Lecture I : Discrete differential operators

Lecture II : 3+1 split

Lecture III : Formulations of Einstein eqs

Lecture IV : Gauges, 1D, GW extraction

Lecture V : Relativistic hydrodynamics

Lecture VI : Gravit. Collapse

Lecture VII : Binary BHs

Lecture VIII : Binary NSs

L. REZZOLLA

$$a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} + f(x, y, u, \partial_x u, \partial_y u) = 0$$

$a_{11} a_{22} - a_{12}^2 < 0$: Hyperbolic eq.

$$\partial_t^2 u = v^2 \partial_x^2 u$$

$$\frac{\partial}{\partial x} = \partial_x$$

$a_{11} a_{22} - a_{12}^2 = 0$: Perabolic eq.

$$\partial_t u = D \partial_x^2 u$$

Evolution eqs.

> 0 : Elliptic eq.

$$\nabla^2 \phi = \rho$$

$\mathcal{L}(u) - f = 0$: continuum

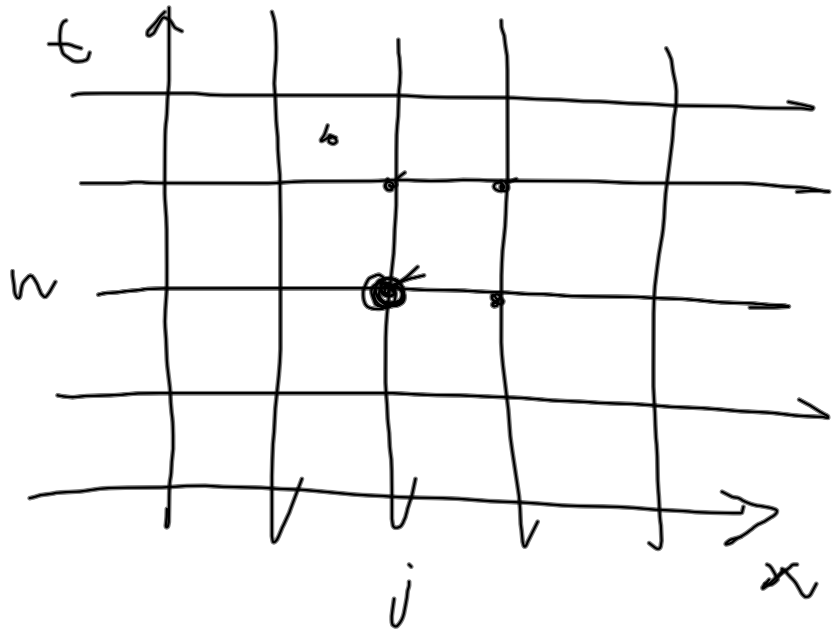
$\mathcal{L}(u)$: differential operator

$u(x, t)$

$\downarrow$

$u(x = x_j, t = t^n)$

$= u_j^n$



$$\mathcal{L}(u) \longrightarrow L_{\Delta}(u_j^n)$$

$$\mathcal{L}(u) - f = 0$$

$$L_{\Delta}(u_j^n) - \tilde{f} = 0 + \epsilon_T$$

ϵ_T ; truncation error and is consequence of discretization

ϵ_{mp} : machine precision error

$$a \,\bar{=}\, a + c$$

$$c \neq 0$$

$$1.\overbrace{0}^{15\times 0} = 1.\overbrace{0}^{15\times 0} + 0.\overbrace{0000\ldots}^{15\times 0}1$$

It's the ability of distinguishing 2 nos!

