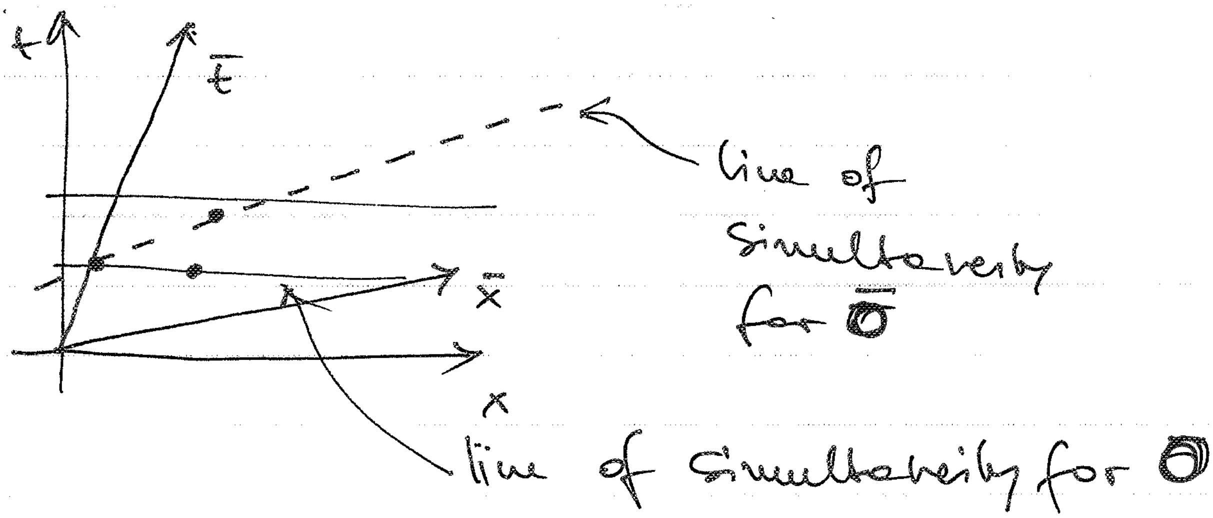


An angle ϕ is present between the two and of course the reverse is also true.



What is important is that the "distances" between events measured in the two systems is the same even if the concept of simultaneity is lost

~~$\Delta s^2 =$~~



Another fundamental quantity is the "distance" between two events

Recall $\frac{dx}{dt} = \pm 1 \implies \Delta x^2 - \Delta t^2 = 0 = \Delta s^2$

ie the distance between two events connected by a light beam is zero.

This distance is the same in all coords systems

$$\Delta \bar{s}^2 = -(\Delta \bar{t})^2 + (\Delta \bar{x})^2 = 0$$

It is therefore sensible to define

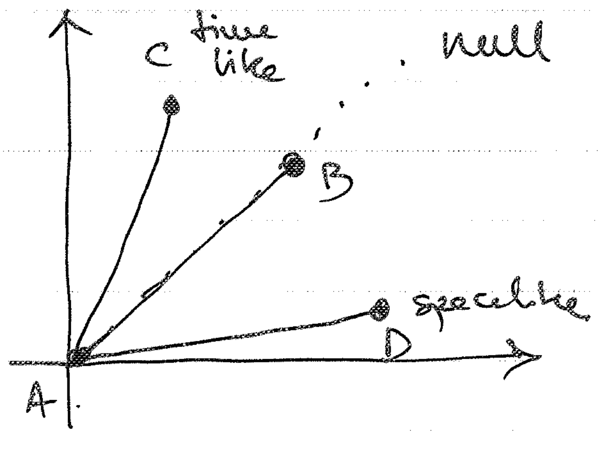
$= \Delta s^2$: distance between two events

$$= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

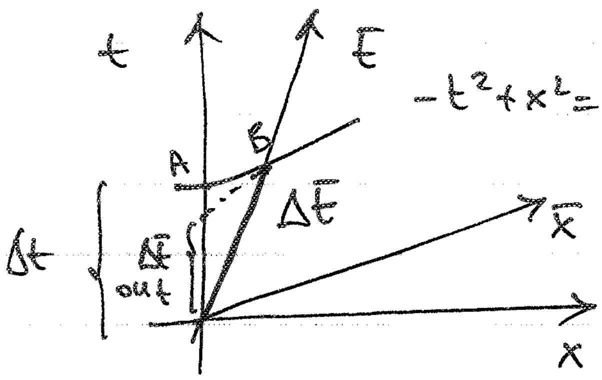
$\Delta s^2_{AC} < 0$ timelike interval

$\Delta s^2_{AB} = 0$ null "

$\Delta s^2_{AD} > 0$ spacelike interval



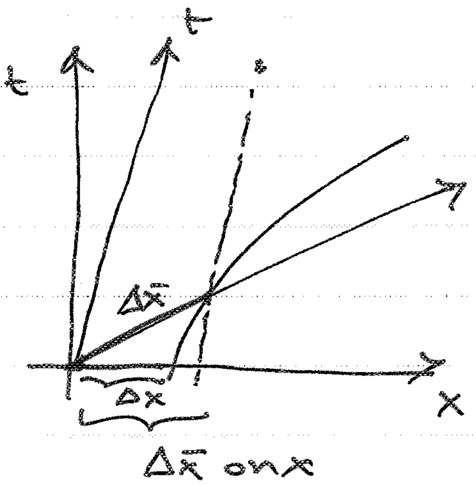
We can now relate the coords in our system to the coords in the other one



$-t^2+x^2=$ const: events are equally distant on this hyp.

A & B : simultaneous

$\Delta t > \Delta \bar{t}$: time dilation!

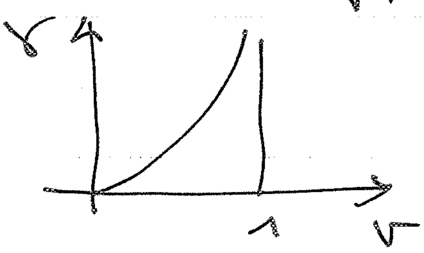


$\Delta x < \Delta \bar{x}$: Lorentz contraction!

In general the transformation is called Lorentz transf. and is given by

$$(*) \quad \begin{cases} \bar{t} = \gamma (t - v x) \\ \bar{x} = \gamma (x - v t) \\ \bar{y} = y \\ \bar{z} = z \end{cases}$$

where $\gamma = \dfrac{1}{\sqrt{1-v^2}}$ — Lorentz factor



to obtain

$x = f(\bar{x})$ all is needed is $v \to -v$

Composition of velocities Galileian

$$W = \frac{dx}{dt} = \frac{\bar{W} + v}{1 + \bar{W} v} \simeq \bar{W} + v$$

only $\bar{W} \ll 1$

In general the transformation (*) can be written as

$$x^{\bar{\alpha}} = \sum_{\beta=0}^{3} \Lambda^{\bar{\alpha}}{}_{\beta}\, x^{\beta} = \Lambda^{\bar{\alpha}}{}_{\beta}\, x^{\beta}$$

and similarly

$$x^{\alpha} = \Lambda^{\alpha}{}_{\bar{\beta}}\, x^{\bar{\beta}}$$ inverse transformation

$\Lambda^{\bar{\alpha}}{}_{\beta} = \Lambda^{\bar{\alpha}}{}_{\beta}(v)$: Lorentz transf.

$\Lambda^{\alpha}{}_{\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\beta}}(v)$: inverse of $\Lambda^{\bar{\alpha}}{}_{\beta}$

$= \left(\Lambda^{\bar{\alpha}}{}_{\beta}\right)^{-1}$

We have seen that what matters are the events and their separation which is independent of coords.

This is generically true: we want to write equations in a covariant (ie coord. independent) manner.

These equations will be valid in any system of coordinates.

To do this we need to introduce the concept of tensors which make the derivation of covariant expressions very simple.

As a matter of fact we already have introduced them, although not explicitely

We can use this notation also to measure distances

$\sum_{\alpha=0}^{3}$

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

$\sum_{\beta=0}^{3}$

where $\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

is another (4×4) matrix

$$\eta_{00} = -1 \;; \quad \eta_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Invariance implies

$$ds^2 = d\bar{s}^2 = \eta_{\bar{\alpha}\bar{\beta}}\, dx^{\bar{\alpha}}\, dx^{\bar{\beta}}$$

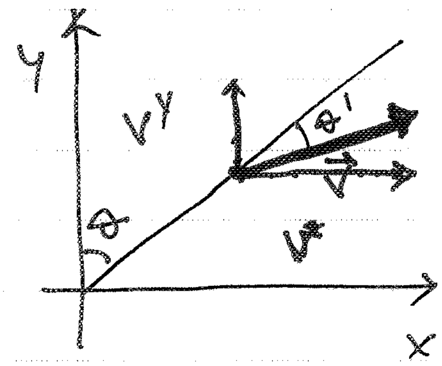
$\underline{\eta}$ is the <u>metric</u> and is a matrix which allows to measure <u>distances</u>

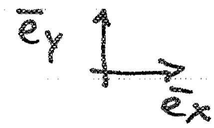
It's the most important quantity in GR/SR.

Because tensors are generalizations of vectors, let's recall some vector calculus you know well

$\vec{V} = \{V^x, V^y, V^z\}$

modulus of $\bar{V}$

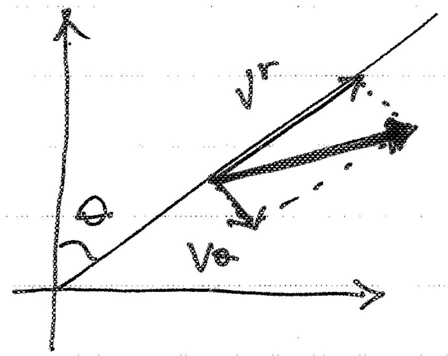


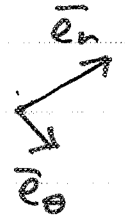


$$V^x = |V| \sin(\theta + \theta')$$
$$V^y = |V| \cos(\theta + \theta')$$

However $\vec{V}$ can be decomposed also in other coordinate system, eg a spherical polar

$\bar{V} = \{V^r, V^\theta, V^\phi\}$





$$V^r = |V| \cos\theta'$$

$$V^\theta = |V| \sin\theta'$$

A bit of algebra shows that

$$\begin{cases} V^x = V^r \cos\theta + V^\theta \sin\theta \\ V^y = V^r \sin\theta + V^\theta \cos\theta \end{cases}$$

so that we can think of a transformation matrix Λ

$$V^{\bar{k}} = \Lambda^{\bar{k}}{}_{\ell} \, V^{\ell} \qquad (2)$$

Note that this is not related to a boost

$$\begin{pmatrix} V^x \\ V^y \end{pmatrix} = |V| \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V^r \\ V^\theta \end{pmatrix}$$

So far all is well established but for $|V|$. What is $|V|$?

It's a measure of the length of V and we have seen that η, the metric, does exactly this: measures the distance between two points, ie the "tip" and the "end" of the vector $\vec{V}$! It's therefore clear that:

Our considerations are fully generic and indeed we can extend expression (2) to a 4D spacetime

$$V^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} \, V^{\beta} \quad : \text{transf. of vector in 4D}$$

By analogy with ds^2

$$|V| = V^{\alpha} V^{\beta} \eta_{\alpha\beta} \quad = V^{\bar{\alpha}} V^{\bar{\beta}} \eta_{\bar{\alpha}\bar{\beta}}$$

frame independent

the lenght of $|V|$ is the same for all coord. systems.

The metric is therefore an <u>operator</u> (geom. object) ~~take~~ acts on a vector and returns its lenght

$$\underline{\underline{\eta}} = \underline{\underline{\eta}} \; (\underline{V} \, , \underline{V}) = |V|$$

If the two slots are occupied by two different vectors, then η returns the lenght of one in the direction of the other, ie the <u>scalar</u> <u>product</u>

$$\underline{V} \cdot \underline{W} = V^{\alpha} W^{\beta} \eta_{\alpha\beta} = \text{scalar}$$

$$\underline{\underline{\eta}} \, (\underline{V}, \underline{W}) = \text{scalar}$$

the metric is also defined as a $\binom{0}{2}$ tensor and thus as a function that takes as input two vectors and returns a scalar.

The metric is linear in its arguments

$$\underline{\underline{\eta}}\,(\alpha \underline{U} + \beta \underline{V}, \underline{W}) = \alpha \underline{\underline{\eta}}\,(\underline{U}, \underline{W}) + \beta \underline{\underline{\eta}}\,(\underline{V}, \underline{W})$$

$$= \alpha \eta_{\mu\nu} U^{\mu} W^{\nu} + \beta \eta_{\mu\nu} V^{\mu} W^{\nu}$$

To understand what the components $\eta_{\mu\nu}$ are it is useful to consider the definition of a vector

$\underline{V} = V^{\alpha} \underline{e}_{\alpha}$ where $\underline{e}_{\alpha}$: coordinate vector basis

bar goes here

$$\underline{e}_{\alpha} = (\underline{e}_0, \underline{e}_1, \underline{e}_2, \underline{e}_3)$$

$$\underline{e}_0 = \{1, 0, 0, 0\}$$
$$\underline{e}_1 = \{0, 1, 0, 0\}$$
$$\underline{e}_2 = \ldots$$
$$\underline{e}_3 = \{0, 0, 0, 1\}$$

Let then have $\underline{\underline{\eta}}$ act on $\underline{e}_\alpha$

$$\underline{\underline{\eta}}\,(\underline{e}_\alpha, \underline{e}_\beta) = \underline{e}^\mu{}_\alpha\, \underline{e}^\nu{}_\beta\, \eta_{\mu\nu}$$

$$= \delta^\mu{}_\alpha\, \delta^\nu{}_\beta\, \eta_{\mu\nu}$$

$$= \eta_{\alpha\beta}$$

In other words: [illegible]
$\eta_{\alpha\beta}$ are the components of ~~the~~ the metric tensor in a coordinate vector basis.

