Lecture I: Discrete differential operators

Lecture II: 3+1 split

Lecture III: Formulations of Einstein eqs

Lecture IV: Gauges, 1D GW extraction

Lecture V: Relativistic hydrodynamics

Lecture VI: Graviton Collapse

Lecture VII: Binary BHs

Lecture VIII: Binary NSs

L. REZZOLLA
\[ a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} + f(x, y, u, \partial_x u, \partial_y u) = 0 \]

\[ a_{11} a_{22} - a_{12}^2 < 0 : \text{Hyperbolic eq.} \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \]

\[ a_{11} a_{22} - a_{12}^2 > 0 : \text{Parabolic eq.} \]

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \]

\[ > 0 : \text{Elliptic eq.} \]

\[ \nabla^2 \phi = c \]
\[ L(u) - f = 0 : \text{continuum} \]
\[ L(u) : \text{differential operator} \]

\[ u(x, t) \]
\[ \downarrow \]
\[ u(x = x_i, t = t^m) \]
\[ = u^n_{ij} \]
\[ L(u) \rightarrow L_{\triangle} (u_n^i) \]

\[ L(u) - f = 0 \]

\[ L_{\triangle} (u_n^i) - f = 0 + \varepsilon_t \]

\( \varepsilon_t \): truncation error and is consequence of discretization
\[ E_{mp} : \text{machine precision error} \]

\[ \theta \approx \alpha + c \]

\[ c = 0 \]

\[ \frac{15 \times 0}{10} \quad \frac{15 \times 0}{10} \quad \frac{15 \times 0}{10} \]

\[ 1.0 = 1.0 + 0.00000 \ldots 1 \]

It's the ability of distinguishing 2 nos.
N floating point operations you build

\[ \epsilon_{R_0} = (N)^{1/2} \epsilon_{mp} \]

\[ \epsilon_{mp} \leftrightarrow \epsilon_{R_0}, \quad \epsilon_T \]

\( \epsilon_T \) is totally under human control and is the result of the
choice made in discretizing $L$

$$\varepsilon_r = \int (\Delta x^n, \Delta t^m)$$
Discretization of hyperbolic eqs

\[ \frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0 \] : wave eq

\[ \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0 \] : advection eq.

\[ u_0 \hspace{1cm} \rightarrow \hspace{1cm} u(x, t) \]
\[ L(u) \rightarrow L_\Delta (u_{n,j}) \]

\[ u^n_{j} = u^n_{j-1} + \frac{\partial u}{\partial x} \bigg|_{n,j} (x_j - x_{j-1}) + o(\Delta x^2) \]

\[ \Rightarrow \]

\[ \frac{\partial u}{\partial x} \bigg|_{n,j} = \frac{u^n_{j} - u^n_{j-1}}{x_j - x_{j-1}} + o(\Delta x) \]

1st order approximation to \[ \partial_x u \]

FINITE DIFFERENCE
\[ \frac{u^{n+1}_{j} - u^{n}_{j}}{\Delta t} = \sqrt{a} \frac{u^{n}_{j+1} - u^{n}_{j}}{\Delta x} + O(\Delta t, \Delta x) \]

\[ u^{n+1}_{j} = u^{n}_{j} + \alpha \left( u^{n}_{j+1} - u^{n}_{j} \right) \]

\[ \alpha = \sqrt{\frac{\Delta t}{\Delta x}} \]
\[ u_{j}^{n+1} = u_{j}^{n} + \Delta t (u_{j}^{n} - u_{j}^{n-1}) \]

These algorithms are called upwind or downwind according to the sign of \( \Delta t \).
\[ \frac{\partial u}{\partial x} \bigg|_{n_{ij}} = \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta x} + O(\Delta x^3) \]

\[ u_{ij}^{n+1} = u_{ij}^n + \frac{\alpha}{2} (u_{ij+1}^n - u_{ij-1}^n) + O(\Delta t, \Delta x^2) \]

FTCS: forward time, central space
\[ un(n) = \ldots \]

\[ \begin{array}{l}
\text{do } i = 1, N \\
\text{UNpo}(i) = UN(i) + \alpha * (UN(i+1) - UN(i-1)) \\
\text{end do}
\end{array} \]
Hyperbolic eqs are IVP
(initial value boundary problems)

\[ t \]

\[ \Delta t \]

\[ \Delta t \]

\[ \Delta t \]

initial conditions
Lecture notes

\[ \text{computer code to solve simple 1D - 2D hyperbolic eqs.} \]
\( \Delta t \) has to be chosen so that 
\( c (\Delta t, \Delta x) \) is small enough. However, different algorithms may impose stronger restrictions.

\[ u^n_j = \xi^n e^{ikx_j} \quad \xi^n \in \mathbb{C} \]

\[ u^{n+1} = T (U^n) : \text{change } \xi^n \]
$e^k$: amplification factor

$|\xi|^2 = e^k \leq 1$: this implies the solution is bounded.

$\rightarrow$ each algorithm need to be checked for stability.