N floating point operations you build

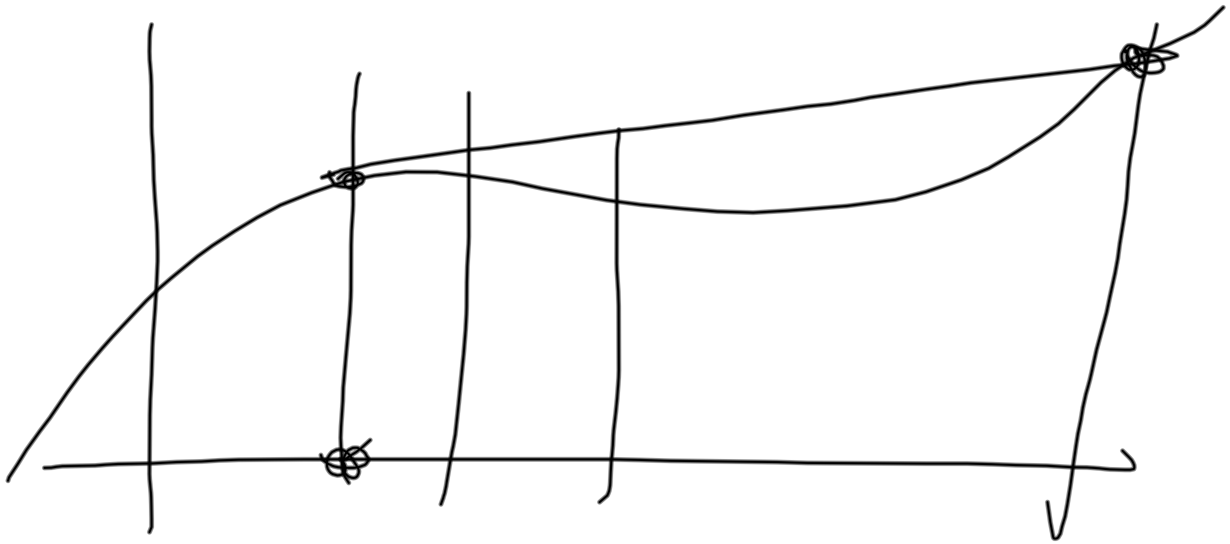
$$\epsilon_{RO} = (N)^{1/2} \epsilon_{MP}$$

— . —

$\epsilon_{MP}, \longleftrightarrow \epsilon_{RO}, \quad \epsilon_T$

ϵ_T is totally under human control and is the result of the choice made in discretising $\mathcal{L}$

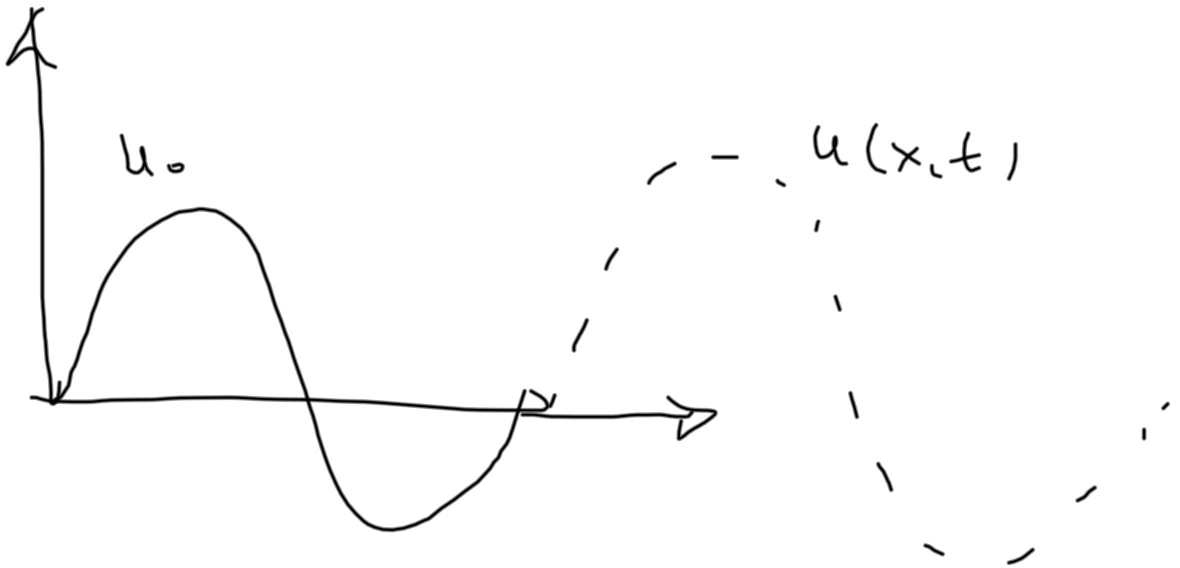
$$\epsilon_T = f\left(\Delta x^n, \Delta t^m\right)$$



Discretization of hyp. eqs

$\partial_t^2 u = v^2 \partial_x^2 u$: wave eq

$\partial_t u = v \partial_x u$: advection eq.



$$\mathcal{L}(u) \longrightarrow L_\Delta(u^n_j)$$

$$u^n_j = u^n_{j-1} + \left.\frac{\partial u}{\partial x}\right|_{n,j} (x_j - x_{j-1}) + O(\Delta x^2)$$

$$\Longrightarrow$$

$$\left.\frac{\partial u}{\partial x}\right|_{n,j} = \frac{u^n_j - u^n_{j-1}}{x_j - x_{j-1}} + \underbrace{O(\Delta x^{1})}_{\epsilon_T}$$