Clearly, the metric tensor is symmetric in its arguments

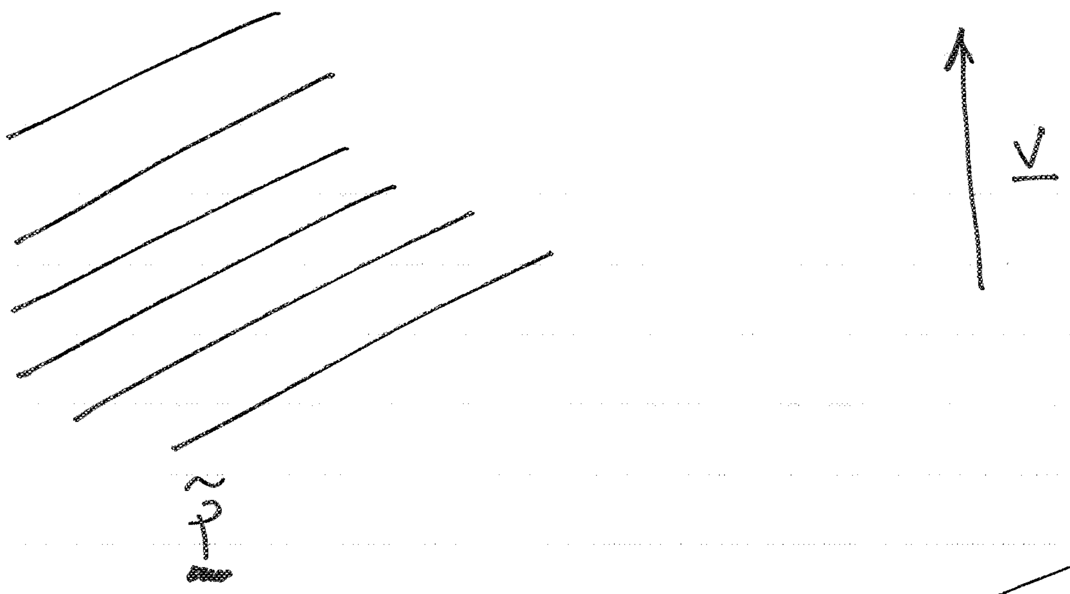
$$\underline{\underline{\eta}}\,(\underline{A}, \underline{B}) = \underline{\underline{\eta}}\,(\underline{B}, \underline{A})$$

$\square$

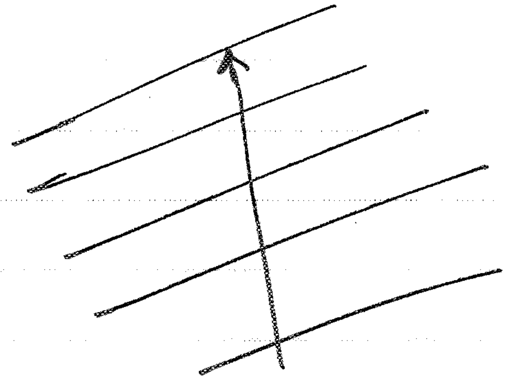
In a similar way we can define a $\binom{0}{1}$ form as a function (operator) that acting on a vector returns a number (scalar)

$$\tilde{p}\,(\underline{V}) = p_\alpha V^\alpha$$

The graphical representation of a $\binom{0}{1}$ form or <u>one-form</u> is via surfaces



So that $\tilde{p}(\underline{V})$ is



the number of surfaces crossed by $\underline{V}$. The larger this number, the smaller the spacing among surfaces and the larger the magnitude of the one-form.

Just like vectors, one-forms have components and basis

$$\tilde{p}(\underline{V}) = p_\alpha V^\alpha$$: p_α are the components of $\tilde{p}$

$$\tilde{p}(\underline{e}_\alpha) = p_\mu \underline{e}^\mu{}_\alpha = p_\mu \delta^\mu{}_\alpha = p_\alpha$$

ie p_α are the components of $\underset{\sim}{p}$ in the coordinate vector basis

I can also write $\underset{\sim}{p} = p_\alpha \tilde{\omega}^\alpha$ basis one-form

and derive that

$$\underset{\sim}{p}(\underline{V}) = p_\alpha V^\alpha = p_\alpha \tilde{\omega}^\alpha(\underline{V}) = p_\alpha \tilde{\omega}^\alpha(V^\beta \underline{e}_\beta)$$

$$= p_\alpha V^\beta \tilde{\omega}^\alpha(\underline{e}_\beta)$$

$$\Longrightarrow \tilde{\omega}^\alpha(\underline{e}_\beta) = \delta^\alpha{}_\beta$$

$\tilde{\omega}^0 = \{1, 0, 0, 0\}$
$\tilde{\omega}^1 = \{0, 1, 0, 0\}$
$\tilde{\omega}^2 = \ldots$
$\tilde{\omega}^3 = \{0, 0, 0, 1\}$

in other words $\tilde{\omega}^\alpha$ are dual to $\underline{e}_\alpha$

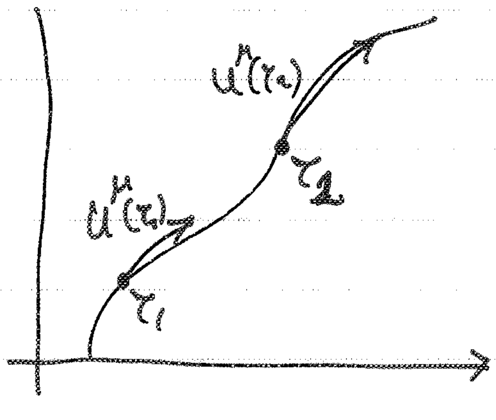
At this point it's clear that one-forms are dual to vectors and just different ~~integers~~ faces of the same coin.

Note that a one-form can be applied also to a function, in which case it represents the <u>gradient</u> of that function.

Consider a curve $\mathcal{C}: \{X^\mu(\tau)\}$:

$X^\mu(\tau)$ could be the worldline of an observer and τ the proper time

$\frac{dX^\mu}{d\tau} = U^\mu(\tau)$: tangent vector of $\mathcal{C}$ and four-velocity of the observer



Let $\phi = \phi(X^\mu)$ a scalar function along $\mathcal{C}$ and thus

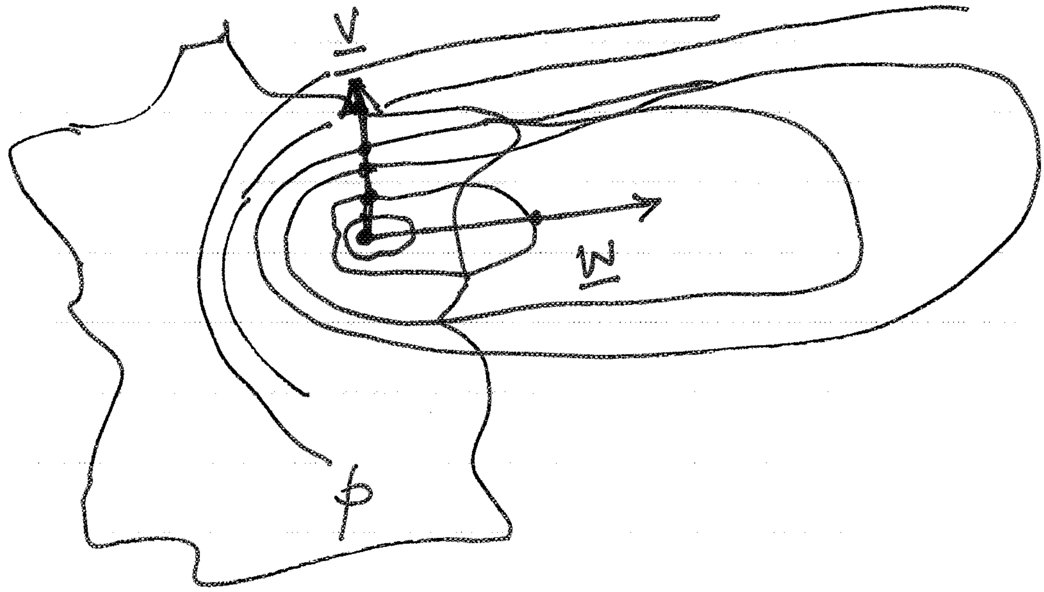
$$\phi(X^\mu) = \phi(X^\mu(\tau)) = \phi(\tau)$$

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial X^\mu}\frac{\partial X^\mu}{\partial\tau} = \partial_\mu\phi\,\frac{dX^\mu}{d\tau} = \partial_\mu\phi\, U^\mu$$

$$= \phi_\mu U^\mu$$

Thus $\phi_\mu = \frac{\partial}{\partial X^\mu}\phi$ = gradient of ϕ is a one-form

$$\tilde{d}\phi = (\partial_t\phi, \partial_x\phi, \partial_y\phi, \partial_z\phi)$$



ϕ could be a pressure at a given altitude

$\tilde{d}\phi$ has little sense

$$\tilde{d}\phi(\underline{V}) = \partial_\mu\phi\, V^\mu = V^\mu \partial_\mu \phi$$

= gradient of ϕ along V is a meaningful number

In the example above

$$V^\mu \partial_\mu \phi >> W^\mu \partial_\mu \phi \iff$$

steep gradient in the North direction

Recap

- $\underset{=}{\eta}$: $\binom{0}{2}$-form, eg metric tensor

$$\underset{=}{\eta}(\underline{V}, \underline{W}) = V^\alpha W^\beta \eta_{\alpha\beta}$$

- $\tilde{p}$: $\binom{0}{1}$-form, eg gradient

$$\tilde{p}(\underline{V}) = p_\alpha V^\alpha = V^\alpha \partial_\alpha \phi$$

- $\underline{V}$: $\binom{1}{0}$-form, ie vector and dual to one-form

$$\underline{V}(\tilde{p}) = V^\alpha p_\alpha = \tilde{p}(\underline{V})$$

- $\binom{M}{N}$-form : operator combining M one-forms with N vectors

$$\underset{\equiv}{R}(\tilde{p}, \underline{U}, \underline{V}, \underline{W}) = R^\alpha{}_{\beta\delta\gamma}\, p_\alpha U^\beta V^\delta W^\gamma$$

Raising-lowering indices

We have seen that the metric tensor acts on vectors to yield a scalar but it can also act on one vector to yield a one-form

$$\underline{\underline{\eta}}(\underline{V}, \underline{W}) = \eta_{\alpha\beta} V^\alpha W^\beta$$

$$\underline{\underline{\eta}}(\underline{V}, \quad) = \eta_{\alpha\beta} V^\alpha = W_\beta \quad : \text{one-form } \tilde{W}$$

If we act $\tilde{V}$ on another vector, $\underline{W}$ then we must end up with $V_\alpha W^\alpha$

$$\cancel{\underline{\underline{\eta}}(\underline{V}, \quad)(\underline{W}) = \tilde{V}(\underline{W}) = V_\beta W^\beta = \eta_{\alpha\beta} V^\beta}$$

$$\underline{\underline{\eta}}(\underline{V}, \quad)(\underline{W}) = \eta_{\alpha\beta} V^\alpha W^\beta = V_\beta W^\beta \Rightarrow$$

$$\boxed{\eta_{\alpha\beta} V^\alpha = V_\beta}$$: the metric has mapped a vector into a one-form (lowered the index)

$$\eta_{\alpha\beta} V^\alpha = V_\beta$$

The inverse is also true

$$\boxed{V^\beta = \eta^{\alpha\beta} V_\alpha}$$

metric maps one-form into vectors: <u>raising index</u>

$\eta^{\alpha\beta}$ is inverse of $\eta_{\alpha\beta}$

$$\eta^{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

This can be done over and over again

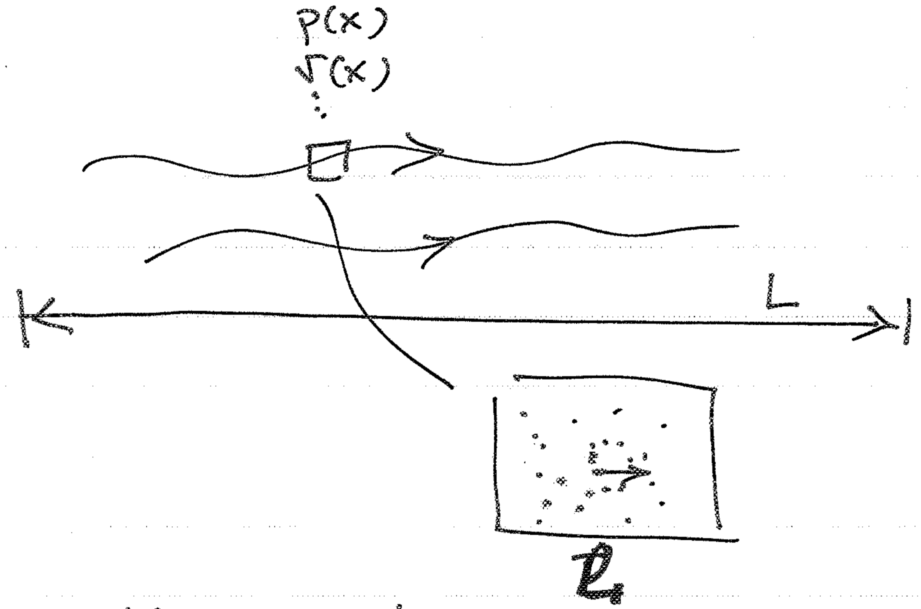
$$R^\alpha{}_{\beta\gamma\delta}\, p_\alpha U^\beta V^\gamma W^\delta =$$

$$R^\alpha{}_{\beta\gamma\delta}\, \underbrace{p^\mu \eta_{\mu\alpha}}_{p_\alpha}\; \underbrace{U_\nu \eta^{\nu\beta}}_{U^\beta}\; V^\gamma W^\delta =$$

$$= R_\mu{}^\nu{}_{\gamma\delta}\; p^\mu u_\nu V^\gamma W^\delta$$

Fluids

A collection of particles whose properties can be defined in terms of averages performed on scales sufficiently large to contain a large (statistically significant) sample but as small as possible to allow for a continuum description



ℓ sufficiently large to contain a large number of particles

VERY IMPORTANT $\ell \gg \lambda$: mean free path and this implies that properties[⊙] are constant in ℓ

$\ell \ll L$: scale of the problem.

These definitions are clearly weak and mathematically vague but the very large number of

⊙ pressure, mean velocity, density, etc...

particles of ordinary fluids make them
very unstable

particles of ordinary fluids make them very natural. Note that I consider "fluids" also what is usually referred to as a gas

Given a single particle of the fluid, there are two important associated quantities

1) $\underline{u}$: four velocity

$$u^\alpha \equiv dx^\alpha / d\tau$$

τ: proper time

then the 4-velocity obeys the normalization condition:

$$u^\alpha u_\alpha = -1$$

$$ds^2 = -d\tau^2$$

$dx^i = 0$

this is the time measured by a clock at rest in a given frame

$$\boxed{\begin{aligned} ds^2 &= \eta_{\alpha\beta}\, dx^\alpha dx^\beta \\ &= \eta_{00} (dx^0)^2 \\ &= -dt^2 = -d\tau^2 \end{aligned}}$$

Proof

$$ds^2 = -d\tau^2 \Rightarrow \frac{ds^2}{d\tau^2} = -1$$

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \Rightarrow \frac{ds^2}{d\tau^2} = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

$$= \eta_{\alpha\beta} u^\alpha u^\beta = u^\alpha u_\alpha = -1$$

qed

Note

$\underline{u}$ is called 4-velocity because the spatial components are those of a 3-velocity for small speeds.