$FTCS\ |\xi|^2 = 1 + (\alpha \sin(\Delta x k))^2 > 1$

$\Rightarrow$ $FTCS$ is unstable.
Never use unstable algorithms!
Uprwind

$$\left| \phi \right|^2 = 1 - 2|\alpha| (1 - |\alpha|) (1 - \cos(k\Delta x))$$

$$\leq 1$$ \iff \alpha \leq 1$$

$$\sqrt{\frac{\Delta t}{\Delta x}} \leq 1 \implies \Delta t \leq \frac{\Delta x}{\sqrt{\gamma}}$$

$$\Delta t = c_{FL} \frac{\Delta x}{\sqrt{\gamma}}$$
\[ u_{j,j+1}^{n+1} = u_{j,j+1}^n + \frac{\alpha}{2} (u_{j+1,j+1}^n - u_{j,j}^n) : \text{FTCS} \]

\[ u_{j,j+1}^{n+1} = \frac{1}{2} (u_{j+1,j+1}^n + u_{j,j-1}^n) + \frac{\alpha}{2} (u_{j+1,j+1}^n - u_{j,j-1}^n) \]

\[ \text{Lax-Friedrichs} \]
LF: stable!

\[ \left| \frac{\partial T}{\partial x} \right|^2 = 1 - \sin^2 (k \Delta x) \left( 1 - \alpha^2 \right) \leq 1 \]

if \( \alpha \ll 1 \)

(2) is a 1st order approx. to

\[ \tau u = \eta \Delta x \]

but a 2nd order approx to

\[ \tau u = \eta \Delta x + \epsilon \Delta^2 x u \]

\[ \text{diffusive} \]
\[ \epsilon = \epsilon \left( \Delta x^3 \right) \]

\[ \lim_{\Delta x \to 0} (1) = (2) \]
2nd order in space/time

\[ \partial_t u |_{n,j} = \frac{u_{n+1}^{j} - u_{n-1}^{j,j}}{2 \Delta t} \]
\[ u^{n+1}_j = u^{n-1}_j + \alpha \left( u^n_{j+1} - u^n_{j-1} \right) + o(\Delta t^2, \Delta x^4) \]

\[ |\mathcal{E}|^2 = 1 \quad : \quad \text{has no dissipation} \]
\( \partial_t u = \sqrt{\partial_x^2} u \) \hspace{2cm} (1)

\[ \partial^2_t u = \partial_x^2 \partial_x^2 u \] \hspace{2cm} (4)

\[ S = \partial_t u \]

\[ r = \sqrt{\partial_x} u \]

\((4) \Leftrightarrow \)

\[ \partial_2 u = \partial_t (\partial_x u) = \partial_x S \]

\[ \partial_x u = \partial_x (\partial_x u) = \partial_x r \]

\[ \begin{cases} 
\partial_t r = \sqrt{\partial_x} S \\
\partial_t S = \sqrt{\partial_x} r \\
\partial_t u = S
\end{cases} \]
Even a wave equation, i.e., a 2nd order PDE can be easily recast into a 1st order form.

Einstein's egg ≠ wave eggs ≠ advection eq.: We know how to solve advection eq.
\( \partial_t u = \sqrt{\partial_x u} \)

\( s = \partial_t u \)

\( r = \sqrt{\partial_x u} \Rightarrow \partial_t r = \frac{1}{r} \sqrt{\partial_x (\partial_x u)} \)

\( \text{RHS} \iff \partial_x (\partial_t u) = \partial_t s \)

\( \text{LHS} \iff \sqrt{\partial_x u} = \sqrt{\partial_x (\sqrt{\partial_x u})} \)

\[ v = \text{const.} \]

\[
\begin{cases}
\partial_t s = \sqrt{\partial_x v} \\
\partial_t r = \sqrt{\partial_x s} \\
\partial_t u = s
\end{cases}
\]
Behind cool movies there is a lot of maths.
That's what we will see these days.
- covering the spacetime
- 3+1 split
- Einstein eqs
Spectrum diagram of Schwarzschild BH
Conformal diagram
(Carter–Penrose)
$\theta = 0, \phi \in [0, 2\pi]$
All slices reach spatial infinity

t=const slices
outer boundary is at finite distance

Most standard choice
Null foliation are very powerful to study CWS. However, they have many complications (initial data, constraints)
\[ ds^2 < 0 : \text{time} \]
\[ ds^2 = 0 : \text{null} \]
\[ ds^2 > 0 : \text{space} \]

Cauchy characteristic extraction:
Combine finite spacelike grid with a null foliation
\[ ds^2 = g_{\mu \nu} \, dx^\mu \, dx^\nu \]
\((g^{\mu\nu}, M)\)

\(t\)

Manifold

Event \(x^m\)

\(\Sigma\)s are hypersurfaces of constant \(t\)
$t = t_0 + dt$

$I$ need the normal to $\Sigma$

$\Omega^\mu = \nabla^\mu t$
\[ |\Omega|^2 = \Omega^m \Omega_m = g^{\mu \nu} \nabla_{\nu} \nabla_{\mu} t \]

The lapse defines \( \Xi = -\alpha^2 \)

the normal to \( \Sigma \)

Now I can build

the normal vector to \( \Sigma \)

\[ n^\mu = -\alpha g^{\mu \nu} \nabla_{\nu} = -\alpha g^{\mu \nu} \nabla_{\nu} t \]
\[ \mathbf{V} = \mathbf{V}_x e_x + \mathbf{V}_y e_y \]

\[ (e_x)_j = \mathbf{\nabla}_j \cdot \mathbf{x} \]

\[ = \{1, 0\} \]
\( n^\mu n_\mu = -1 \)

Once I have \( n^\mu \) I can calculate the spatial metric

\[ \gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu \]

\( \gamma \) is the spatial metric
With \( n^\mu \) and \( \gamma_{\mu} \) I can build a 3+1 Split

I can define a spatial projector and a timelike projector.

projection on \( \gamma \)
Spatial projector

\[ T^\mu_\nu = g^{\mu\alpha} \, Y_{\alpha\nu} = g^{\mu\alpha} (g_{\alpha\nu} + n_\alpha n_\nu) \]

\[ = \delta^\mu_\nu + n^\mu n_\nu \]
Timelike projector

\[ N^\mu_\nu = -n^\mu n_\nu \quad N^\mu_\mu = 1 \]

\[ N^\mu_\nu \chi^\nu_\mu = 0 \]
In what we will do we will transform 4-D objects into 3D ones by taking spatial projections.

\[ D_\gamma T_\alpha^\beta = \gamma_\alpha \gamma_\beta \nabla_\gamma T_\alpha^\beta \]

3D covariant deriv.