1st order approximation to $\partial_x u$

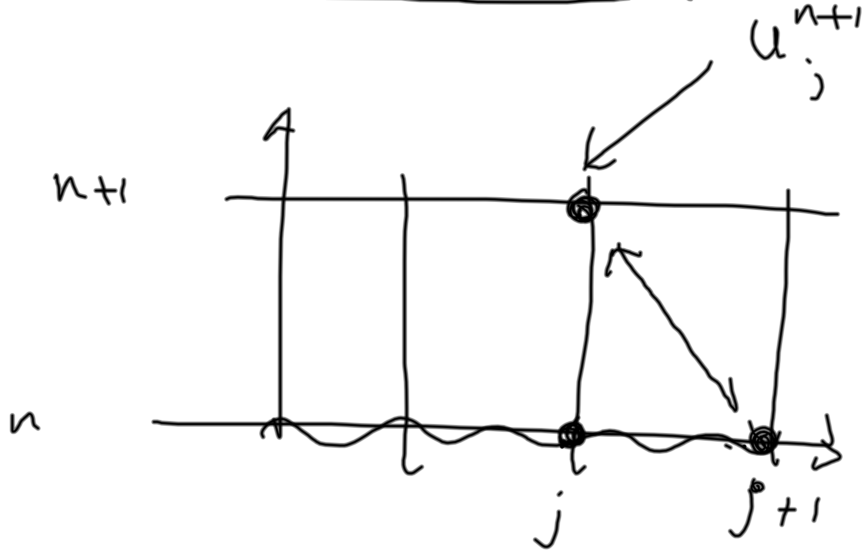
FINITE DIFFERENCE

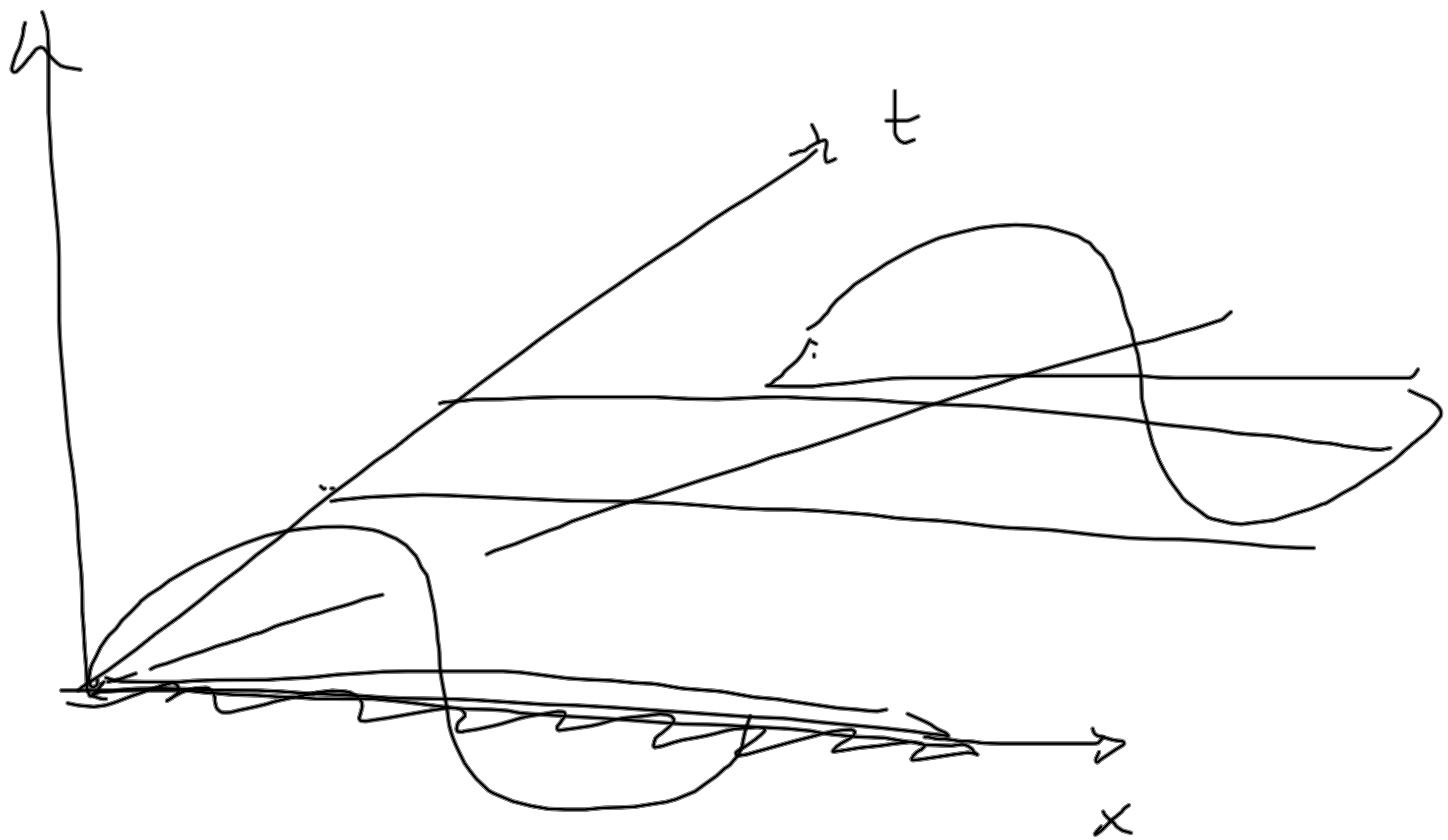
$$\partial_t u = v\, \partial_x u \qquad u = u(x,t)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = v \frac{u_{j+1}^n - u_j^n}{\Delta x} + O(\Delta t, \Delta x)$$