In a momentarily comoving reference frame (MCRF)

$$u^{\mu}_{MCRF} = \{+1, 0, 0, 0\} \qquad u^{\alpha} u_{\alpha} = -1$$

$$\underline{u} = u^{0} \underline{e}_{0} \quad \text{: along time direction} \qquad (u^{0})^{2} \eta_{00} = -1$$

In a frame which is moving at speed v in the x-direction, then (ie a general frame at speed v)

$$u^{\mu} = \{\gamma, \gamma v, 0, 0\} \qquad \text{as deduced by Lorentz transf.}$$

$$u^{x} = \gamma v = \frac{v}{\sqrt{1-v^{2}}} \cong v\left(1+\frac{v^{2}}{2}\right) \sim v \quad \text{reqd}$$

2) $\underline{p}$: four-momentum

$$\underline{p} \equiv m_0 \underline{U}$$

$$p^{\mu}_{MCRF} = m_0 U^{\mu}_{MCRF} = m_0 \{1, 0, 0, 0\}$$

m_0 is the "rest-mass" (ie the mass in the frame in which it is at rest)

in general, for a particle moving in x-direction

$$p^{\mu} = m_0 \{\gamma, \gamma v^x, 0, 0\}$$

$$p^0 = m_0 \gamma = \frac{m_0}{\sqrt{1-v^2}} \cong \underbrace{m_0}_{\text{rest mass energy}} + \underbrace{\frac{m_0 v^2}{2}}_{\text{kinetic energy}}$$

because of this

$p^0 = E$: energy of the particle

$$p_0 = \eta_{00} p^0 = -E$$

(*) Note that $\underline{u}$ cannot be defined for a photon

$$ds^2 = 0 = -d\tau^2$$

$$\underline{u} = \frac{d\bar{x}\cdot d\bar{x}}{d\tau\, d\tau} = \frac{0}{0}$$

This doesn't imply that there isn't a tangent to the trajectory of a photon, which is indeed given by $d\bar{x}$.

Rather it stresses that it is not for an observer to boost to a speed in which the photon is at rest. The speed of light is c for all.

It's easy to show that the energy of the particle relative to our observer with four-velocity $\underline{u}$

is $E = -\underline{p}\cdot\underline{u} = -p^\alpha u_\alpha$

(*) must be valid for any observer and so also for a MCRF one

$$= -p^\alpha u^\beta_{MCRF}\,\eta_{\alpha\beta}$$

$$= -p^0 u^0_{MCRF}\,\eta_{00} = p^0 u^0_{MCRF}$$

$$= p^0 = E \quad \text{qed}$$

(*)

$\square$

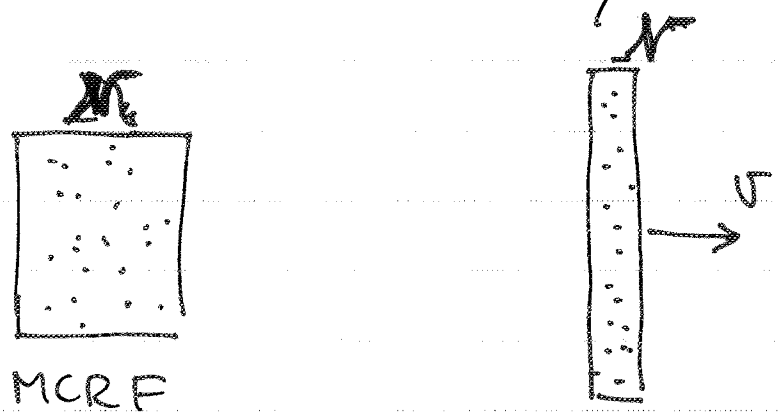
If $\underline{u}$ and $\underline{p}$ are useful quantities for a single particle, there is one which is useful also in terms of a collection of particles: number-flux density

To simplify our analysis, let's consider a very special fluid: dust, ie a collection of identical particles which

are all at rest in one inertial frame (this is also referred to as a zero-pressure fluid; more later)
Let N be the number of particles and n its number density

$$n = \frac{N}{V}$$

In another inertial frame moving at speed v the number density will be $\bar{n} = \gamma n$

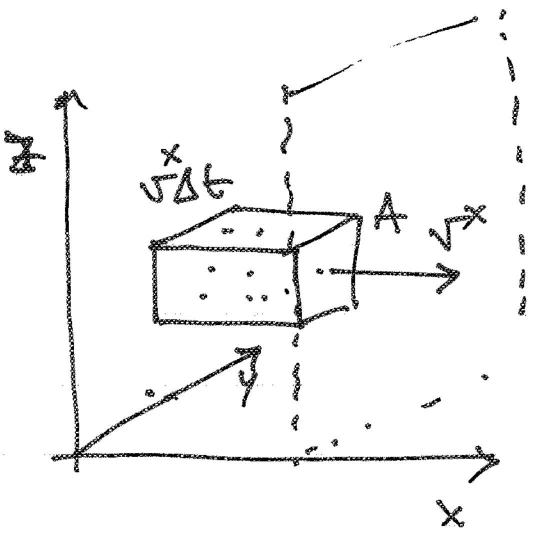


n

$\bar{n} = \gamma n$
ie the number density increases because of Lorentz contraction while N remain the same

Similarly we can measure the number flux

$$(\text{flux})^x = \frac{\text{number of particles}}{\text{unit area} \cdot \text{unit time}}$$



$$(\text{flux})^x = \frac{\gamma n \cdot v^x \Delta t A}{\Delta t A} = \gamma n v^x$$

I can therefore define the $\underline{N}$: number flux ~~density~~ 4-vector

$$\underline{N} = n \underline{u}$$

$$N^\mu = n\{\gamma, \gamma v^x, 0, 0\}$$

γ: number density

γv^x: number flux across surface $x = \text{const}$

Note that $\underline{N} \cdot \underline{N} = N_\mu N^\mu = -n^2$

n: number density in the MCRF

$$n = (-\underline{N} \cdot \underline{N})^{1/2}$$

just like rest mass is the one in the MCRF

If a vector $\underline{N}$ is sufficient to measure the number of particles moving in a certain direction, it's clear that to measure the flux of momentum component

a given

in a specified direction we need a tensor of rank 2

$$T^{\alpha\beta} = e\, u^{\alpha} u^{\beta}$$: stress-energy tensor for dust

$$T^{00} = e\, u^{0} u^{0} = \gamma^2 e = \frac{e}{\sqrt{1-v^2}^{\,2}}$$: energy density

$$T^{0x} = e\, u^{0} u^{x} = \gamma^2 e\, v^{x} = \frac{\gamma^2 e\, v^{x}}{1-v^2}$$: energy flux

$$T^{xx} = e\, u^{x} u^{x} = \gamma^2 e\, (v^{x})^2$$: momentum flux

Note that v and v^x are NOT distinct

v : velocity of observer ⟺

v^x: " of dust particles

What is e?

$$T^{00}_{MCRF} = e u^0 u^0 = e \text{ : energy density}$$

$$= \rho(1+\epsilon)$$

specific internal energy

Note that the momentum flux is equivalent to pressure, indeed the pressure is the manifestation of a flux of momentum across a surface

$$\text{pressure} = \frac{\text{force}}{\text{area}} = \frac{\text{momentum}}{\text{time area}} = \frac{\text{momentum}}{\text{time} \cdot \frac{\text{volume}}{\text{distance}}}$$

$$= \frac{\text{momentum}}{\text{Volume}} \cdot \text{velocity}$$

$$= (\text{mom. dens}) \cdot \text{velocity}$$

$$= \text{mom. flux} \;\square$$

More generically

T^{0j} : energy flux in x^j direction

T^{ij} : (mom. density flux)^i in the x^j direction

The extension of the stress-energy tensor to a <u>perfect</u> <u>fluid</u> (ie a fluid with zero viscosity and heat losses) is straightforward

$p \neq 0$ and contributes to energy density (same dimensions)

$$T^{\alpha\beta} = (e+p)\, u^\alpha u^\beta + p\, g^{\alpha\beta}$$

$$T^{00} = (e+p)\, u^0 u^0 + p\, g^{00}$$

$$= \gamma^2 (e+p) - p$$

$$\sim (1+v^2)(e+p) - p = e(1+v^2) + p v^2$$

$\epsilon \ll \rho$

$p \ll \rho$

$$= \rho\,(1+v^2)$$

$$\cong \frac{\rho}{(1-v^2)^{1/2}} + \rho \frac{v^2}{2}$$

$$T^{0k} = (e+p)\, u^0 u^k = \underbrace{(e+p)}_{\text{inertial mass-en. density}} \gamma^2 v^k$$

$$T^{jk} = (e+p)\, \gamma^2 v^j v^k + p\,\delta^{jk}$$

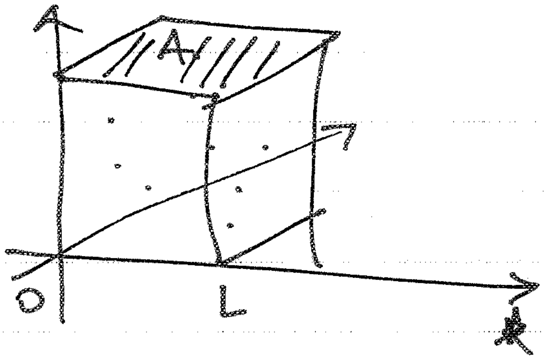
$j=k$

$$= (e+p)\, \gamma^2 (v^k)^2 + p$$

isotropic pressure contribution

Conservation laws

We want to derive expressions that quantify the conservation of baryon number (particle) energy and momentum



let's consider a MCRF

$$\Delta t \, \frac{\partial}{\partial t} \underbrace{(n \bar{V})}_{\text{tot. number}} = (\text{flux-in}) - (\text{flux-out})$$

$$= (n v^x A \Delta t)_{x=0} - (n v^x A \Delta t)_{x=L}$$

$$+ (\quad)_{y=0} - (\quad)_{y=L}$$

$$+ (\quad)_{z=0} - (\quad)_{z=L}$$

$$= - \frac{\partial}{\partial x}(n v^x) \underbrace{A \Delta t L}_{\text{vol.}} - \frac{\partial}{\partial y}(n v^y) \ldots$$

$$\partial_t n = - \partial_x (n v^x) - \partial_y (n v^y) - \partial_z (n v^z) \iff$$

$$\boxed{\frac{\partial}{\partial x^\alpha}(n u^\alpha) = 0}$$ rest mass conservation

similarly one can write for the energy and momentum

$$\frac{\partial}{\partial x^\alpha}(T^{\alpha\beta}) = 0$$

$\beta = 0$: energy conservation

$\beta = k$: mom. conserv. (Euler eq.)

These equations are those of special-relativistic hydrodynamics and are also written as

$$\partial_\alpha (n u^\alpha) = 0 \iff (n u^\alpha)_{,\alpha} = 0$$

$$\partial_\alpha (T^{\alpha\beta}) = 0 \iff (T^{\alpha\beta})_{,\alpha} = 0$$

$\square$

The same conservation laws can be defined for EM

$$T^{\alpha\beta}_{EM} = \frac{1}{\mu_0}\left[F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right]$$

where $F_{\alpha\beta} = 2 u_{[\alpha}E_{\beta]} + \eta_{\alpha\beta\gamma\delta}u^\gamma B^\delta$
Faraday tensor

Symmetric and antisymetric tensor

$T_{\alpha\beta} = T_{\beta\alpha}$: sym.

$T_{\alpha\beta} = -T_{\beta\alpha}$: anti sym.

$T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right)$: anti/sym by construction

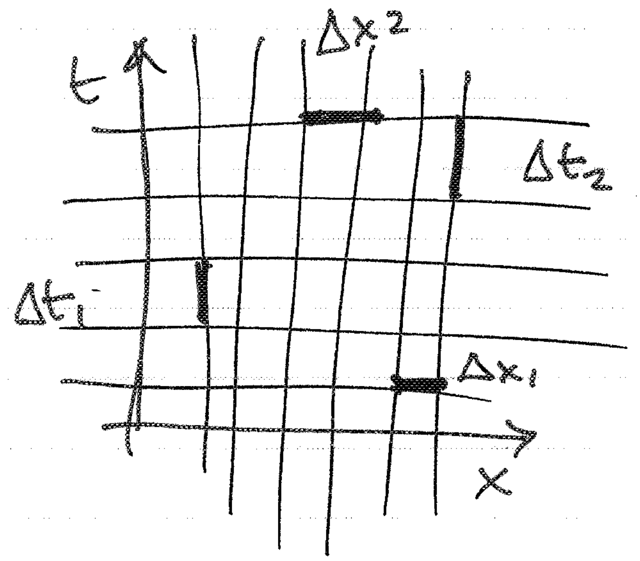
$T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right)$

Ex.

$$F_{\alpha\beta} = 2u_{[\alpha}E_{\beta]} + \eta_{\alpha\beta\gamma\delta}\, u^{\gamma} B^{\delta}$$

$$= u_{\alpha}E_{\beta} + u_{\beta}E_{\alpha} + \eta_{\alpha\beta\gamma\delta}\, u^{\gamma} B^{\delta}$$

SR relied on the existence of inertial frames, ie set of coordinates such that the separation among points does not change in space and the time intervals are the same everywhere



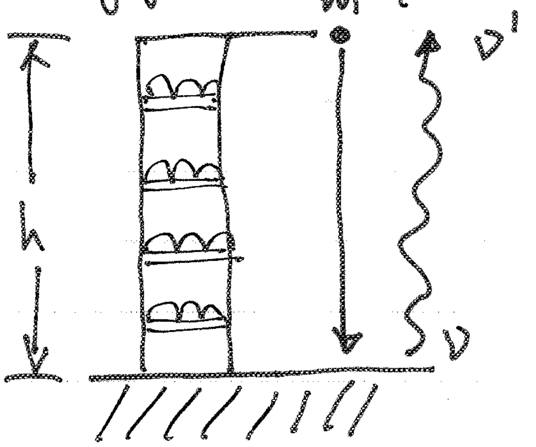
$\Delta x_1 = \Delta x_2$
$\Delta t_1 = \Delta t_2$

Different observers in relative motion will each have his own inertial frame and a Lorentz transformation takes us from one to the other

$$x^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\nu}\, x^{\nu}$$

This mental construction is incompatible with the presence of a gravitational field. However it is not totally useless...

GPS system is already a clear proof but to be convinced one can perform the "gravitational redshift" experiment first suggested by Einstein



- take particle of mass m and let it fall from height h

- on the ground it will have energy
$$m + \frac{m v^2}{2} = m + mgh$$

- convert all the energy in a single photon of energy $E = h\nu = m + mgh$ and send it back!