4D cov. deriv.
\[ \nabla_\mu g^{\mu \nu} = 0 \]

3-metric

\[ D_\lambda g^{\alpha \beta} = 0 \]

(spacial)

is compatible with covariant deriva.

3D Christoffel symbols

\[ \Gamma^\lambda_{\rho \delta} = \frac{1}{2} g^{\lambda \mu} \left( \delta_{\mu \rho} g_{\delta \mu} + \delta_{\mu \delta} g_{\rho \mu} - \delta_{\mu \rho} g_{\delta \mu} \right) \]
In the same way you can define the 3D Riemann tensor

\[ 2 \mathcal{D}_\alpha \mathcal{D}_\beta W_\gamma = R^\mu_{\alpha \beta \gamma} W_\mu \]
\[ D_\alpha (D_\beta W^\mu) = \]
\[ D [\alpha, D_\beta] W^\mu = D_\alpha D_\beta W^\mu - D_\beta D_\alpha W^\mu \]
\[ R^\mu_{\delta \alpha \rho} \eta^\mu = 0 \]

\[ R^\alpha_{\beta \gamma \delta} = \Gamma^\alpha_{\beta \gamma} - \Gamma^\mu_{\beta \gamma} \Gamma^\alpha_{\mu \delta} + \Gamma^\mu_{\beta \delta} \Gamma^\alpha_{\mu \gamma} \]

\[ R_{\alpha \rho} = R^\delta_{\alpha \delta \rho} \quad ; \quad R = R^\alpha_{\alpha} \]
\( R^M_{\alpha \beta \sigma} \) : 3D Riem.

\( R^M_{\alpha \beta \sigma} \) : 4D Riem.

intrinsic curvature on \( \Sigma \)

curvature in 4D

The missing information is in the
extrinsic curvature $K_{\mu\nu}$

There are a no. of different ways to compute $K_{\mu\nu}$

$$K_{\alpha\beta} = - \gamma^\mu_{\ ;\ \mu} \nabla_{\mu \nu}$$  (1)

$$2T_{(\alpha \beta)} = T_{\alpha \beta} + T_{\beta \alpha}$$

$$2T_{[\alpha \beta]} = T_{\alpha \beta} - T_{\beta \alpha}$$
in terms of the acceleration of normal observers

\[ \ddot{a}^m = u^\nu \nabla_\nu u^m : \text{fluid accel} \]

(\( u \) fluid 4-vel)

\[ a^m = n^\nu \nabla_\nu n^m \]

\[ K_{\alpha \beta} = - \nabla_\alpha n_\beta - n_\alpha a_\beta \]
\[ K_{\alpha \beta} = -\frac{1}{2} \ln \chi_{\alpha \beta} \]

\[ L_n : \text{Lie derivative along } n \]

\[ V^\mu \text{ directional derivative along } W^\mu \]

\[ Z^\mu = W^\nu \nabla_{\nu} V^\mu \]
$$L_x \phi = x^m \partial_m \phi = x^m \partial_m \phi$$
$$L_x \nu^\alpha = x^m \partial_m \nu^\alpha - \nu^m \partial_m x^\alpha$$
Intrinsic curvature tells me the difference between red/blue.
Extrinsic curvature tells me the difference between orange/green.
RECAP

1. Take 4-D spacetime $g_{\mu\nu}$
2. Introduce time coord $t$
3. Normal to $t=const$ surf:
   \[ n^\mu = -\alpha g^{\mu\nu} \nabla_\nu t \quad n^\mu n_\mu = -1 \]
4. Define spatial metric
   \[ g_{\mu\nu} = g_{\mu 0} + n_\mu n_\nu \]
construct 3D objects by projecting with $Y^\mu$ and $N^\mu = -h^\mu \nu$

Eg

\[ D_a V^\mu = \Gamma^\mu_{\nu \lambda} Y^\nu \nabla_\mu V^\lambda \]

3D spatial Covariant deriv.
\[ R_{\mu\nu\rho} \leftrightarrow (4) R^{\alpha}_{\mu\nu\rho} \]

\[ K_{\mu\nu} = - \gamma_{\mu} \gamma_{\nu} \nabla (\alpha \eta_{\rho}) \]

Extrinsic curvature

Properties

1) Spatial tensor \( K_{\mu\nu} \)

2) Sym. tensor \( K_{\mu\nu} = K_{\nu\mu} \)
\[ K = K^{\mu} = -\nabla^{\mu} n_{\mu} \]

\[ \nabla_{\mu} \Gamma^{\nu\mu} = 0 = \nabla_{\mu} T^{\mu\nu} \]

\[ \Gamma^{\mu\nu} \]

\[ \Rightarrow \text{TT}^{\mu\nu} \]

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \]

\[ \underbrace{\text{curvature}}_{\text{energy}} \]
\[ \alpha \Gamma_{\mu
u,\beta} \propto \partial_{(01)} \]

\[ s = \gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} R_{\mu
u,\rho\sigma} \]

\[ R_{\rho\sigma,\sigma} + k_{\rho\beta} K_{\gamma\delta} - K_{\delta\gamma} K_{\rho\beta} \]