$$\boxed{u_j^{n+1} = u_j^n + \alpha \left(u_{j+1}^n - u_j^n \right)}$$

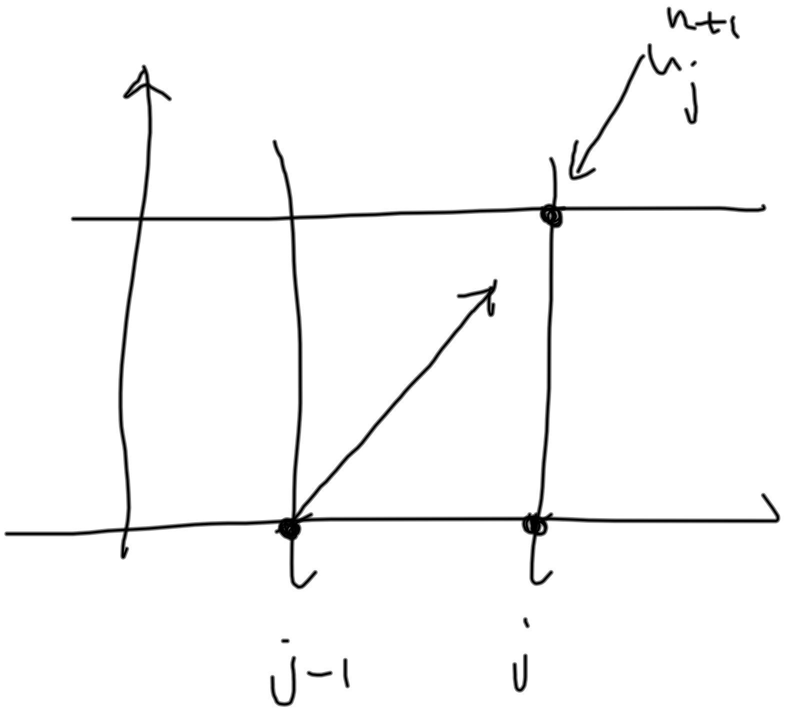
$$\alpha \equiv v \frac{\Delta t}{\Delta x}$$




u
t
x

$$u_j^{n+1} = u_j^n + \alpha \left(u_j^n - u_{j-1}^n \right)$$

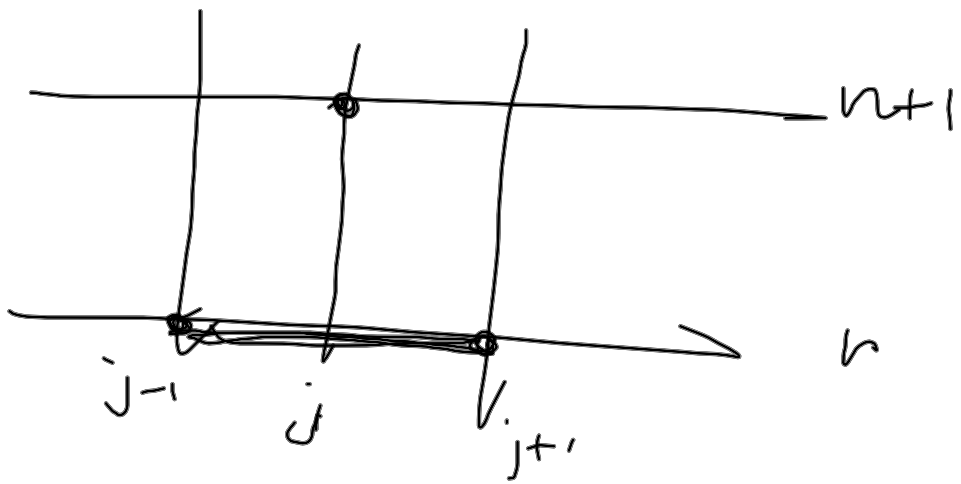
These algorithms are called upwind or downwind according to the sign of v.