- convert the photon into a particle of mass $m' = E' = h\nu'$

Because we have not created energy
$m' = m$

$$\frac{E'}{E} = \frac{h\nu'}{h\nu} = \frac{m}{m + mgh} \simeq 1 - gh + o(v^4)$$

$E' < E$ or $\boxed{\nu' < \nu}$ the photon at the top of the tower has been redshifted

This result is incompatible with the existence of an inertial frame on Earth. In such a frame, in fact the clocks at the bottom and top of the tower should tick at the same rate :

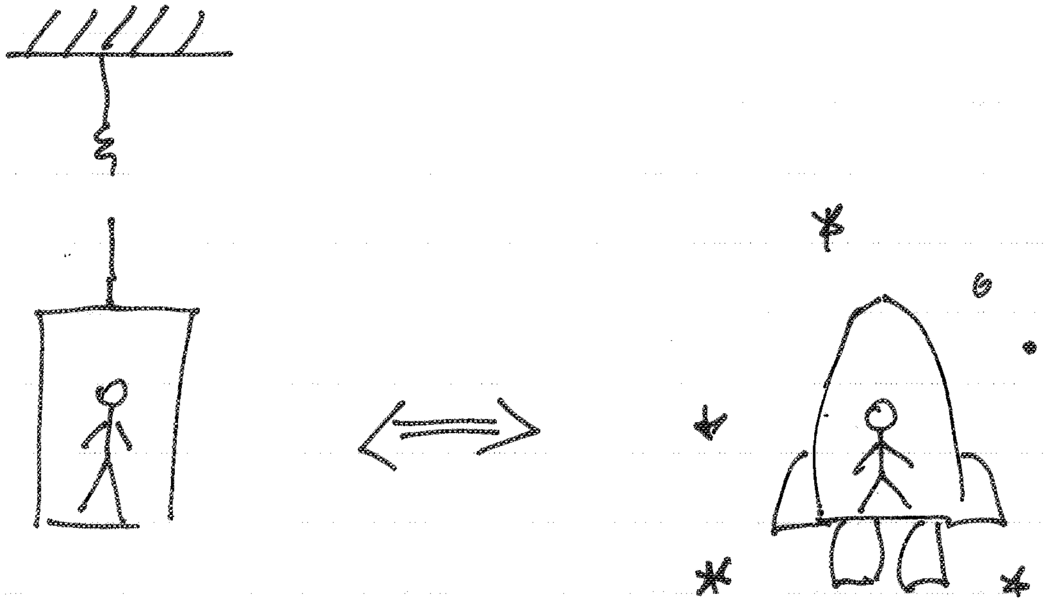
$$\frac{\Delta t'}{\Delta t} = 1 = \frac{\nu}{\nu'} :$$

but we have seen $\nu' < \nu \Rightarrow \Delta t' > \Delta t$

This shows that a <u>global</u> inertial frame is incompatible with a gravitational field but does not exclude the existence of a <u>local</u> inertial frame, i.e. the possibility of having an inertial frame in the neighborhood of any event

This is indeed what expressed by the equivalence principle

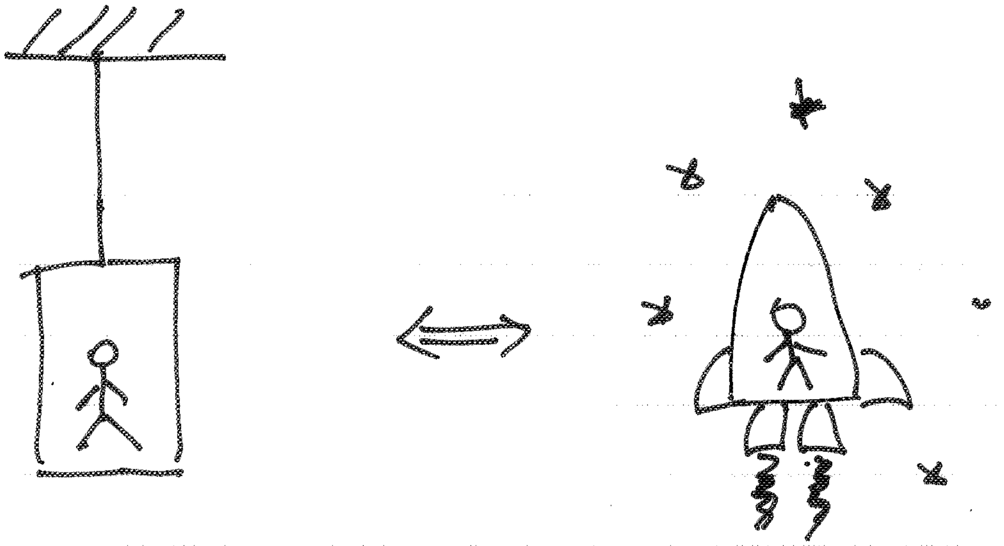
If gravity makes an inertial frame impossible, the way to produce an inertial frame is to remove gravity, ie to be in free fall



The two frames are equivalent and inertial. The problem is that in both cases they are not global (ie they exist only in a portion of the spacetime)

This is the strong equivalence principle: the laws of physics in a free falling frame are the same as in an inertial frame (ie as in SR)

Note that there is the opposite (weak) equivalence principle starting the equivalence between an accelerated frame and gravity



the "weight" in the two frames is the same.

How do we reveal the existence of gravitational fields?

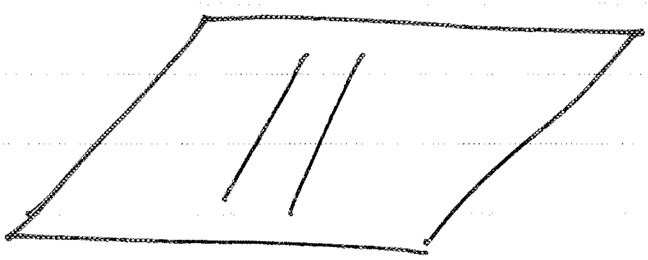
Free particles in SR move on straight lines so that two particles on initially parallel lines do not ever intersect.

Loss of an inertial frame in a gravitational field will imply loss of parallelism and hence we can associate gravity with curvature

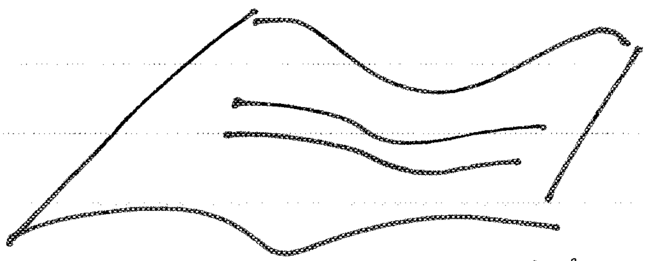
The reason for this is that parallel lines intersect in a curved space.

The logic is therefore:

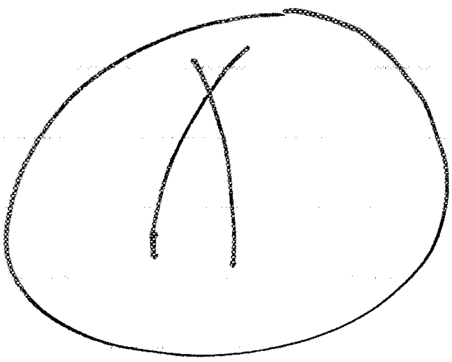
$$\begin{tikzcd} \left(\text{gravitational field}\right) \arrow[r, Leftrightarrow, "1"'] \arrow[d, Leftrightarrow, "4"] & \left(\text{loss of global inertial frame}\right) \arrow[d, Leftrightarrow, "2"'] \\ \left(\text{presence of curvature}\right) \arrow[r, Leftrightarrow, "3"] & \left(\text{loss of parallelism for free particles}\right) \end{tikzcd}$$



parallel



parallel



intersect

Before discussing the relation between gravity and curvature (ie Einstein eqs), we need to define the mathematical tools necessary for its measurement

The most important of these tools is the covariant derivative, ie a derivative that accounts also for the fact that the coords can change locally.

Note that we do not really need a curved space to define the covariant derivative.

Suppose we want to compute

$$\frac{\partial}{\partial x^\alpha} \bar{V} = \frac{\partial}{\partial x^\alpha} \left(V^\beta \underline{e}_\beta \right)$$

$$= \underbrace{\frac{\partial V^\beta}{\partial x^\alpha} \underline{e}_\beta}_{\substack{\text{standard} \\ \text{derivative} \\ \partial_\alpha V^\beta \underline{e}_\beta \\ V^\beta{}_{,\alpha} \underline{e}_\beta}} + \underbrace{V^\beta \frac{\partial \underline{e}_\beta}{\partial x^\alpha}}_{\substack{\text{new term:} \\ \text{accounting for} \\ \text{change of basis} \\ \text{vectors}}}$$

Let's restrict to 2D, cartesian coords $\{x, y\}$

$\underline{e}_x \cdot \underline{e}_x = 1 = \underline{e}_y \cdot \underline{e}_y$

$0 = \underline{e}_x \cdot \underline{e}_y$

$$ds^2 = g_{xx}\, dx^2 + g_{yy}\, dy^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad (g_{ij})^{-1} = g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$V^\beta \frac{\partial}{\partial x^\alpha} \underline{e}_\beta \rightsquigarrow V^j \frac{\partial}{\partial x^i} \underline{e}_j$$

If $i = x$: $V^x \frac{\partial}{\partial x} \underline{e}_x + V^y \frac{\partial}{\partial x} \underline{e}_y = 0$

because both $\underline{e}_x$ and $\underline{e}_y$ do not vary in space (they're 1).

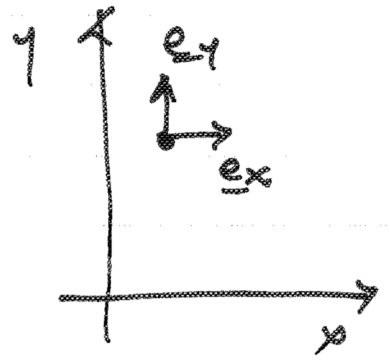
Let's consider in the same plane a polar coordinate system $\{r, \theta\}$

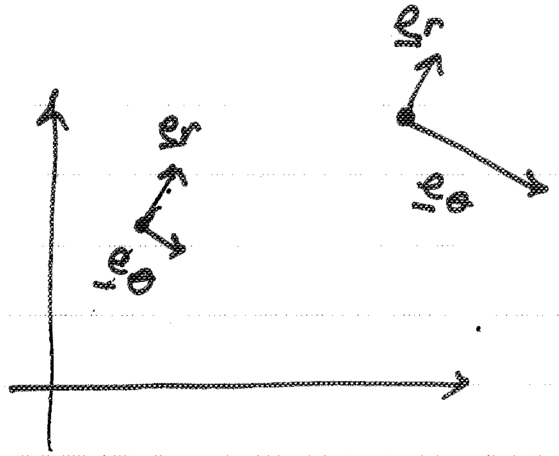
$$ds^2 = g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 = dr^2 + r^2 d\theta^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad ; \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$$

$\Rightarrow \quad \underline{e}_r \cdot \underline{e}_r = 1 \qquad \underline{e}_\theta \cdot \underline{e}_\theta = r^2$

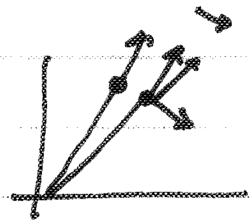
$\underline{e}_r \cdot \underline{e}_\theta = 0$





A bit of algebra shows that

$$\frac{\partial}{\partial r}\underline{e}_r = 0 \quad ; \quad \frac{\partial}{\partial \theta}\underline{e}_r = \frac{1}{r}\underline{e}_\theta$$



$$\frac{\partial}{\partial r}\underline{e}_\theta = \frac{1}{r}\underline{e}_\theta \; ; \quad \frac{\partial}{\partial \theta}\underline{e}_\theta = -r\underline{e}_r$$

What matters here is that $\frac{\partial}{\partial x^i}\underline{e}_j \neq 0$

so the derivative has a new term accounting for the change in the curvilinear coords.

let's go back to 41)

$= \frac{\partial}{\partial x^\alpha} \underline{e}_\beta$ will be a vector and hence expressed in terms of a basis

$= (\Gamma^\mu_{\alpha\beta})\, \underline{e}_\mu$

so

$$\frac{\partial}{\partial x^\alpha} \bar{V} = \frac{\partial V^\beta}{\partial x^\alpha} \underline{e}_\beta + V^\beta \Gamma^\mu_{\alpha\beta} \underline{e}_\mu$$

$$= \frac{\partial V^\beta}{\partial x^\alpha} \underline{e}_\beta + V^\mu \Gamma^\beta_{\alpha\mu} \underline{e}_\beta$$

$$= \underbrace{\left(\frac{\partial V^\beta}{\partial x^\alpha} + V^\mu \Gamma^\beta_{\alpha\mu} \right)}_{V^\beta{}_{;\alpha} = \nabla_\alpha V^\beta} \underline{e}_\beta$$

ie

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\mu} V^\mu = \nabla_\beta V^\alpha$$

A corresponding derivative can be derived for a one-form

$$\nabla_\beta V_\alpha = V_{\alpha;\beta} = V_{\alpha,\beta} - \Gamma^{\mu}_{\alpha\beta} V_\mu$$

or a rank-2 tensor

$$\nabla_\alpha V^{\mu\nu} = V^{\mu\nu}{}_{,\alpha} + \Gamma^{\mu}_{\alpha\beta} V^{\beta\nu} + \Gamma^{\nu}_{\alpha\beta} V^{\mu\beta}$$

$\Gamma^{\alpha}_{\beta\gamma}$: Christoffel symbol (affine connection)

$\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$: sym. on lower indices

Note that $\Gamma^{\alpha}_{\beta\gamma}$ is not a tensor as it does not transform as a tensor.

Let's consider an example : 2D, $\{r, \theta\}$

$= V^{\alpha}{}_{;\alpha}$: divergence

$= V^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}_{\beta\alpha} V^{\beta}$

$= V^{r}{}_{,r} + V^{\theta}{}_{,\theta} + \Gamma^{\alpha}_{r\alpha} V^{r} + \Gamma^{\alpha}_{\theta\alpha} V^{\theta}$

$$\boxed{\begin{aligned} \Gamma^{\alpha}_{r\alpha} &= \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} = \frac{1}{r} \\ \Gamma^{\alpha}_{\theta\alpha} &= \Gamma^{r}_{\theta r} + \Gamma^{\theta}_{\theta\theta} = 0 \end{aligned}}$$

$$= V^r{}_{,r} + V^\theta{}_{,\theta} + \frac{1}{r} V^r$$

$$= \frac{1}{r} \partial_r (r V^r) + \partial_\theta V^\theta$$

: well-known expression of the divergence in polar coords.

A very important covariant derivative is that of the metric

$$g_{\mu\nu;\alpha} = 0 \qquad (3)$$

This is easy to show and there are two different ways at least:

$$V_\alpha{}_{;\beta} = (V^\alpha g_{\alpha\mu})_{;\beta} = V^\alpha{}_{;\beta}\, g_{\alpha\mu} + V^\alpha g_{\alpha\mu;\beta}$$

$$= \cancel{(V^\mu{}_{;\beta})\, g_{\mu\alpha}} \qquad = \cancel{V^\mu{}_{;\beta}\, g_{\mu\alpha}} + V^\alpha g_{\alpha\mu;\beta} \Rightarrow$$

$$V^\alpha g_{\alpha\mu;\beta} = 0 \Rightarrow g_{\alpha\mu;\beta} = 0 \qquad \text{qed}$$

The second one is that (3) is a tensor equation and thus valid in all frames.