**Gauss's eqs**
\[ S \Rightarrow t \]

\[ \delta \varepsilon \equiv \eta \sigma \equiv R_{\mu \nu \sigma} \]

\[ D_\alpha k_{\beta \gamma} - D_\beta k_{\alpha \gamma} \]

*Codazzi* eqs
\[ R_{\text{calc}} \text{ eqs} \]

\[ = \gamma_{\mu}^s \gamma_{\nu}^s \Gamma^{s \nu}_{\mu} \]

\[ \ln K^{x \beta} - \frac{1}{\alpha} D \alpha D \beta \alpha - K^x_{\beta} K_{x \gamma} \]

\[ h^m = -\alpha g^{m \nu} \nabla_\nu t \]

\[ U^m n_m = 0 : U \text{ is spatial} \quad D \mu U^\nu = \gamma^\rho_{\mu} \nabla_\rho U^\nu + \]
Kre unter

\[ A \]

Gauss, Codazzi and Ricci eqs give 3 different splitting of the Riemann tensor

\[ A: \begin{array}{c} \text{Y Y Y R} \end{array} \]

\[ B: \begin{array}{c} \text{Y Y R n} \end{array} \]

\[ C: \begin{array}{c} \text{Y Y n n} \end{array} \]
\[ T^\mu_\nu = (e + p) u^\mu u^\nu + p g^\mu_\nu \]

\[ u^\mu : \text{fluid four-velocity} \]

\[ u^\mu u_\mu = -1 : \text{timelike unit vector} \]

\[ \nu^\mu n_\mu = +1 : \text{// 4-vector} \]

\[ e : \text{energy density} \]

\[ p : \text{pressure} \]

\[ h : \text{specific enthalpy} \quad h = \frac{e + p}{p} \]
$e$: rest mass density

\[
\begin{align*}
\rho & = m_0 \upsilon_n = e \\
\text{no density} & \\
\end{align*}
\]

\[e = e (1 + \epsilon) = \text{rest mass + internal energy} \]

\[\text{specific internal energy} \]
\[ e = n^\mu n^\nu T_{\mu\nu} \]

\[ j^\alpha = -\delta^\mu_\alpha n^\nu T_{\mu\nu} \]

---

momentum density
We have derived all the relevant elements to proceed to a 3+1 splitting of the Einstein eqs.

Anticipate the results

\[ G_{\mu \nu} = 8\pi T_{\mu \nu} \]

10 eqs
10 eqs: 2nd order PDEs

12 evolution eqs: 1st order

6 evolution eqs: 2nd order

\[ \frac{\partial^2 u}{\partial t^2} = \nabla^2 \frac{\partial^2 u}{\partial x^2} \rightarrow \]

\[ \frac{\partial r}{\partial t} = r \frac{\partial}{\partial x} s \]

\[ \frac{\partial s}{\partial t} = r \frac{\partial}{\partial x} r \]

4 constraint eqs are not evolution
\[ \partial_t \mathbf{E} = \ldots \]
\[ \partial_t \mathbf{B} = \]
\[ \nabla \cdot \mathbf{E} = 4\pi \rho \]
\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \square \mathbf{E} = 0 \]

\{ evolution e\qs \}
\{ constraint e\qs \}
$2 \eta^m \eta^n G_{mn} = R + k^2 + k_{\mu \nu} k^{\mu \nu}$

Hamiltonian
Constraint
(no time derivatives)

$= 2 e \cdot 8 \pi = 16 \pi e$

$\nabla^2 \phi = 4 \pi e$
\[-Y^\mu_2 n^\nu \text{ (4)} G_{\mu\nu} = -R_{\lambda\nu} n^\lambda + \frac{1}{2} n^\nu n R \]

\[= -D_\mu K^\mu_\lambda + D_\alpha K \]

\[= -8\pi j_\lambda \]

**Momentum constraint eq 30**
\Omega_\mu = \nabla_\mu t

n^\mu = -\alpha g^{\mu\nu} \nabla_\nu t

So far we have only defined the normal \( n \) to \( \Sigma \) but \( n \) is not the natural direction of time evolution (\( n \) is not dual to \( \mathbb{R} \)).

\[ n^\mu \Omega_{\mu\nu} = \frac{1}{\alpha} \neq 1 \]
We need a different 4-vector which is still time-like

\[ t^\mu = \alpha n^\mu + \beta^\mu \]

\( \beta^\mu \) is purely spatial

\[ t^\mu \Sigma_\mu = (\alpha n^\mu + \beta^\mu) \Sigma_\mu \]

\[ = \alpha n^\mu \Sigma_\mu + \beta^\mu \Sigma_\mu \]

\[ = 1 \]
$t^\mu$ is dual to $\Sigma_\mu$
normal world line

coordinate world line

\( n^i_1 \)

\( n^m_2 \)
\[ n^\mu \Omega_{\mu} \neq 1 \]
\[ t^\mu \Omega_{\mu} = 1 \]
Recall one def. of ext. curr.

\[ K_{\alpha \beta} = -\frac{1}{2} \, L_\nu \, F_{\alpha \beta} \]

\[ t^\nu = \alpha \, n^\nu + \beta^\nu \implies \alpha \, L_\nu = L_t - L_\beta \]

\[ \implies \begin{vmatrix} L_t \delta_{\mu \nu} = -2\alpha \, k_{\mu \nu} + L_\beta \delta_{\mu \nu} \end{vmatrix} \]

definition of ext. curvature

but gives evolution eq for \( k_{\mu \nu} \)
$\Sigma_{\mu \nu} \rightarrow \left( \begin{array}{c} 10 \\ 0 \end{array} \right)$