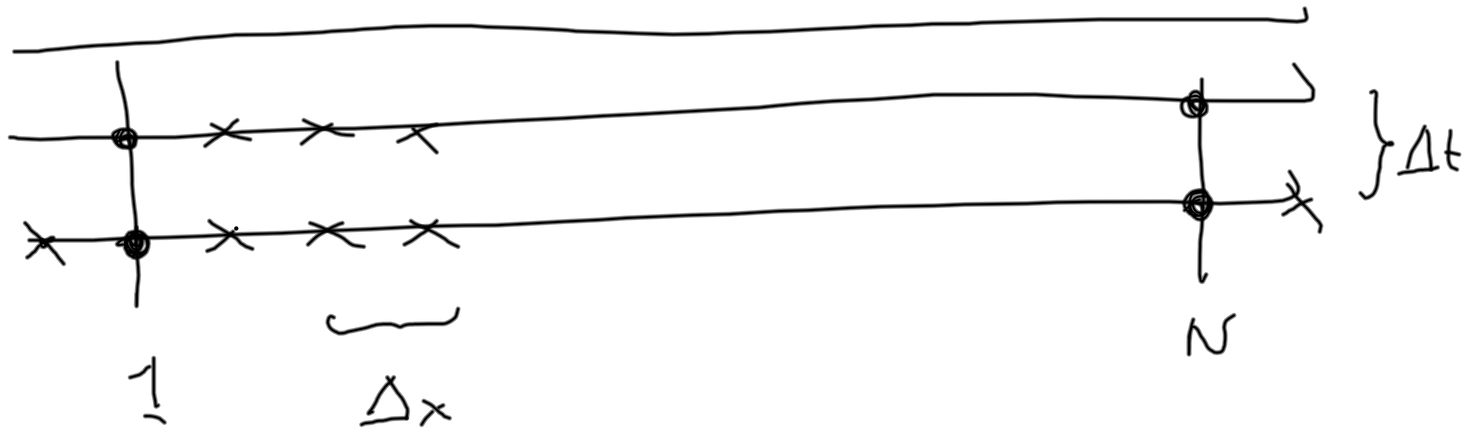
$$\left.\frac{\partial u}{\partial x}\right|_{n,j} = \frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x} + O(\Delta x^3)$$

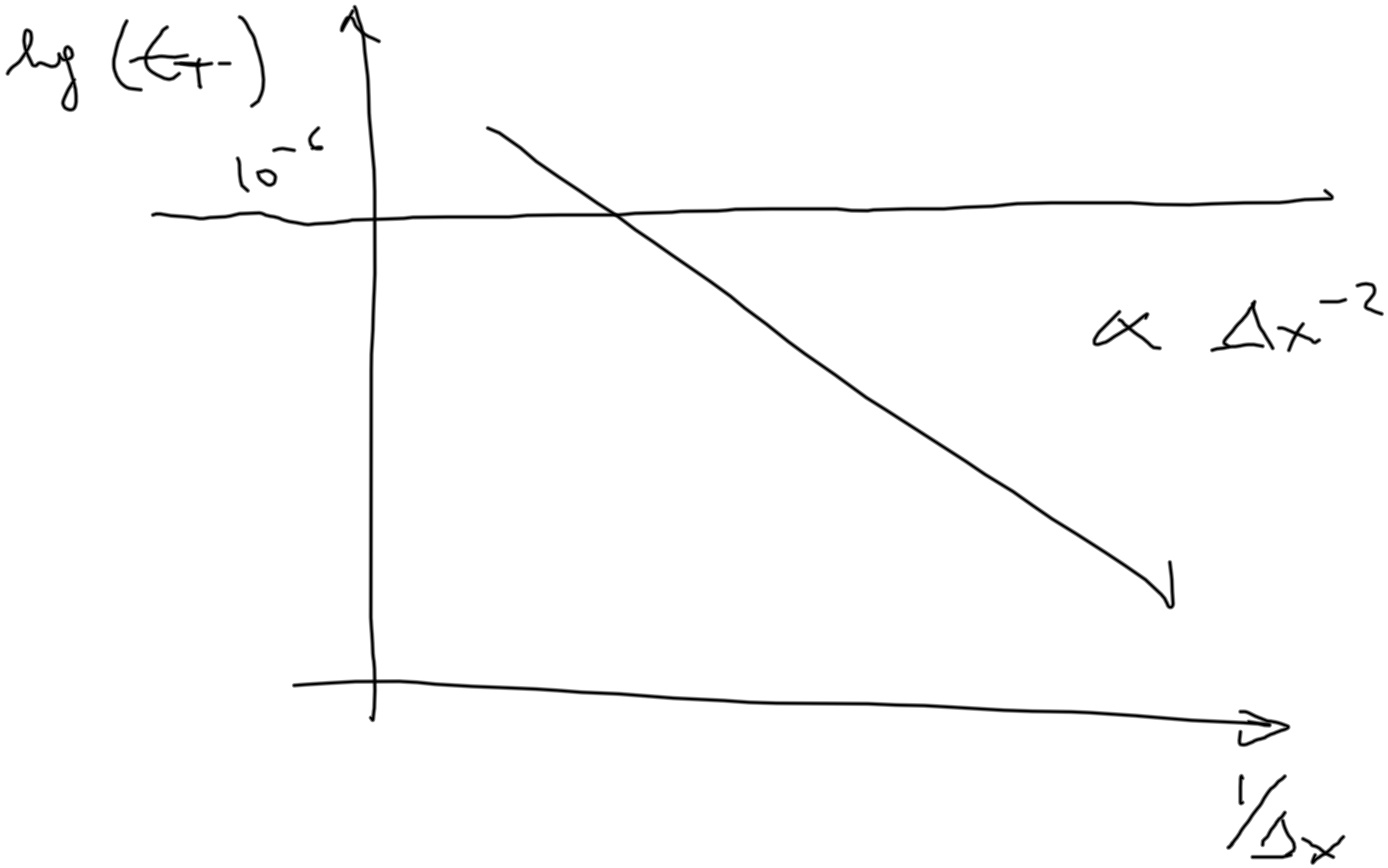
$$\boxed{u^{n+1}_j = u^n_j + \frac{\alpha}{2}\left(u^n_{j+1} - u^n_{j-1}\right) + O(\Delta t, \Delta x^2)}$$