If ϕ go to a cartesian coord. system then $g_{\alpha\beta,\mu} = 0$; $\Gamma^\mu{}_{\alpha\beta} = 0 \Rightarrow g_{\alpha\beta;\mu} = 0$

Another possibility is to see $g_{\mu\nu;\alpha}$ as the components of a tensor ∇g and if zero in one frame they are zero in all frames.

How do we calculate the Γs ?

Exploiting (3)!

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^{\rho}_{\mu\lambda} g_{\rho\nu} - \Gamma^{\rho}_{\nu\lambda} g_{\mu\rho} = 0 \quad (a)$$

$$g_{\nu\lambda;\mu} = g_{\nu\lambda,\mu} - \Gamma^{\rho}_{\nu\mu} g_{\rho\lambda} - \Gamma^{\rho}_{\lambda\mu} g_{\nu\rho} = 0 \quad (b)$$

$$g_{\lambda\mu;\nu} = g_{\lambda\mu,\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} - \Gamma^{\rho}_{\mu\nu} g_{\lambda\rho} = 0 \quad (c)$$

$(b) + (c) - (a) \implies$

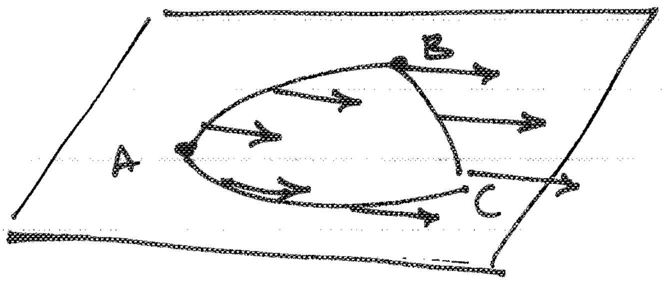
$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\gamma\delta,\beta} + g_{\delta\beta,\gamma} - g_{\beta\gamma,\delta} \right)$$

It looks complicated but it's just algebra. Just remember that $\Gamma \rightarrow$ partial derivatives of the metric

~~if these are zero in one frame, they are zero in all frames.~~

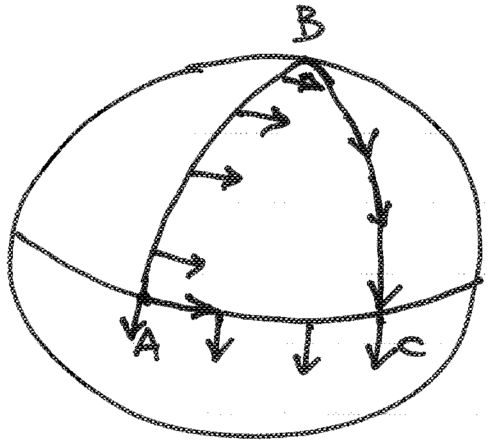
At this point we need a "curvature detector", ie an object that allows us to detect that there is curvature.

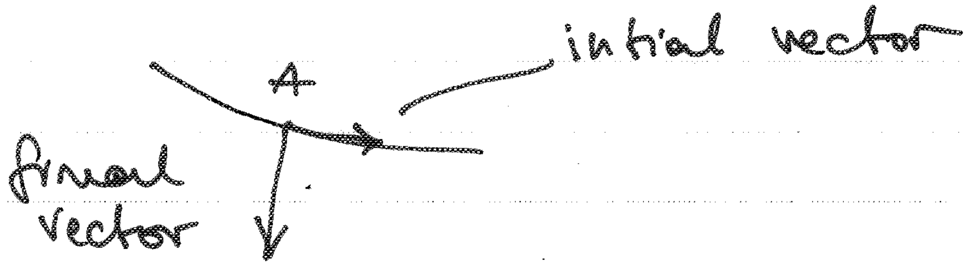
Any idea?



I can move a vector along a closed loop making sure to keep it always parallel to the same direction.
This is a "parallel transport". In a flat spacetime the result of this operation yields the same vector but this is clearly not the case if the surface is curved

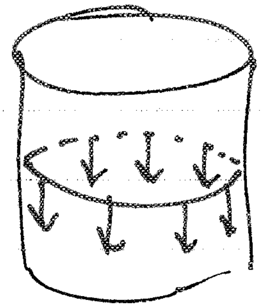
The point is that one must always "stay" on the surface (be tangent) and have the vector drawn as parallel as possible to the previous one.





<u>I</u>nitial and final vectors are clearly different.

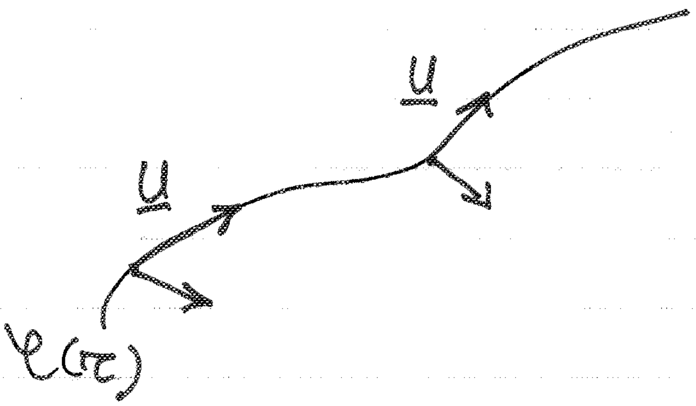
Careful on your intuition of curved!



A cylinder is <u>intrinsically</u> flat: you can open and lay on a flat surface without creases.

We need now a mathematical definition of "parallel transport".

We want to move a vector along a curve without changing it.
A curve is defined by its tangent vector



$$\underline{u} = \frac{dx^\mu}{d\tau}\underline{e}_\mu$$

(4) $$\nabla_{\underline{u}} \underline{V} = 0$$: the variation of $\underline{V}$ along the curve with $\underline{u}$ tg vector are zero

In coordinate form

$$V^\alpha{}_{;\beta} U^\beta = 0 = V^\alpha{}_{,\beta} U^\beta + \Gamma^\alpha_{\beta\mu} U^\mu U^\beta$$

The concept of parallel transport allows also to define a "straight" curve

A "straight" curve is a curve that parallel transports its tangent vector

$$\nabla_{\underline{u}} \, \underline{u} = 0$$

$$U^\alpha{}_{;\beta} U^\beta = U^\alpha{}_{,\beta} U^\beta + \Gamma^\alpha_{\beta\mu} U^\mu U^\beta$$

but $U^\alpha = \dfrac{dx^\alpha}{d\tau}$ $\qquad U^\beta \partial_\beta = \dfrac{dx^\beta}{d\tau} \dfrac{d}{dx^\beta} = \dfrac{d}{d\tau}$

$$= \frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) + \Gamma^\alpha_{\beta\mu} \frac{dx^\mu}{d\tau} \frac{dx^\beta}{d\tau} = 0 \quad (4)$$

This is a curve of "extremal" (smallest) length, ie a <u>geodesic</u>

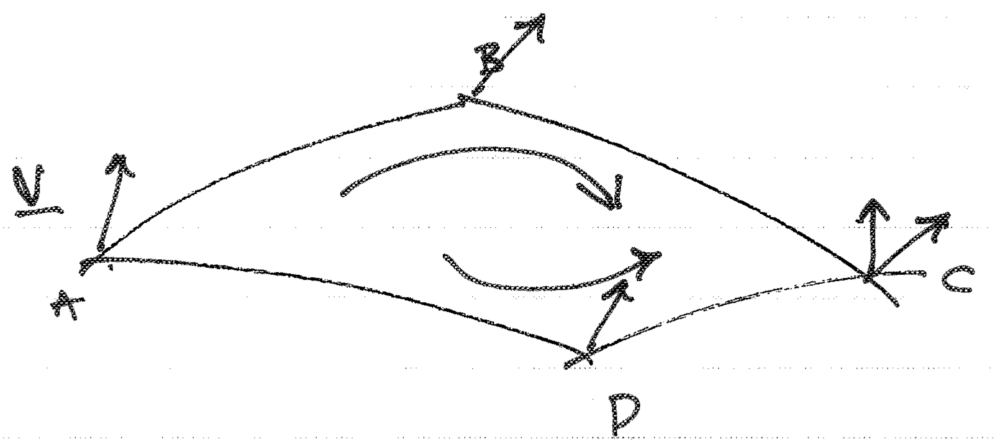
Note that a geodesic is "straight" only in flat spacetime and remain the curve of minimal distance in any spacetime.

If $\Gamma^\mu_{\alpha\beta} = 0$, (4) $\iff$

$(x^\alpha)'' = 0 \Rightarrow (x^\alpha)' = \text{const} \Rightarrow x^\alpha = k\tau$

straight line

We can finally define the "curvature" following the same logic: move along a closed loop parallelly transporting a vector



Alternatively you can compare the vectors transported in two different directions

$$\nabla_{[\alpha}\nabla_{\beta]} V^{\mu} = V^{\mu}{}_{[;\alpha\beta]} = \tfrac{1}{2}\left(V^{\mu}{}_{;\alpha\beta} - V^{\mu}{}_{;\beta\alpha}\right)$$

$$= \frac{1}{2} R^{\mu}{}_{\nu\beta\alpha} V^{\nu}$$

$\underline{\underline{R}}$ is the Riemann tensor.

Note that while $V^{\mu}{}_{,\alpha\beta} = V^{\mu}{}_{,\beta\alpha}$

2 cov. deriv do not commute $\quad V^{\mu}{}_{;\alpha\beta} \neq V^{\mu}{}_{;\beta\alpha}$

R is just a combination of Γs, ie 1st and 2nd - order derivs of the metric

$$R^{\mu}{}_{\gamma\beta\alpha} = \Gamma^{\mu}_{\gamma\alpha,\beta} - \Gamma^{\mu}_{\gamma\beta,\alpha} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\gamma\beta}$$

Note $R^{\alpha}{}_{\beta\gamma\delta} = 0 \iff \nabla_{[\alpha}\nabla_{\beta]} V^{\mu} = 0$

$\updownarrow$ $\downarrow$

(zero curvature) $\iff$ (no difference in // transp. vector)

Properties of Riemann tensor

1) Antisym. on last pair of indices

$$R^{\alpha}{}_{\beta\gamma\delta} = - R^{\alpha}{}_{\beta\delta\gamma}$$

2) Sym. on exchange of 1st and 2nd pair of indices

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

3) Anti sym. on last 3 indices

$$R_{\alpha[\beta\gamma\delta]} = 0$$

4) Bianchi identity

$$R^{\alpha}{}_{\beta[\gamma\delta;\mu]} = 0$$

$$R^{\alpha}{}_{\beta\gamma\delta;\mu} + R^{\alpha}{}_{\beta\mu\gamma;\delta} + R^{\alpha}{}_{\beta\delta\mu;\gamma} = 0$$

The Bianchi identities also define another tensor equation

$G^{\alpha\beta}{}_{;\beta} = 0$ where

$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = G^{\alpha\beta}$: Einstein tensor

$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha}$: Ricci tensor

$R = R^{\alpha}{}_{\alpha} = R^{\mu\nu}{}_{\mu\nu}$: Ricci scalar

single number containing all the information on curvature.

Let's consider a concrete example
a 2-sphere

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} ; \quad g^{ij} = \begin{pmatrix} R^{-2} & 0 \\ 0 & (R^2 \sin^2\theta)^{-1} \end{pmatrix}$$

Calculating the corresponding $\Gamma^{\alpha}_{\mu\nu}$ one finds that the only non zero ones are

$$\Gamma^{1}_{22} = -\sin\theta\cos\theta \, ; \quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \cot\theta$$

from which the only non zero

$$\left.\begin{aligned} R^{1}{}_{221} &= -\sin^2\theta = -\frac{1}{R^2} \, g_{22} \\ R^{2}{}_{121} &= \frac{1}{R^2} \, g_{11} = 1 \end{aligned}\right\} \quad R_{ic} = \frac{1}{R^2} \text{ and}$$

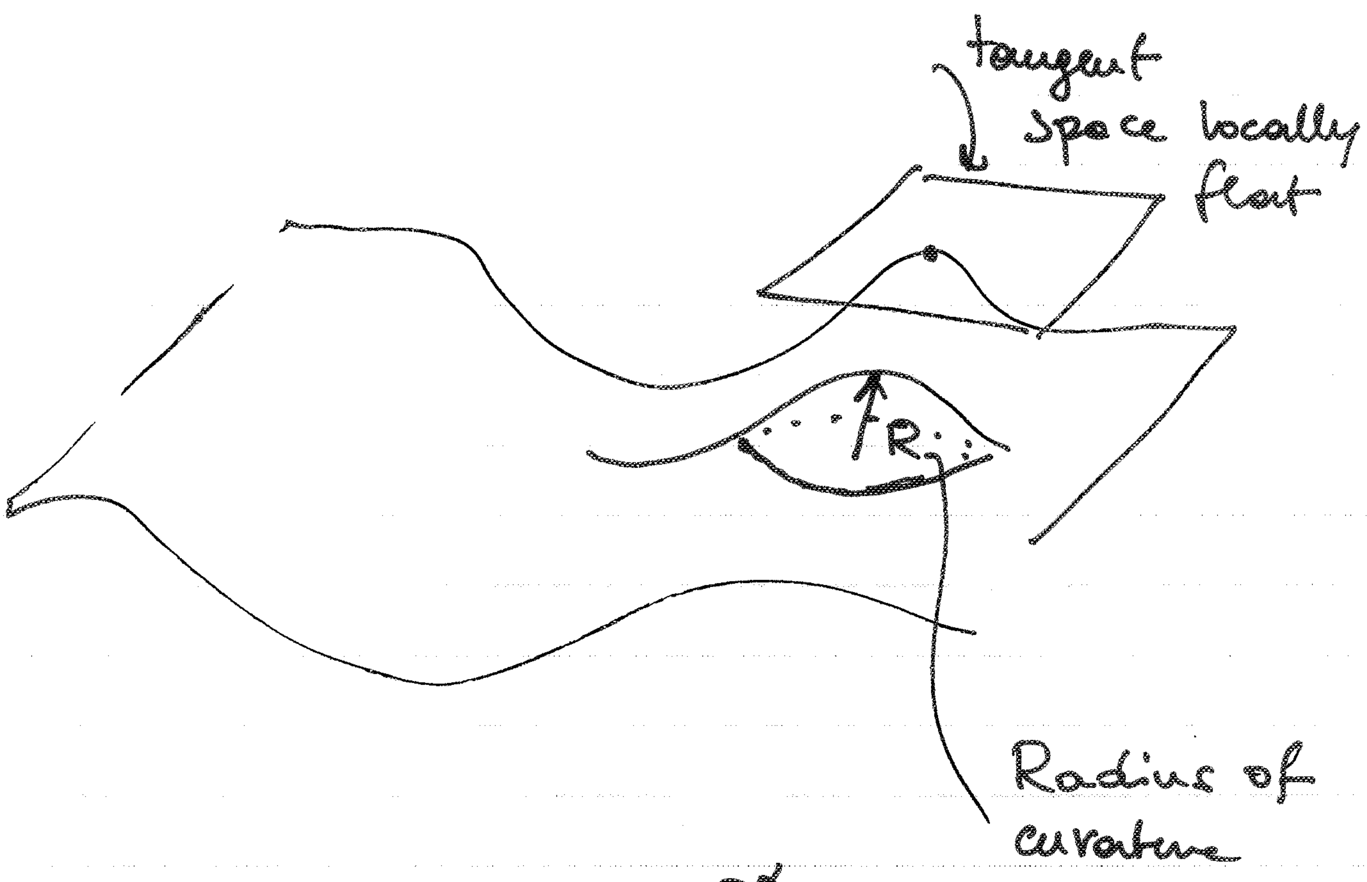
Ricci <u>scalar</u>

this is a single measure of the curvature of the 2-sphere

In general, for any 2-surface when $R = \text{const}$

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{R^2} \left(\delta^{\alpha}_{\beta} \, g_{\mu\nu} - \delta^{\alpha}_{\nu} \, g_{\mu\beta} \right)$$

Note that $\boxed{R^{\alpha}{}_{\mu\beta\nu} \propto 1/R^2}$ curvature tensor is proportional to the inverse of the square of the curvature radius



$R^{\alpha}_{\beta\mu\nu} \propto 1/R^2$

Clearly : • $R \to \infty$: flat spacetime

• can always find a tangent space which is locally flat

Important : if all the components of the Riemann tensor vanish, the manifold is flat

Note : for this to happen you must have $\Gamma, \Gamma_{,\alpha} = 0$. In other words not only the first derivatives have to be zero, but also the second ones.