$\gamma_{\mu \nu} \rightarrow \left( \begin{array}{c} \chi \chi \chi \\ \chi \chi \chi \chi \chi \end{array} \right)$

6 spectral components of $\gamma$
\[ T_{\mu \nu} \stackrel{(4)}{\Rightarrow} \text{elliptic eq} \]

\[ \gamma^\nu_{\mu p} q_{\mu \nu} = 8\pi G \Psi_\beta = 8\pi \gamma^\nu_{\mu p} T_{\mu \nu} \]

Using Ricci eq

\[ S \quad K_{\mu \nu} = D_{\mu} D_{\nu} \alpha + \alpha (\ldots) \]

\[ + \quad S_{\beta} K_{\mu \nu} \]

6 evol. eqs for \( K_{\mu \nu} \)
\( n n^{(4)G} \rightarrow \) Ham. eq
\( Y n^{(4)G} \rightarrow \) mom. const
\( X X^{(4)} G \rightarrow \)
\[
\begin{align*}
L_t K_{\mu\nu} &= D_\mu D_\nu \alpha + \ldots \\
L_t \gamma_{\mu\nu} &= -\alpha K_{\mu\nu} + \ldots
\end{align*}
\]
\( A r n o w i t t \)
\( D e s e r \)
\( M i s n e r \) eqs
\( 6 + 6 = 12 \)
Choosing a proper coordinate basis to make the eqs more transparent

\[ L_t \rightarrow \partial_t \]

We want a set of unit vectors \((e_i)^m\) which highlight the special nature of \(g_{\mu\nu}, K_{\mu\nu}\)
Requirements:

$(e_j)^m$ have to be spatial

$\{e_1\}^m = \{0, 1, 0, 0\}$

$\{e_3\}^m = \{0, 0, 0, 1\}$

\[ n^m (e_j)^m = 0 \]

$(e_2)^m = t^m = (1, 0, 0, 0)$
\[
\Rightarrow \quad L_t \quad \rightarrow \quad \partial_t \\
\eta^M = \frac{1}{\alpha} (1, -\beta^i) ; \quad \eta_\mu = (-\alpha, 0, 0, 0) \\
\beta^\mu = (0, \beta^i) ; \quad \beta_\mu = (0, \beta^i) \\
\gamma^{\alpha \alpha} = 0 \quad \text{for } \gamma \text{ spatial} \\
\gamma^{\mu \nu} = \delta^{\mu \nu} - n^M n^\nu
\]
\[ g^{\mu \nu} = \begin{pmatrix} -1/d^2 & \beta_i/d^2 \\ \beta_i/d^2 & \delta^{ij} - \beta_i \beta_j/d^2 \end{pmatrix} \]

\[ \gamma_{ij} \gamma_{ij} = \delta_{ij} \]

\[ g_{\mu \nu} = \begin{pmatrix} -d^2 + \beta_i \beta_i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} \]

\[ d\ell^2 = \gamma_{ij} dx_i dx_j \]  

Spatial 3 metric
\( Y \) can be used to lower and raise the indices.

\[
V_i = Y_{ij} V^j
\]

\[
V^i = Y^{ij} V_j
\]
\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu \]
\[ = -(x^2 - \beta^i \beta_i) dt^2 + 2 \beta^i dx^i \frac{dx^i}{dt} \]
\[ + \gamma_{ij} dx^i dx^j \]
\[ dl^2 = \gamma_{ij} dx^i dx^j \]
\[ d\tau^2 = - (\alpha^2 - (\beta^i \beta_i) dt^2) \]
\[ = - \alpha^2 dt^2 \]
The lapse function measures (in newt)
how time changes from one slice to the next
Let's compare Einstein with Maxwell

$\mathbf{\nabla} \times \mathbf{B} = 4\pi \mathbf{J}$

$\mathbf{\nabla} \times \mathbf{E} = 0$

$\mathbf{\nabla} \cdot \mathbf{B} = 0$

$\mathbf{\nabla} \cdot \mathbf{E} = 4\pi \rho$

$\partial_t E_i = \epsilon_{ijk} B_j E_k$

$\partial_t B_i = -\epsilon_{ijk} E_j E_k$

$\partial_i E^i = 4\pi \rho$

$\partial_i B^i = 0$

$1 + 1$

Const.
\[ 4D : \quad G_{\mu \nu} = 8\pi T_{\mu \nu} \]

10, 2nd-order PDEs

Introduce 3+1 split and obtain

\[ (3+1) + (6 + 6) \]

\[ \frac{4}{12}, 1st \text{ order eqs} \]

constraint eqs

\[ \text{evolution eqs} \]
\[ n \cdot n \cdot G \Rightarrow R + k^2 - K_{ij} K^{ij} = 16\pi e \]

\[ \gamma \cdot n \cdot G \Rightarrow D_j K^{ij} - D_i K = 8\pi j_i \]

\[ \gamma \cdot \gamma \cdot G \Rightarrow \partial_t \delta_{ij} = -2\alpha K_{ij} + \mathcal{P}_\beta Y_{ij} \]

\[ \partial_t K_{ij} = -D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K^{ik} + K K_{ij}) - 8\pi \alpha (R_{ij} - \frac{1}{2} \delta_{ij} (S - e)) + \mathcal{P}_\beta K_{ij} \]
\[ \partial_t \vec{E} = \nabla \times \vec{B} - 4\pi J \]
\[ \partial_t \vec{B} = -\nabla \times \vec{E} \]
\[ \nabla \cdot \vec{E} = 4\pi \rho_e \]
\[ \nabla \cdot \vec{B} = 0 \]
$A_\mu = (-\phi, A_i)$