FTCS : forward time, centered space



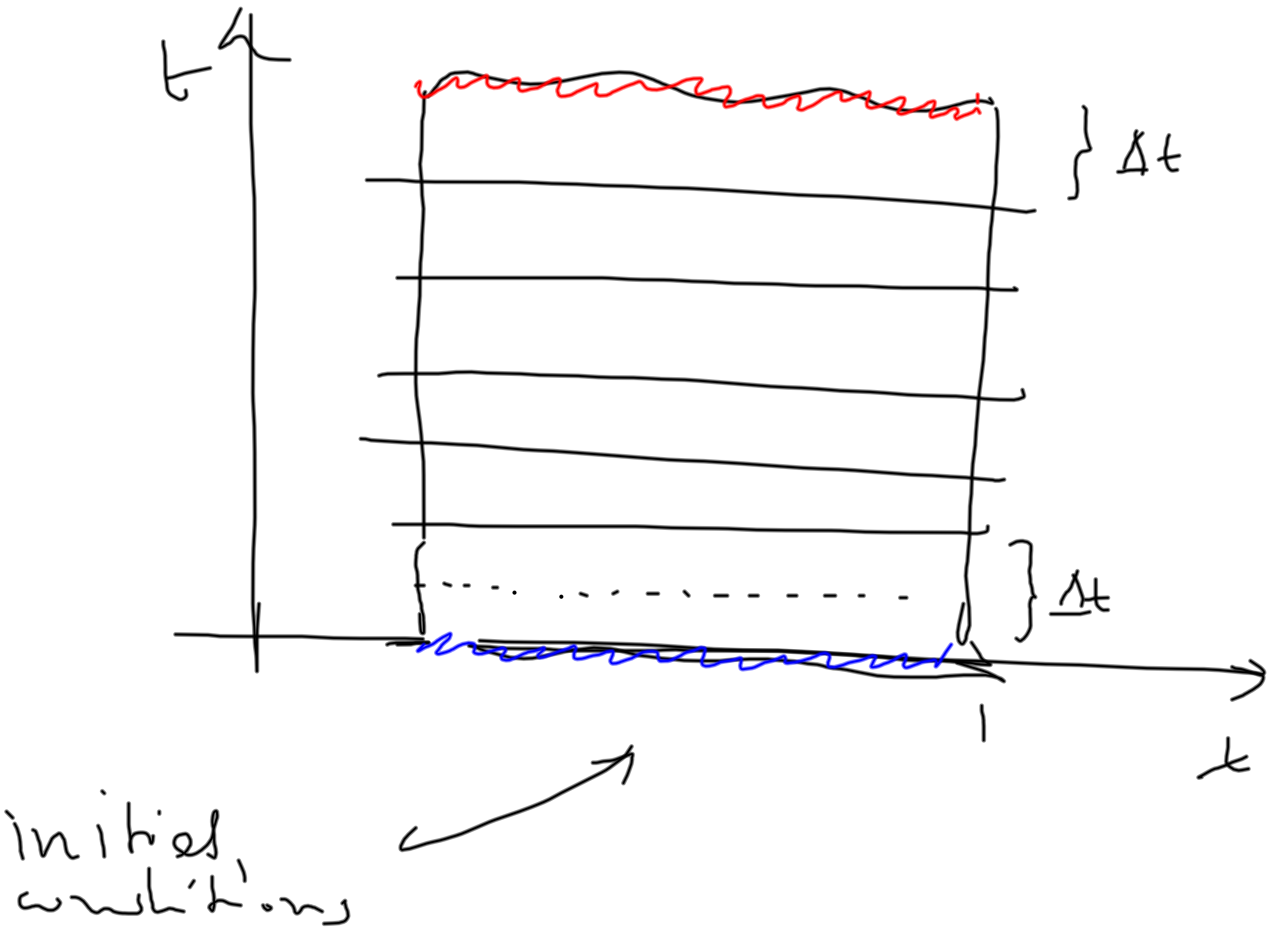
```
UN(:) = ...
do i=1, N

  UNPO(I)= UN(I) + α * (UN(I+1)- UN(I-1))

end do
```