The Riemann tensor is a $4 \times 4 \times 4 \times 4$ matrix and hence it has 256 components! However they are not all independent.

Indeed the independent ones are only 20

$D=4$	$R^{\alpha}_{\cdot\beta\gamma\delta}$	$R^{\alpha}_{\ \beta}$	R
ind. comp.	20	10	1

$\square$

STRONG EP

We have discussed that there are no global inertial frames but that at least locally it is possible to find a reference frame which is inertial, ie freely falling.

In this frame the laws of physics will be those of SR. Stated differently: any law which can be expressed in tensor notation in SR has exactly the same form in a locally inertial frame of a curved spacetime

This is the "comma-to-semicolon" rule

Ex

$$N^{\alpha}{}_{,\alpha} = 0 \qquad T^{\alpha\beta}{}_{,\beta} = 0 \qquad (5)$$

In a local inertial frame this law is the same as in a curved spacetime

$$N^{\alpha}{}_{;\alpha} = 0 \qquad T^{\alpha\beta}{}_{;\beta} = 0 \qquad (6)$$

Note that (6) are valid in the whole spacetime. This is why we don't really trash SR and its extension to GR is very simple.

Another example is a geodesic. In a local inertial frame particles move on straight lines, ie

$$u^{\alpha}{}_{,\mu} u^{\mu} = 0 \qquad \Longrightarrow$$

$\boxed{\text{This is eq. to } x^{\alpha} = k\tau}$

$$u^{\alpha}{}_{;\mu} u^{\mu} = 0$$

and hence freely falling particles move on timelike geodesic of the spacetime this is an important result because geodesics reflect curvature of spacetime and freely falling particles are obeying gravitational field's produced by matter.

followed by Einstein

The logic is therefore

$$\left(\text{CURVATURE} \right) \rightleftarrows \left(\text{MATTER} \right)$$

we know that the RHS is given by $T^{\alpha\beta}$ and that $T^{\alpha\beta}{}_{;\beta} = 0$

I need a tensor on the LHS that has these properties, eg

$$R^{\alpha\beta} + c_1 g^{\alpha\beta} R + c_2 g^{\alpha\beta}$$

This is indeed a generalization of Einstein's tensor for which $c_1 = -\frac{1}{2}$, so that the field eqs

$$\boxed{R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R + c_2 g^{\alpha\beta} = K T^{\alpha\beta}}$$

We will see that $K = \frac{8\pi G}{c^4}$ and $c_2 = \Lambda$: cosmol. const.

We still need to compute the constant in the Einstein eqs. which hereafter we will consider in the case $\Lambda=0$

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = k T^{\alpha\beta}$$

We need to obtain in the weak-field limit the standard Newtonian expression

$$\nabla^2 \phi = 4\pi G \rho$$

Let's contract with $g_{\alpha\beta}$ $\Rightarrow$

$$R^{\alpha\beta} g_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta} R = k T^{\alpha\beta} g_{\alpha\beta}$$

$$R^{\alpha}{}_{\alpha} - \frac{4}{2} R = k T^{\alpha}{}_{\alpha} = kT$$

$$R = -kT \quad \Rightarrow$$

$$R^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} T\right) \qquad (6)$$

In weak field $T^{00} \gg T^{0j} \gg T^{ij}$; using the covariant components, the 00 component of (6) is

$$R_{00} = k\left(T_{00} - \frac{1}{2} g_{00} T\right)$$

Note

$$\eta_{00} = -1 \; ; \; \eta^{00} = \frac{1}{\eta_{00}} = -1$$

$$h_{00} = h^{00} \neq -1$$

In flat spacetime

$$T_{00} \neq \eta^{00} T_{00}$$

$$= \eta^{0\alpha} \eta^{0\beta} T_{\alpha\beta} = (\eta^{00})^2 T_{00}$$

$$g_{00} T = g_{00} T^{\mu}{}_{\mu} = g_{00} g^{\mu\nu} T_{\mu\nu}$$

$$\simeq g_{00} g^{00} T_{00} = T_{00}$$

$$T_{00} - \frac{1}{2} g_{00} T = \frac{T_{00}}{2} \simeq \eta_{0\beta} \eta_{0\alpha} \frac{T^{\alpha\beta}}{2}$$

$$= (\eta_{00})^2 \frac{T^{00}}{2} = \frac{\rho c^2}{2}$$

$$R_{00} = \frac{k}{2} \rho c^2$$

The LHS can be computed easily to be

$$R_{00} = -\frac{1}{2} h_{00,ii} = -\frac{1}{2} \nabla^2 h_{00}$$

where we have introduced $h_{\mu\nu}$ as a correction to the flat spacetime metric, ie

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

We need to relate h_{00} to ϕ and we can do this by using the geodesic equation which we know is the equation of a freely falling particle and the eq. of motion in the presence of gravity

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \qquad \text{vs} \qquad \ddot{\vec{x}} = -\vec{\nabla}\phi$$

consider spatial components and neglect terms $O(v^2)$ $\Rightarrow$ $\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \sim \Gamma^\mu_{00} u^0 u^0$
$= \Gamma^\mu_{00}$

$$\ddot{x}^k = -\Gamma^k_{00} = \underbrace{-\frac{1}{2}\nabla^k h_{00}}_{\text{bit of algebra}}$$

re-introduced c

$$\ddot{x}^k = \underbrace{-\frac{c^2}{2}\nabla^k h_{00}}_{\text{weak limit of geod. eq.}} = \underbrace{-\nabla^k\phi}_{\text{Newtonian}}$$

$$\Rightarrow \frac{\phi}{c^2} = -\frac{h_{00}}{2} \qquad \text{or} \qquad h_{00} = -2\frac{\phi}{c^2}$$

let's go back to the LHS of Einstein eqs

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00} = -\frac{1}{2}\nabla^2\left(-2\frac{\phi}{c^2}\right) = \frac{\nabla^2\phi}{c^2}$$

putting thing together : Einstein eqs in weak field limit

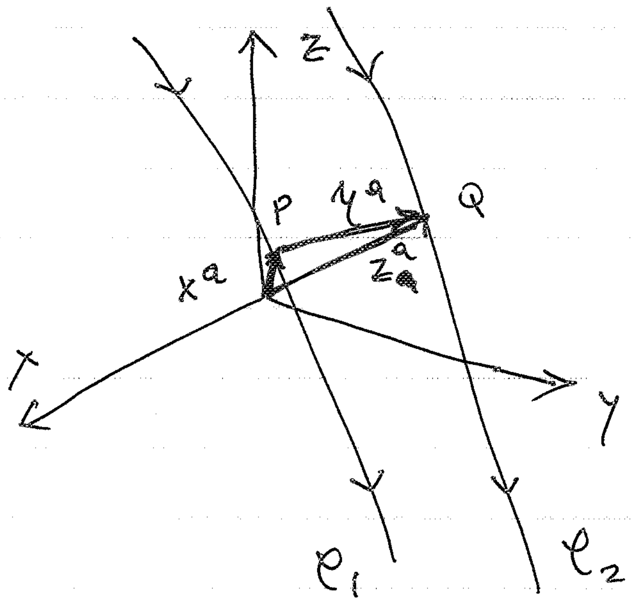
$$\frac{\nabla^2\phi}{c^2} = \frac{k\rho c^2}{2} \qquad \text{vs} \qquad \nabla^2\phi = 4\pi G\rho$$

$$\boxed{k = \frac{8\pi G}{c^4}}$$

Geodesic deviation

We have seen that geodesics are the trajectories of free particles and that these are parallel in an inertial frame and not in general.

Now we compute how the separation between two geodesics varies and relate this change to the curvature of spacetime. Before doing this in GR let's do this calculation in Newtonian gravity



$x^a = x^a(t)$ along ℓ_1

$z^a = z^a(t)$ " ℓ_2

$= x^a(t) + \eta^a(t)$

The equations of motion will be those of bodies in free fall

(7) $$\ddot{x}^a = -(\partial^a\phi)_P$$

$(\phi)_Q = (\phi)_P + \eta^b(\partial_b\phi)_P + \ldots$ Taylor exp.

(8) $$\ddot{z}^a = -(\partial^a\phi)_Q = \ddot{x}^a + \ddot{\eta}^a =$$

$$= -\left[\partial^a\left[(\phi)_P + \eta^b\partial_b(\phi)_P\right]\right]$$

$$= -(\partial^a\phi)_P - \eta^b\partial^a\partial_b(\phi)_P = -(\partial^a\phi)_P - K^a{}_b\eta^b$$

where $K^a{}_b \equiv \partial^a \partial_b \phi$: 2nd deriv. of potential

Comparing (7) and (8)

$$\boxed{\ddot{\eta}^a = - K^a{}_b \eta^b} \qquad (9)$$

This is the eq. of motion of the vector η seperating the two trajectories.

If the field is uniform $\partial^a \phi = 0 = \partial^b \partial_a \phi$
$\Rightarrow \ddot{\eta}^a = 0 \Rightarrow \eta^a = \text{const}$

Thus even in Newtonian gravity two freely falling particles will meet eventually

$\phi = -\frac{M}{r} \Rightarrow \partial^a \partial_b \phi = + \partial^r \left(+ \frac{M}{r^2} \right) = -2 \frac{M}{r^3}$

ie $\ddot{\eta}^a = \ddot{\eta}^r = -\frac{2M}{r^3}$

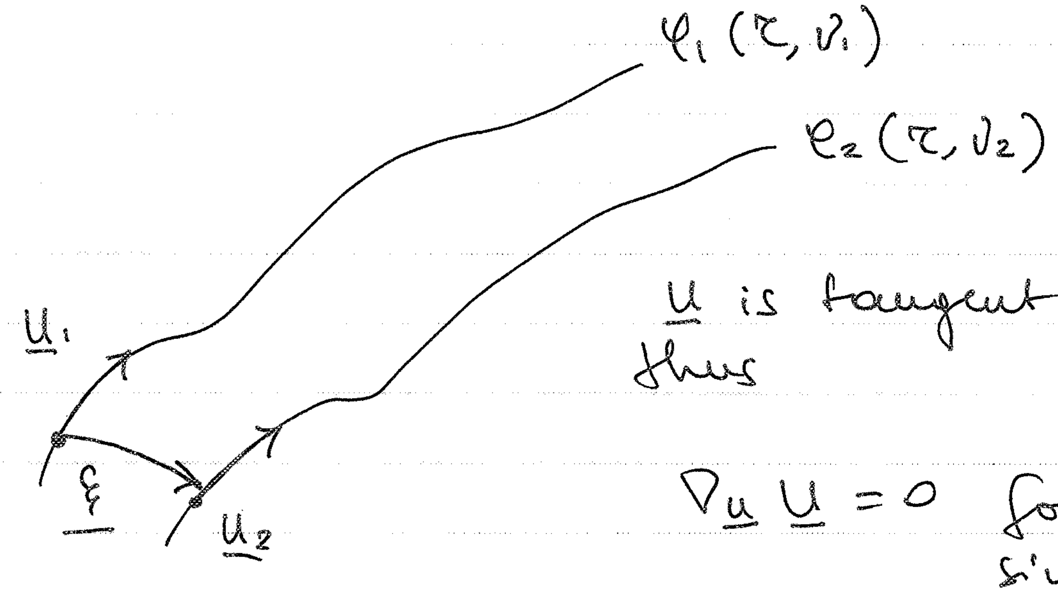
these are tidal fields and scale like r^{-3}

$\square$

Back to GR

We have seen that deviations from $\parallel$ transport are measured by the curvature tensor

$$2\nabla_{[\alpha}\nabla_{\beta]}V^{\mu} = R^{\mu}{}_{\nu\alpha\beta}V^{\nu}$$



$\underline{u}$ is tangent vector and thus

$\nabla_{\underline{u}}\,\underline{u} = 0$ for both curves since they are geodesics

$\underline{\xi}$ vector connecting the two geodesics

$$\xi^{\alpha} = \frac{dx^{\alpha}}{d\nu}$$

parameter used to distinguish the geodesics

It is now easy to show that the covariant derivative of $\underline{u}$ along $\underline{\xi}$ is the same as the covariant derivative of $\underline{\xi}$ along $\underline{u}$

$$\nabla_{\underline{\xi}}\, \underline{u} = \nabla_{\underline{u}}\, \underline{\xi}$$

Proof

$$\nabla_{\underline{\xi}}\, u^\alpha = u^\alpha{}_{;\beta}\, \xi^\beta = u^\alpha{}_{,\beta}\, \xi^\beta + \Gamma^\alpha_{\beta\mu}\, \xi^\beta u^\mu$$

$$\nabla_{\underline{u}}\, \xi^\alpha = \xi^\alpha{}_{;\beta}\, u^\beta = \xi^\alpha{}_{,\beta}\, u^\beta + \Gamma^\alpha_{\beta\mu}\, u^\beta \xi^\mu$$