$\partial_t A_i = -E_i - D_i \phi$

$\partial_t E_i = -\partial^j D_j A_i + D_i D^j A_j - 4\pi J_i$

$\partial_t \gamma_{ij} = -2\alpha k_{ij} + \Phi_\beta \gamma_{ij}$

$\partial_t k_{ij} = -D_i D_j \alpha + \alpha (R_{ij} + \ldots)$

ADM
The ADM eqs, as the Maxwell eqs we have written are weekly hyperbolic and hence ill-posed: the solution can grow unbounded.

In contrast, a strongly hyperbolic formulation is well posed.
We've eq is a typical example of a strongly hyperbolic eq.
Let's try to write Maxwell as wave eq

\[ \partial_t A_i = -E_i - D_i \phi \]

take time derivative

\[ -\partial^2 A_i + D_j D_j A_i - D_i D^j A_j = D_i \partial_t \phi - 4\pi J_i \]

mixed derivatives
\[ u = 0 \quad \partial_t^2 u - \partial_i \partial^i u = 0 \]

The system is weakly hyperbolic, but we can introduce a new variable

\[ \Gamma \equiv D^i A_i \]

\[ \partial_t E_i = -D^j D_j A_i + D_i \Gamma - 4\pi \mathcal{J}_i \]

\[ A_i = -D_i \Gamma - D_i \partial_t \phi + 4\pi \mathcal{J}_i \]

wave eq. principal part
The price to pay is an additional eq. for \( \tau \)

\[
\partial_t \tau = \partial_t \text{Di} A_i = \text{Di} \partial_t A_i
\]

\[
= -4\pi \epsilon_0 - \text{Di} \text{Di} F
\]

\[
\partial_t \tau = \partial \tau 
\quad \Rightarrow \quad \int \partial_t \tau = \int \tau \quad \text{ex}
\]

\[
\partial s = \int \partial \tau
\]

Adding a new eq. is not a problem.
\[ \partial_t K_{ij} = - D_i D_j \alpha + \lambda (R_{ij} + \ldots) \]

contains mixed deriv and spoils were stree.

We do the same done for Maxwell and introduce a number of new quantities: conformal, trace-free formulation (BSSNOK)
\[ \phi = \frac{1}{12} \ln \left( \det (\delta_{ij}) \right) = \frac{1}{12} \ln (\delta) \]

: conformal factor

\[ ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \]

\[ ds^2 = \tilde{g}_{\mu\nu} \, dx^\mu \, dx^\nu \]  

\[ \tilde{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu} \]

\[ \text{conformal factor} \]
\[ \tilde{\gamma}_{ij} = e^{-4\phi} \gamma_{ij} \]

\[ \tilde{\gamma}_{ij} \text{ con formal} \]

3-metric

\[ \overset{\sim}{A}_{ij} = e^{-4\phi} \left( k_{ij} - \frac{1}{3} \gamma_{ij} k \right) \]

\[ \text{trace free conf. extrinsic curv} \]
\[ \Gamma = D_i A_i \]
\[ \Gamma^i = \gamma^{jk} \Gamma^i_{jk} \]
\[ \tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk} \]

After doing all this we obtain the following set of eqs.
\[ \Delta_t \tilde{r}_{ij} = -2\lambda \tilde{A}_{ij} \quad \Delta_t = \partial_t - \mathcal{L}_p \]

\[ \Delta_t \phi = -\frac{1}{6} \lambda k \]

\[ \Delta_t \tilde{A}_{ij} = e^{-4\phi} \left[ -\nabla_i \nabla_j \lambda + D^i D_j \tilde{r}_{ij} + \ldots \right] \]

\[ \Delta_t k = -2\tilde{A}_{ij} \nabla_i \nabla_j \lambda + \ldots + e \nabla^6 \tilde{A}_{ij} \]

\[ \Delta_t \tilde{F}_i = -2\tilde{A}_{ij} \partial_j \lambda + \ldots \]

BSSNOK is hyperbolic and therefore well-posed

\[ |\epsilon| \leq 1 \]
\[ \partial_t \tilde{\gamma}_{ij} \]
\[ \partial_t \tilde{\alpha}_{ij} \]

\[ H = R + K^2 - K_{ij} K_{ij} = 0 \quad (\star) \]
\[ M_j = D_j (K_{ii} - \tilde{g}_{ij} K) = 0 \]

At each time level we compute the left-hand side of (\star) and monitor them.

\[ \| H \|_2 = \frac{1}{N} \sum_{ijk} (H(x,y,z))^2 \]
We check that $\|H\|_2 \leq \varepsilon$

The constraints are not solved but monitored.
\[ \| \|_2 = \frac{1}{N} \sum_{i,j,k} H_{ijk} \]

\[ \| \|_\infty = \max_{i,j,k} H_{ijk} \]
There are alternative formulations to BSSNOK and these are called generalized harmonic (GH).