log(ε_T...)
10^-6
∝ Δx^-2
1/Δx

Hyperbolic eqs are IVBP

(initial value boundary problems)



www.aei.mpg.de/~rezzolla

Lecture notes

→ computer code to solve simple 1D - 2D hyperbolic eqs.

Δt has to be chosen so that

$\epsilon_T(\Delta t, \Delta x)$ is small enough. However different algorithms may impose stronger restrictions.

$\Downarrow$ Stability of discretized operators

$$u_j^n = \xi^n e^{ikx_j} \qquad \xi^n \in \mathbb{C}$$

$$u^{n+1} = T(u^n) \quad : \text{change } \underline{\underline{\xi^n}}$$

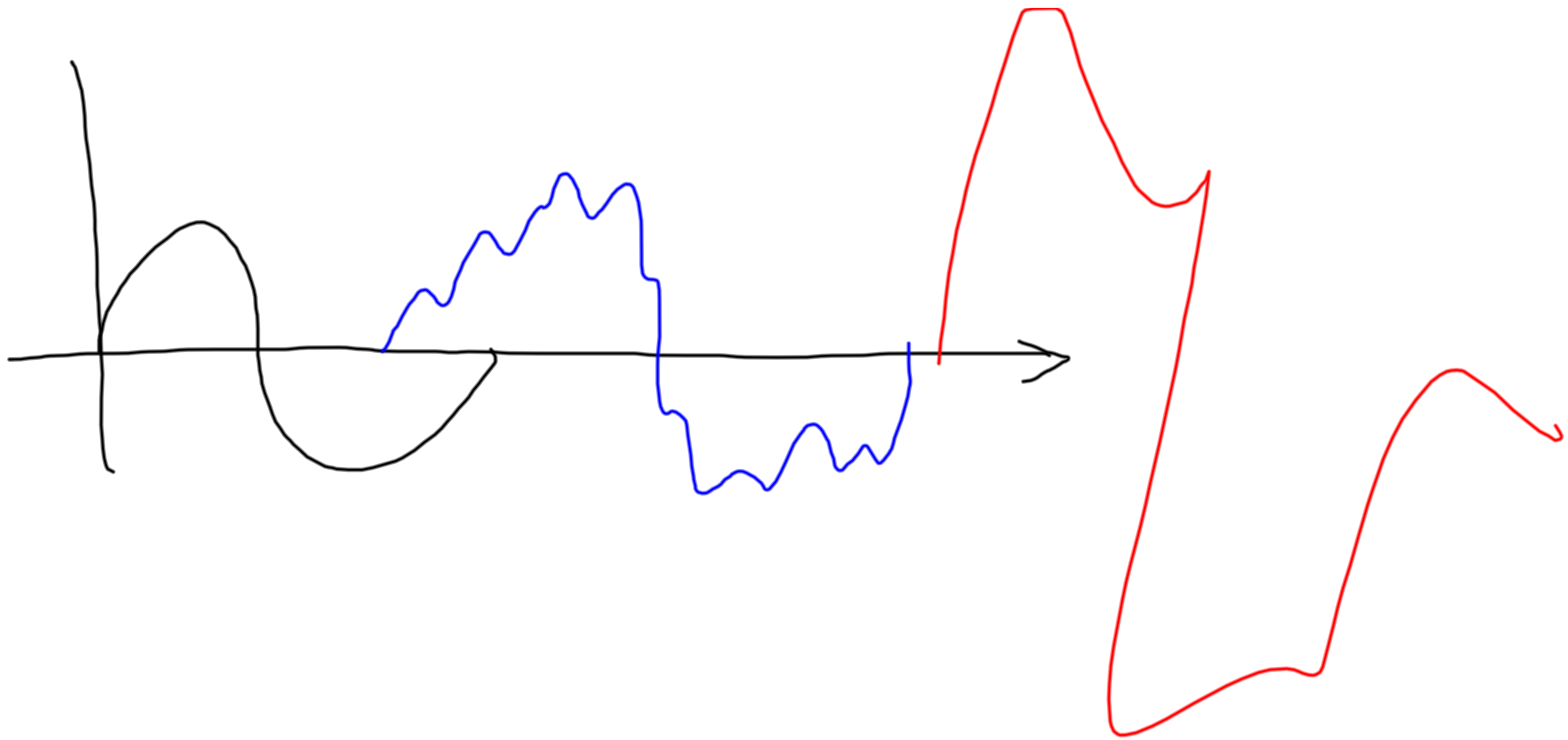
ξ^n : amplification factor

$|\xi|^2 = \xi \xi^* \leq 1$: this implies the solution is bounded

$\Longrightarrow$ each algorithm need to be checked for stability

FTCS $|\xi|^2 = 1 + (\alpha \sin(\Delta x k))^2 > 1$