$(-) \Rightarrow$

$$\underbrace{u^\alpha{}_{,\beta}\, \xi^\beta - \xi^\alpha{}_{,\beta}\, u^\beta}_{=0} + \underbrace{\Gamma^\alpha_{\beta\mu}\left(\xi^\beta u^\mu - u^\beta \xi^\mu\right)}_{=0}$$

$= 0$ because it is equal to

$$= \frac{dx^\beta}{d\nu}\frac{d}{dx^\beta}\frac{dx^\alpha}{d\tau} - \frac{dx^\beta}{d\tau}\frac{d}{dx^\beta}\frac{dx^\alpha}{d\nu}$$

$$= \frac{d^2x^\alpha}{d\nu\, d\tau} - \frac{d^2x^\alpha}{d\tau\, d\nu}$$

$= 0$ because contraction of sym. and antisym. tensor $T_{(\alpha\beta)}T^{[\alpha\beta]} = 0$

I next want to take another cov. derivative of $\underline{\xi}$ along $\underline{u}$

$$\nabla_{\underline{u}} \nabla_{\underline{u}} \xi^\alpha = \nabla_{\underline{u}} \nabla_{\underline{\xi}} u^\alpha$$

$$= 2 \nabla_{[\underline{u}} \nabla_{\underline{\xi}]} u^\alpha \qquad (9)$$

proof

$$2 \nabla_{[\underline{u}} \nabla_{\underline{\xi}]} u^\alpha = \nabla_{\underline{u}} \nabla_{\underline{\xi}} u^\alpha - \cancel{\nabla_{\underline{\xi}} \nabla_{\underline{u}} u^\alpha}$$

$$= \nabla_{\underline{u}} \nabla_{\underline{u}} \xi^\alpha$$

on the other hand it is possible to calculate (good exercise)

$$2 \nabla_{[\underline{u}} \nabla_{\underline{\xi}]} u^\alpha = R^\alpha{}_{\beta\mu\nu} u^\mu \xi^\nu u^\beta \qquad (10)$$

(9), (10) $\Rightarrow$

$$\boxed{\nabla_{\underline{u}} \nabla_{\underline{u}} \xi^\alpha = R^\alpha{}_{\beta\mu\nu} u^\mu \xi^\nu u^\beta}$$

This is the eq. of geodesic deviation (second cov. der. along γ of $\underline{\xi}$)

Let's search the analogy with Newtonian

$$\nabla_{\underline{u}} = \frac{D}{D\tau}$$

then (10) $\Leftrightarrow$

$$\frac{D^2 \xi^\alpha}{d\tau^2} = R^\alpha{}_{\beta\mu\nu} U^\beta U^\mu \xi^\nu = K^\alpha{}_\nu \xi^\nu$$

where $K^\alpha{}_\nu = R^\alpha{}_{\beta\mu\nu} U^\beta U^\mu$

this is to be compared with the corresponding Newtonian expression

$$\ddot{\xi}^a = - K^a{}_b \, \xi^b$$

As in Newtonian physics, if

$K^\alpha{}_\nu = 0$ (ie $R^\alpha{}_{\beta\mu\nu} = 0$) then

$\frac{D^2}{D\tau^2} \xi^\alpha = 0$: geodesics that are initially parallel remain parallel.

Alternative view: the convergence/divergence of geodesics initially parallel are telling us about the curvature of spacetime.

I can monitor two freely falling particles and check if their distance changes. This is indeed a GW detector!

Let's consider a concept already mentioned: tidal fields in the exterior of a black hole

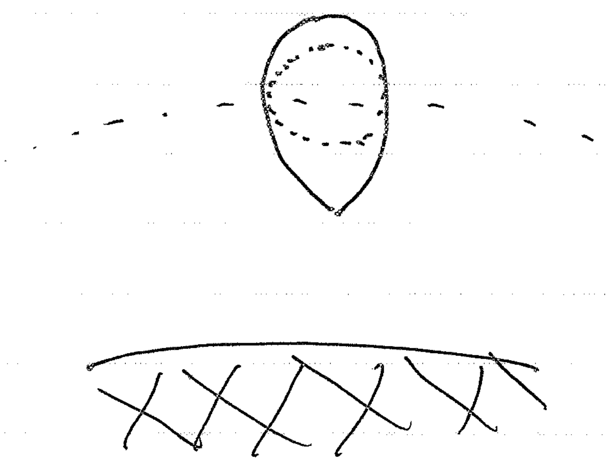
It is possible to calculate the geodesic deviation for particles freely falling onto a black hole and obtain

$$\frac{D^2\eta^r}{D\tau^2} = \frac{2M}{r^3}\eta^r \qquad \eta^\alpha = h^\alpha{}_\beta \xi^\beta$$

$$\frac{D^2\eta^\theta}{D\tau^2} = -\frac{M}{r^3}\eta^\theta \qquad h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$$

projector

$$\frac{D^2\eta^\phi}{D\tau^2} = -\frac{M}{r^3}\eta^\phi$$



$$\frac{D^2\eta^r}{D\tau^2} > 0$$

$$\frac{D^2\eta^\theta}{D\tau^2} < 0$$

GRAVITATIONAL WAVES

The starting point are EFEs

$$\boxed{G_{\mu\nu} = 8\pi T_{\mu\nu}} \quad . \qquad (1)$$

but we need to look for solutions in a linearized theory of gravity, ie in a limit in which the fields and hence the curvature is small.

Let's start from flat space-time and move away from there

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and

$$\left| h_{\mu\nu} / \eta_{\mu\nu} \right| \ll 1$$

Ex

$$h_{\mu\nu} \sim \phi \sim \frac{M_\odot}{R_\odot} \sim 10^{-6}$$

If we want to rewrite (1) we need to work out the affine connections

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(g_{\nu\alpha,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right)$$

$g_{\nu\alpha,\beta} = \eta_{\nu\alpha,\beta} + h_{\nu\alpha,\beta} = h_{\nu\alpha,\beta} \quad \Rightarrow$

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} \eta^{\mu\nu} \left(h_{\nu\alpha,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu} \right).$$

$$= \frac{1}{2} \left(h^{\mu}{}_{\alpha,\beta} + h^{\mu}{}_{\beta,\alpha} - h_{\alpha\beta,}{}^{\mu} \right)$$

Note that the indices of $h_{\mu\nu}$ are raised and lowered using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ and NOT $g^{\mu\nu}$

Similarly, the linearized Ricci tensor

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu}$$

$$= \frac{1}{2} \left(h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} \right)$$

where $h \equiv h^{\alpha}{}_{\alpha} = \eta^{\mu\alpha} h_{\mu\alpha}$ trace of $h_{\mu\nu}$

(*)

$$\bar{h} = -h$$

proof

$$\bar{h} = \bar{h}^{\mu}{}_{\mu} = h^{\mu}{}_{\mu} - \frac{1}{2} \eta^{\mu}{}_{\mu} h$$
$$= h - \frac{1}{2} \cdot 4 h = -h$$

$$R = g^{\mu\nu} R_{\mu\nu} \cong \eta^{\mu\nu} R_{\mu\nu}$$

$$\Rightarrow \quad 2 G_{\mu\nu} = 16\pi T_{\mu\nu} \quad \Longleftrightarrow$$

$$h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta,}{}^{\alpha\beta} - h_{,\beta}{}^{\beta}\right)$$
$$= 16\pi T_{\mu\nu} \qquad (2)$$

Expression (2) can be made more compact if we use the trace-free part of $h_{\mu\nu}$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

where the bar can be applied to any sym. tensor:

$$G_{\mu\nu} = \bar{R}_{\mu\nu} \; ; \; \bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$$

trace of $\bar{h}_{\mu\nu}$ (*)

Proof
$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \overset{(*)}{=}$$
$$= \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\left(\eta^{\alpha\beta}\bar{h}_{\alpha\beta}\right)$$
$$= \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\left(\eta^{\alpha\beta}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h\right)\right)$$
$$= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h - \frac{1}{2}\eta_{\mu\nu}\left(h - \frac{4}{2}h\right)$$
$$= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h + \frac{1}{2}\eta_{\mu\nu} h = h_{\mu\nu}$$

In this case the EFEs take the form

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 16\pi T_{\mu\nu} \qquad (3)$$

$$-\Box \bar{h}_{\mu\nu} = -\left(\partial_t^2 + \partial_x^2 - \partial_y^2 - \partial_z^2\right)\bar{h}_{\mu\nu}$$

We can now exploit the gauge freedom and specify $h_{\mu\nu}$ so that

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0 \qquad (4)$$

(4) is the equivalent of the Lorentz gauge condition and as natural as that

DIGRESSION

we have already seen that if we make an infinitesimal coordinate transformation

$x^{\alpha'} = x^{\alpha} + \xi^{\alpha}$, then the new "perturbed" metric

$$g_{\mu'\nu'} + h_{\mu'\nu'} = g_{\mu\nu} + h_{\mu\nu} + \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

ie $h_{\mu\nu}^{new} = h_{\mu\nu}^{OLD} + \xi_{\mu,\nu} - \xi_{\nu,\mu}$

where ξ^{μ} are arbitrary but small functions.

For any of these all of the observables are unmodified.

Think of electromagnetism

$$A_\mu \longrightarrow A_{\mu'} = A_\mu + \Psi_{,\mu}$$

then $F^{NEW}_{\mu'\nu'} = A_{\mu',\nu'} - A_{\nu',\mu'} = A_{\mu,\nu} + \Psi_{,\mu\nu}$

$$- A_{\nu,\mu} - \Psi_{,\nu\mu} = F^{OLD}_{\mu\nu}$$

$\square$

Recap: the linearized theory of gravity with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

satisfies the field equations

$$\boxed{-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 16\pi T_{\mu\nu}} \qquad (5)$$

Gauge transformations and coordinate transformations in linearized gravity

In their linearized form the EFE are invariant to both GLOBAL Lorentz (coord.) transformations and LOCAL ones

1) Global Lorentz transf.

$$x^\mu \longrightarrow x^{\mu'}$$

$$x^{\mu'} = \Lambda^{\mu'}{}_\alpha x^\alpha ;$$

$$\eta_{\alpha'\beta'} = \Lambda^\mu{}_{\alpha'} \Lambda^\nu{}_{\beta'} \eta_{\mu\nu}$$

The metric then transforms as

$$g_{\alpha'\beta'} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'} = \Lambda^\mu{}_{\alpha'} \Lambda^\nu{}_{\beta'} (\eta_{\mu\nu} + h_{\mu\nu})$$

ie the perturbations also transform like tensors

$$\boxed{h_{\alpha'\beta'} = \Lambda^\mu_{\alpha'} \Lambda^\nu_{\beta'} h_{\mu\nu}} \qquad (*)$$

2) One can also consider small LOCAL coordinate transformations

$$x^{\mu'}(P) = x^\mu(P) + \xi^\mu \quad \text{where} \quad \frac{\xi^\mu}{x^\mu} \ll 1$$

one can always perform these transformations if these leave $h_{\mu'\nu'} \ll 1$

As an example consider the temperature as a function of position $T = T(x^\mu)$ and perform a small coordinate transf.

$$T(x^{\mu'} = a^\mu) = T(x^\mu + \xi^\mu) = T(x^\mu = a^\mu - \xi^\mu)$$

temp. at new position = temp. at old position displaced by ξ^μ

$$\cong T(x^\mu) - T_{,\mu}\xi^\mu$$

If $T = \cos^2(x^0)$ and $\xi^0 = 0.001 \sin(x^1)$

$$T_{,\mu}\xi^\mu = \partial_0 T \xi^0 = -2\cos(x^0)\sin(x^0)$$

$$\Rightarrow T' = \cos^2(x^{0'}) + \underbrace{0.002\cos(x^{0'})\sin(x^{0'})\sin(x^{1'})}_{\text{ignorable}}$$

these changes cannot be ignored in the metric which is already measuring deviations from flat and therefore gravity.

$$g_{\alpha'\beta'}\left(x^{\alpha'}(P)\right) = g_{\mu\nu}\left(x^{\alpha}(P)\right)\frac{\partial x^{\mu}}{\partial x^{\alpha'}}\frac{\partial x^{\nu}}{\partial x^{\beta'}}$$

so that

$$\boxed{h^{\mu\nu}_{NEW} = h^{\mu\nu}_{OLD} - \xi^{\mu}{}_{,\nu} - \xi^{\nu}{}_{,\mu}} \qquad (**)$$

this is the result of the <u>local</u> coord. transf.

(**) are reminiscent of what happens in EM. there too the eqs are invariant for transformations of the type

$$A_{\mu'} = A_{\mu} + \Psi_{,\mu}$$

so that

$$F^{NEW}_{\mu'\nu'} = A_{\mu',\nu'} - A_{\nu',\mu'} = A_{\mu,\nu} + \Psi_{,\mu\nu}$$

$$- A_{\nu,\mu} - \Psi_{,\nu\mu}$$

$$= F^{OLD}_{\mu\nu}$$

In a similar way one can show that

$$R^{NEW}_{\alpha\beta\mu\nu} = R^{OLD}_{\alpha\beta\mu\nu}$$

because all equations are invariant under the transformation (**) one has the freedom to choose ξ^μ in the most suitable way

Note: fixing the GLOBAL gauge does not fix the LOCAL one.

If one chooses the Lorentz gauge

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$$

the eqs are still invariant for any choice of the ξ^μ such that

$$\xi^{\mu,\alpha}{}_{\alpha} = 0$$

we will exploit this freedom later on.

Let's consider now the linearized eqs of gravity in vacuum

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0 \iff$$

$$\boxed{\Box \bar{h}^{\mu\nu} = 0} \qquad (6)$$

Dalambertian

In the Lorentz gauge the gravitational field propagates in spacetime as a <u>wave</u>

Note : in linearized theory the spacetime is curved although not severely.

Q: What's the simplest solution to (6)?
It's a <u>plane wave</u>

wave vector

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left\{ A_{\mu\nu}\, e^{i k_\alpha x^\alpha} \right\} \qquad (7)$$

Amplitude tensor

where k^α is a <u>null</u> wave vector

$k^\alpha k_\alpha = 0$ and therefore the plane wave described by (7) is propagating in the <u>direction</u>

$\tilde{\underline{k}} = \frac{1}{k^0}(k_x, k_y, k_z)$ with <u>frequency</u> $\omega \equiv k^0 = (k^j k_j)^{1/2}$

symm - 2nd tensor
GW 7

Solutions (7) indicate there are 16-6=10 indipendent components in the amplitude tensor but we know there are only 2 dynamical degrees of freedom in GR (spin-2 fields)

The excess of components is due to the fact that we still have not yet constrained all the possible gauge conditions

A) $\bar{h}^{\mu\nu}{}_{,\alpha} = 0$ Lorentz gauge [4]
$A_{\mu\nu} K^{\nu} = 0$ ($\bar{\bar{A}}$ is orthogonal to $\underline{k}$)

B) Select a global Lorentz frame (as in SR) with 4-vel $\underline{u}$ such that [3]

$$A_{\mu\nu} U^{\nu} = 0 \quad (*)$$

C) Choose the remaining gauge freedom to set $A_{\mu}{}^{\mu} = 0$ [1]

This condition is equivalent to fixing the infinitesimal gauge transformation, ie choosing the displacement vector

$$\xi^{\mu} = -i C^{\mu} e^{i k_\alpha x^\alpha}$$

(*) Note that $A_{\mu\nu} U^{\nu} = 0$ - are just 3 conditions because I also know that

$k^{\mu} A_{\mu\nu} U^{\nu} = 0$ $\quad$ $- A_{00} U^0 + A_{01} U^1 + \dots A_{03} U^3 = 0$
$A_{10} U^0 +$
$A_{30} U^0 \qquad = 0$

If we now choose a Lorentz frame in which $u^\nu = (1, 0, 0, 0)$ ie $u^0 = 1$; $u^j = 0$ the conditions expressed before for $A_{\mu\nu}$ can be translated in conditions for $h_{\mu\nu}$.