This formulation imposes that the coordinates follow a wave eq.

\[ \nabla = \Box X^\mu = \frac{1}{\sqrt{-g}} \partial_{\nu} (\sqrt{-g} g^{\nu \beta} \partial_{\beta} x^\mu) \]
If this is imposed the Einstein eqs have a wave principal part

\[ \Box g_{\mu \nu} = \mathcal{C}(g_{\mu \nu}, \partial g_{\mu \nu}, \ldots) \]

However \( \Box x^\mu = 0 \) is problematic because of caustics

\[ \Box x^\mu = H^\mu \]

generalized harmonic
The specification of \( H^\tau \) amounts to selecting a gauge condition

\[ 3+1 \text{ metric} \]

\[ ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = a^2 (dt^2 - 2 \beta \, dx^i \, dt + \delta_{ij} \, dx^i \, dx^j) \]

\( a, \beta \) (leapse and shift) are arbitrary and reflect our arbitrariness in laying \( \alpha \) and \( \xi \).
Defining "good" gauge condition is an art in NR.

What is "good"?

Oppenheimer-Snyder collapse (collapse of dust $p=0$)

$a=1$, $\beta^i=0$
$a = 1, \beta^i = 0$

Geodesic slicing is not a good idea because grid points end up in the singularity.
The trick is to shear the spacetime avoiding the singularity and this is called sing avoiding shearing.

You do the trick by “slowing” proper time in regions where curvature is large.
choice of $a \iff$ slicing condition

$\Pi \equiv \beta \iff$ spatial gauge conditions.

It is possible to define perfect gauge conditions but these are always the result of elliptic eqs and hence very expensive.
Eg

Maximal slicing: guarantees singularity avoidance but is elliptic

\[ \exists \alpha = 0 \]

\[ D^i D_i \alpha = 2 \left[ k_{ij} k^{ij} + 4\pi (\varepsilon + s) \right] \]

We rather use

\[ \partial_t \alpha - \beta^i \partial_i \alpha = - \alpha^2 \frac{\partial}{\partial t} \left( K - K_0 \right) \]

\[ \frac{\partial}{\partial t} \alpha \]

\[ \partial_t \alpha \]

initial
\[ \frac{dx}{dt} = -x^2 f(x) \ (K - K_0) \]

\[ f(x) = \begin{cases} 
0 & : \text{geodesic slice} \quad (x > 1 \ \text{as} \ t \to \infty) \\
1 & : \text{harmonic} \\
2/x & : \quad \| 1 + \text{kay} \| \\
\to \infty & : \text{maximal slicing} 
\end{cases} \]

\[ \alpha = \alpha_0 e^{-t/\alpha} \]

\[ \alpha = \alpha_0 e^{-t/\alpha} \]
It log suppresses exponentially fast the lapse in the presence of large curvature. A gauge
Spatial condition must avoid distortions
Good gauge spectral conditions need to avoid obstructions (usually via elliptic eqs)

Alternatives are

$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \frac{3}{4} \alpha B^i$$

$$\partial_t \beta^i = \frac{3}{4} \alpha B^i$$

$$\partial_t B^i - \beta^j \partial_j B^i = \partial_t \tilde{F}^i - \beta^j \partial_j \tilde{F}^i$$

$$\partial_t B^i = \partial_t \tilde{F}^i$$
This enforces that $d\tilde{F}_i = 0$

"gamma-driver" condition
Initial Data

GW extraction

Relativistic hydrodynamics
Simplest way is to assume there is an equilibrium \( \Rightarrow \) remove time derivative \( \Rightarrow \) constraint eqs.

\[
\begin{align*}
R + k^2 - K_{ij} \ k_{ij} &= e \\
D_j K_{ij} - D_i K &= \delta T_{ij}
\end{align*}
\]

\[
\delta_{ij} = \gamma^4 \ \tilde{\delta}_{ij} = \gamma^4 \ \delta_{ij}
\]

elliptic eqs

\[
\begin{align*}
\text{CONFORMAL FLAT APPROX.}
\end{align*}
\]
The consequence of this is that the initial data do not contain waves \( \Rightarrow \) NR simulation all start with a little bit of "junk" radiation.
GW extraction

There are different ways of doing this

1) perturbative matching

2) extraction via Weyl scalars
\( M : \frac{\Delta E_{gw}}{M} = \text{few} \times 10^{-2} \)

binary bhs

= \( 10^{-3} \) binary NS

= \( 10^{-6} \)
collapse to bh
1) Assume that at certain distance the spacetime is that of Schwarzschild bh.
\[ g_{\mu \nu} = \tilde{g}_{\mu \nu} + h_{\mu \nu} \]

\[ h_{\mu \nu} = g_{\mu \nu} - \tilde{g}_{\mu \nu} \]

waves computed background

In practice $h_{\mu \nu}$ is decomposed into tensor spherical harmonics
2) Weyl

Computes the radiative part of Weyl tensor (i.e., the trace-free part of the Riemann tensor)

\[ \gamma_4 \propto \frac{C_{\mu\nu}}{r} : \text{outgoing radiation} \]
\[ h^+, h_x \]

2) \[ h_{\mu \nu} \rightarrow Q^+_{em}, Q^x_{em} \]

\[ h^+ - i h_x = \frac{1}{\sqrt{2} r} \sum_{em} \left( Q^+_{em} - \right. \]

\[ + i \int_{-\infty}^{t} Q^x_{em} (t') dt' \]
\[ 2)\]
\[ h_+ - i h x = \int_{0 L} \int_{0 L} -\Psi_4 \]
\[ h \propto \Psi_4 \]
Relativistic hydrodynamics

\[ T_{\mu\nu} = (e + p) u_{\mu} u_{\nu} + p g_{\mu\nu} \]

perfect fluid
\[ = he u_{\mu} u_{\nu} + p g_{\mu\nu} \]

\[ T_{\mu\nu} = T_{\mu\nu}^{A} + T_{\mu\nu}^{B} + \ldots \]
\[ U^\mu U_\mu = -1 \]

(3) \( \gamma \cdot (\nabla T) = 0 \) : conservation of mom.