In particular

(B) $A_{\mu\nu} u^\nu = 0 \iff h_{\mu 0} = 0$: only the spatial components of $h_{\mu\nu}$ are nonzero

(A) $A_{\mu\nu} k^\nu = 0 \iff h_{ij,j} = 0$: the spatial component are divergence free

(C) $A^\alpha{}_\alpha = 0 \iff h = h^k{}_k = 0$: the spatial component are trace-free

CONDITIONS (A), (B), (C) DEFINE THE TT GAUGE

Because $h = 0$ there is no difference between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ in this gauge.

Q: One might wonder: how general is this treatment?

We know from EM that any EM wave can be decomposed as the superposition of planar waves.

The same is true in linearized gravity

Furthermore because the gauge conditions (A)–(C) are all linear in $h_{\mu\nu}$ we can say that any gravitational wave can be written in a TT gauge

Indeed there is a theorem:

Pick a specific global Lorentz frame (ie a specific 4 vel. vector such that $u^\alpha = \delta^\alpha_0$). For any gravitational wave satisfying

$$\Box \bar{h}_{\mu\nu} = 0$$

one can find suitable gauge conditions such that

$$\underbrace{\Box h_{ij} = 0}_{\text{only 6 eqs}} \quad \& \quad h_{\mu\nu} = h^{TT}_{\mu\nu} \iff h_{\mu\nu} \text{ satisfies (A)–(C)}$$

MAKING SENSE OF THE TT-GAUGE

The use of the TT gauge introduces important advantages and simplifications

The most relevant one is the the only nonvanishing time-space components of the Riemann tensor are:

$$R_{j0k0} = R_{0j0k} = -R_{j00k} = -R_{0jk0}$$

where

$$\boxed{R_{j0k0} = -\frac{1}{2}\, h^{TT}_{jk,00}} \qquad (8)$$

(8) indicates that a travelling gw introduces a local oscillation of the curvature tensor

$$h_{ij}^{TT} \propto e^{i\omega t}$$

$$h^{TT}_{ij,00} \sim -\omega^2 e^{i\omega t} \sim R_{i\hat{0}j\hat{0}}$$

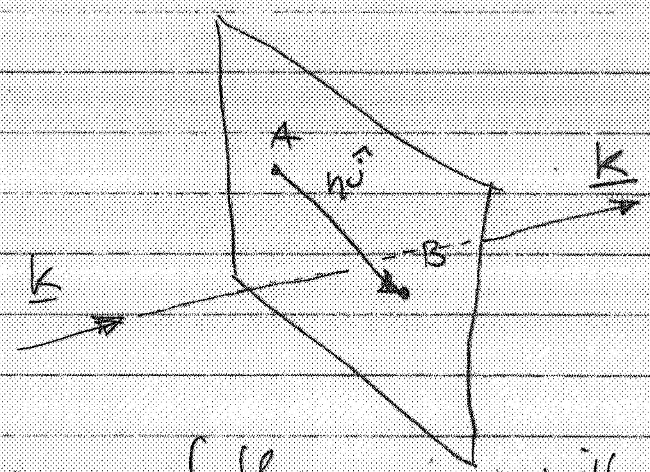
This oscillatory behaviour appears as oscillations in the seperation between two neighbouring particles A and B

Consider a coordinate system $x^{\hat\alpha}$ centred on A and comoving with it: $x^{\hat 0} = \tau$; $x^{\hat j} = 0$

proper time

All along A's worldline the line element is

$$ds^2 = -d\tau^2 + \delta_{\hat i \hat j}\, dx^{\hat i} dx^{\hat j} + O(|x^{\hat j}|^2)\, dx^{\hat\alpha} dx^{\hat\beta}$$



The presence of the gw will alter the geodesic motion of A,B and be and this will manifest in the geodesic deviation equation

$$\boxed{U^{\gamma} U^{\beta} V^{\alpha}{}_{;\beta\gamma} = -R^{\alpha}{}_{\beta\gamma\delta}\, U^{\beta} V^{\gamma} U^{\delta}}$$

$$\boxed{\frac{d^2 V^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dV^{\alpha}}{d\lambda}\frac{dV^{\beta}}{d\lambda} = R^{\alpha}{}_{\beta\gamma\delta}\, U^{\beta} V^{\gamma} U^{\delta}}$$

Define $n^{\hat{j}}_{B} = -x^{\hat{j}}_{A} + x^{\hat{j}}_{B} = x^{\hat{j}}_{B}$

$$\frac{D^2 n^{\hat{j}}_{B}}{D\tau^2} = -R^{\hat{j}}{}_{0\hat{k}0}\, n^{\hat{k}} \qquad (9)$$

Because $\Gamma^{\hat{j}}_{\alpha\beta} = 0$ in the vicinity of A, (9) $\Leftrightarrow$

$$\frac{d^2 x^{\hat{j}}_{B}}{d\tau^2} = -R^{\hat{j}}{}_{0\hat{k}0}\, x^{\hat{k}}_{B} \qquad \Leftrightarrow \quad t = \tau$$

$$\frac{d^2 x^{\hat{j}}_{B}}{dt^2} = \frac{1}{2}\left(\frac{\partial^2 h^{TT}_{\hat{j}\hat{k}}}{\partial t^2}\right) x^{k}_{B}$$

where we have chosen a gauge that at first order in h^{TT}_{jk} moves with A

after the time integration

$$\boxed{x^{\hat{j}}_{B}(\tau) = x^{\hat{k}}_{B}(0)\left[\delta_{\hat{j}\hat{k}} + \frac{1}{2} h^{TT}_{\hat{j}\hat{k}}\right]} \qquad (10)$$

INITIAL POSITION

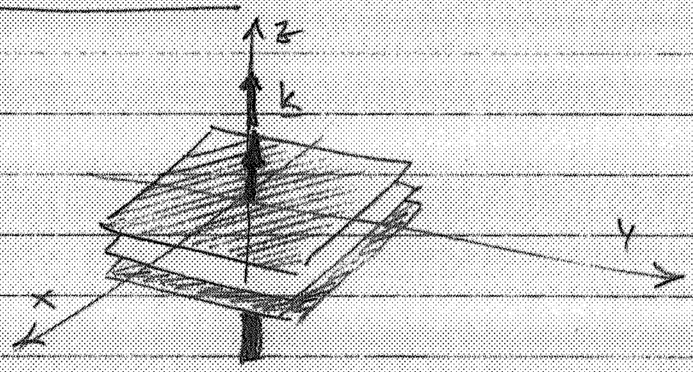
Eq. (10) tells us that in the reference frame comoving with A the particle B is subject to an oscillation with amplitude h^{TT}_{jk}

If the particles are in the direction of the wave (ie $\bar{n} \,//\, \bar{k}$), then no oscillation is recorded (these are transverse waves

$$h^{TT}_{jk}\, x^k_B(0) \propto h^{TT}_{jk}\, k^k = 0$$

ie the wave is really transverse: not only in the mathematical description but also in the physical behaviour (geod. deviation)

let's specify to a concrete example: consider a wave propagating in the positive z-direction



Then

$$\left.\begin{aligned} h^{TT}_{xx} &= -h^{TT}_{yy} = \mathrm{Re}\left\{A_+ e^{-i\omega(t-z)}\right\} \\ h^{TT}_{xy} &= h^{TT}_{yx} = \mathrm{Re}\left\{A_\times e^{-i\omega(t-z)}\right\} \end{aligned}\right\}$$

A_+ and $A_\times$ represent the two independent modes of polarization

Just like in EM, one can decompose a gw in two linearly polarized plane waves or in two circularly polarized ones

The polarization can be expressed in terms of the LINEAR POLARIZATION TENSORS

linear pol. vectors

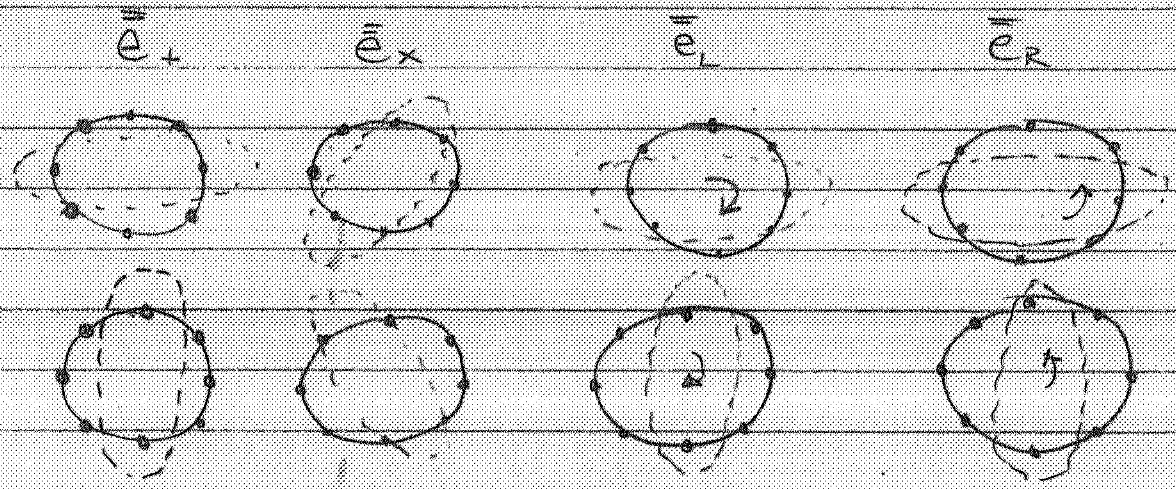
$$\bar{\bar{e}}_+ = \bar{e}_x \otimes \bar{e}_x - \bar{e}_y \otimes \bar{e}_y$$

$$\bar{\bar{e}}_\times = \bar{e}_x \otimes \bar{e}_x + \bar{e}_y \otimes \bar{e}_y$$

Similarly one can define the CIRCULAR POL. TENSOR

$$\bar{\bar{e}}_L = (\bar{e}_x + i\bar{e}_y)/\sqrt{2}$$

$$\bar{\bar{e}}_R = (\bar{e}_x - i\bar{e}_x)/\sqrt{2}$$



Deformations of rings of particles through the action of gws with single polarization, eg

$$h_{jk} = \mathrm{Re}\left\{ A_+ e^{-i\omega(t-z)} (e_+)_{jk} \right\}$$

GENERATION OF GWs

We will estimate the generation of GWs using Newtonian physics. Modulo factors O(10) these estimates are reliable

How do you produce EM waves ? Oscillating EM fields, such as those produced by an oscillating electric dipole

$$L_{\text{electric dipole}} \equiv \frac{\text{energy emitted}}{\text{Unit time}}$$

$$= \frac{2}{3} e^2 a^2 = \frac{2}{3} \ddot{d}^2$$

second time derivative of the electric dipole

$d = ex \rightarrow$
$\ddot{d} = e\ddot{x} = ea$

Can we have dipolar gws

$$\underline{d} = \sum_A m_A \underline{x}_A$$

$$\dot{d} = \sum_A m_A \dot{\underline{x}}_A = \underline{p}$$

$$\ddot{d} = \dot{\underline{p}} = 0 \quad (\text{Euler-eq. (mom. cons.)})$$

$$\Rightarrow L_{\text{mass dipole}} = 0$$

(this is equivalent to the impossibility of having monopolar EM radiation)

Next in EM ~~D~~ we can calculate the energy loss due to varying electric current (oscillating magnetic dipole)

$$\underline{\mu} = \underline{B}\, i \quad \xleftarrow{} \text{electric current}$$

$$L \propto \ddot{\mu} \propto \underline{B}\, \underline{\ddot{i}}$$

$\ddot{i}$ — second time derivative of the electric current

the gravitational analogue is

$$\underline{\mu} = \sum_A m_A (\underline{r}_A) \times (\underline{v}_A) = \underline{J}$$

$$\underline{\dot{\mu}} \propto \underline{\dot{J}} = 0$$

Next we need to consider changing quadrupoles

$$L_{\substack{\text{electric}\\ \text{quadrupoles}}} = \frac{1}{20} \dddot{Q}^2 = \frac{1}{20} \dddot{Q}_{ij} \dddot{Q}_{ij}$$

where $Q_{ij} = \sum_A e_A \left((x_A)_i (x_A)_j - \frac{1}{3} \delta_{ij} (x_A)_k (x_A)^k \right)$

similarly one can show that the GW luminosity is

$$L_{\text{mass quadrupole}} = \frac{1}{5} \langle \dddot{I}^2 \rangle = \frac{1}{5} \langle \dddot{I}_{jk} \dddot{I}_{jk} \rangle$$

Where

$$I_{jk} = \sum_A m_A \left((x_A)_j (x_A)_k - \frac{1}{3}\delta_{jk} (x_A)_\ell (x_A)^\ell \right)$$

$$= \int \rho \left(x_j x_k - \frac{1}{3}\delta_{ij}\, x^\ell x_\ell \right) dV$$

$$\dddot{I} \sim \frac{\begin{pmatrix}\text{mass of the}\\ \text{system in motion}\end{pmatrix}\begin{pmatrix}\text{size of the}\\ \text{system}\end{pmatrix}^2}{(\text{time scale})^3} \sim \frac{MR^2}{\tau^3}$$

$$\sim \frac{M v^2}{\tau}$$

Consider two masses M, coupled by a spring of length ℓ and oscillating at ω

$$L_{GW} \sim \frac{1}{24} M \ell^2 \omega^3 \sim \frac{M}{10^8 g} \cdot \left(\frac{\ell}{20m}\right)^2 \left(\frac{\omega/2\pi}{4Hz}\right)^3 10^{-23} \text{ erg s}^{-1}$$

only if one considers very large masses, & relativistic velocities will

$\dddot{I}_{ij}$ be sufficiently large

$$L_{GW}\Big|_{\substack{\text{binary}\\ NS}} \sim 10^{30} - 10^{50} \text{ erg s}^{-1} \; !$$