(1) \( n \cdot (\nabla T) = 0 \) : cons. of energy

(1) \( (\nabla \rho u) = 0 \) : cons. of baryon

\[ p = p(e, \varepsilon, \ldots) \]
Euler eqs: describe fluid motion in Newtonian gravity.

\[ \partial_t (e \xi) + \partial_i (e \xi \xi_j + p g_{ij}) = - \partial_i \phi \]

\[ \partial_t \phi = \tau_i = \nabla \cdot \mathbf{F} \]
\[ \partial_t u = \Delta u + uu \] : hyperb.

Hydrodynamic eqs. are also hyperbolic but nonlinear.

\[ \partial_t u = u \partial_x u \] : Burgers equation

The evolution of a nonlinear hyp. eq. leads to development
of discontinuities

t = \pi \tau x v n

\text{graphical representation}
\[ \triangledown u = u \triangledown^2 u \]
1) Why does that happen?
2) Why do we care?
We care because the
by the sky maonic egs are nonlinear
even smooth initial data will produce discontinuities (shocks)

\[ \dot{u} = u \partial_x u + \varepsilon \partial_x^2 u \]

Because of this nonlinear feature of hydrodynamics, they can easily develop shock
but we have to be able to follow them properly.

In other words we need to write the eqs. in flux-consen form.

\[ \partial_t \mathbf{u} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{S}(\mathbf{u}) \]
1) Only a FC formulation of the eggs leads to correct weak solution

2) A non FC formulation leads to the incorrect weak solution
1) \( \exists t \ u = n \times u \quad : \quad \text{Non} \quad \not\in \quad \mathbb{C} \)

2) \( \exists t \ u = 2\times \left( \frac{u^2}{2} \right) \quad : \quad \text{FC} \)

\[ \sqrt{\text{}} \]
\[ \forall \underline{u} + \partial x F (\underline{u}) = 0 \]

\[ A = \partial F / \partial \underline{u} \]

\[ \forall \underline{u} + \lambda \partial x \underline{u} = 0 \]

\[ \Lambda = R^{-1} A R : \text{diagonal} \]

\( R \): right eigenvector

\[ \Lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \]
$W$: characteristic variable

$W = R^{-1} u$

$\partial_t W + \bigwedge^2 x W = 0$

This is an advection eq,
along the directions given by

$\chi$
\[ \frac{d}{dt} W = 0 \quad \text{along} \quad \frac{dx}{dt} = \pm \lambda; \]

characteristic directions
Then what happens in the Burgers eq is that the characteristics merge (caustics)
Recap

- Hydro eqs are nonlinear

- Hydro eqs develop shock
  even from smooth II

- Eq with discontinuities
  "must" be written in a FC
  form
We have to write the rel. hydro eqs in a F C: Valencia formulation

\[ v^i = \frac{\delta^i_r u^r}{-\mathbf{n} \cdot \mathbf{u}} = \frac{1}{\alpha} \left( \frac{u^i}{u^0} + \beta^i \right) \]

\[ \sqrt{i} = \gamma v^j \sqrt{j} \]

\[ W = \alpha u^0 = \frac{1}{\sqrt{1 - v^i \sqrt{i}}} : \text{Lorentz factor} \]
$$T_{\mu \nu} = T_{\mu \nu} \left( \mathbf{w}, \mathbf{p}, \mathbf{e}, \mathbf{\epsilon}, . . . \right)$$

These cannot be used

$$D = e \mathbf{W} : \text{conserved rest mass density}$$

$$S_{ij} = e \mathbf{h} W^2_{ij} : \text{conserved mom. flux}$$

$$T = e \mathbf{h} W^2 - e \mathbf{W} : \text{cons. energy}$$
\[
\frac{1}{\sqrt{-g}} \left\{ \partial_t (\sqrt{g} F^0) + \partial_i (\sqrt{g} F^i) \right\} \\
\sqrt{-g} = \alpha Y = S
\]

\( F^0 = U = (D, S_\perp, \tau)^T \)

\( F^i = F^i (\cdots) \)

\( S = S (g, \omega \times g, \cdots) : \) No deriv. of the fluid variables
Solution of hydro eqs.

Finite-difference methods are not adequate for the solution of the hydro eq.

Finite volume methods are instead preferable.
\[ \bar{u}_j^n = \frac{1}{\Delta x} \int_{x_i - \gamma_2}^{x_i + \gamma_2} u(x, t^n) \, dx \]

\[ \forall \text{ fluid cell} \]

\[ \frac{\partial u}{\partial t} + \nabla F = 0 \]
\[ \hat{u}_{i+1} = \hat{u}_i - \frac{\Delta t}{\Delta x} (\hat{F}_{i+1/2} - \hat{F}_{i-1/2}) \]
\[ \hat{F}_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(U(x_{i+\frac{1}{2}})) \, dt \]

(\cdot) is the standard 1st order (Godunov) representation of a finite volume conservative eq.
One can therefore think that a piecewise constant representation of $\text{hypo var}$,

\[ y = x^2 \]

this is a consequence of discretization.
is a series of initial conditions of local Riemann problems

evolution of 2 discontinuous initial states
Godunov realized that and produced a series of methods (high resolution shock capturing, HRS) that allow to solve the hydro eq. to great accuracy even with shock.