$\Longrightarrow$ FTCS is <u>unstable</u>!



Never use unstable algorithms!

Upwind

$$|\xi|^2 = 1 - 2|\alpha|\,(1-|\alpha|)\,(1-\cos(k\Delta x))$$

$$\leq 1 \quad \text{iff}$$

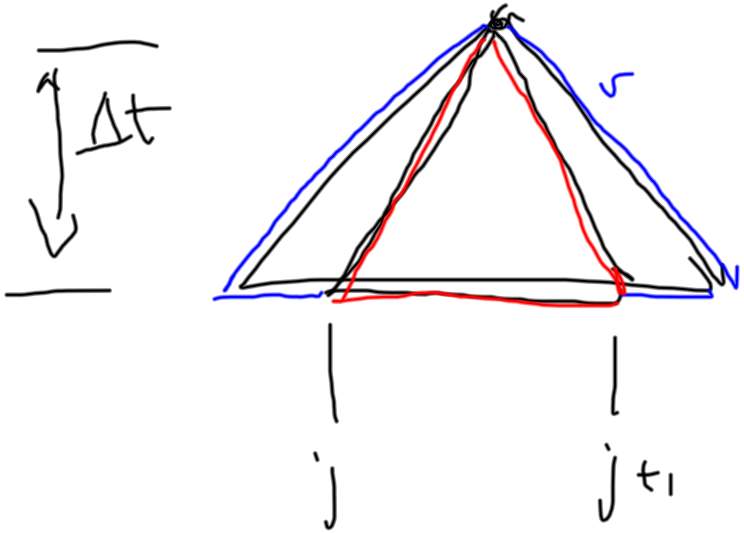
$$\alpha \leq 1$$

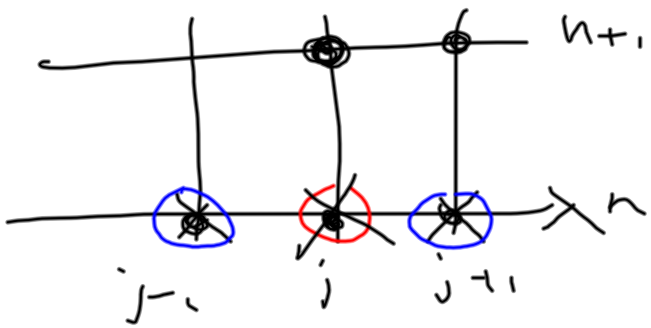
CFL condition

$$v\frac{\Delta t}{\Delta x} \leq 1 \implies \boxed{\Delta t \leq \frac{\Delta x}{v}}$$

$$\Delta t = c_{CFL}\frac{\Delta x}{v}$$

$$\Delta t \leq \frac{\Delta x}{v}$$





$$u_j^{n+1} = u_j^n + \frac{\alpha}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$: FTCS

(*) $$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) + \frac{\alpha}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Lax-Friedrichs

LF : stable!

$$|\xi|^2 = 1 - \sin^2(k\Delta x)(1-\alpha^2) \leq 1$$

if $\alpha \leq 1$

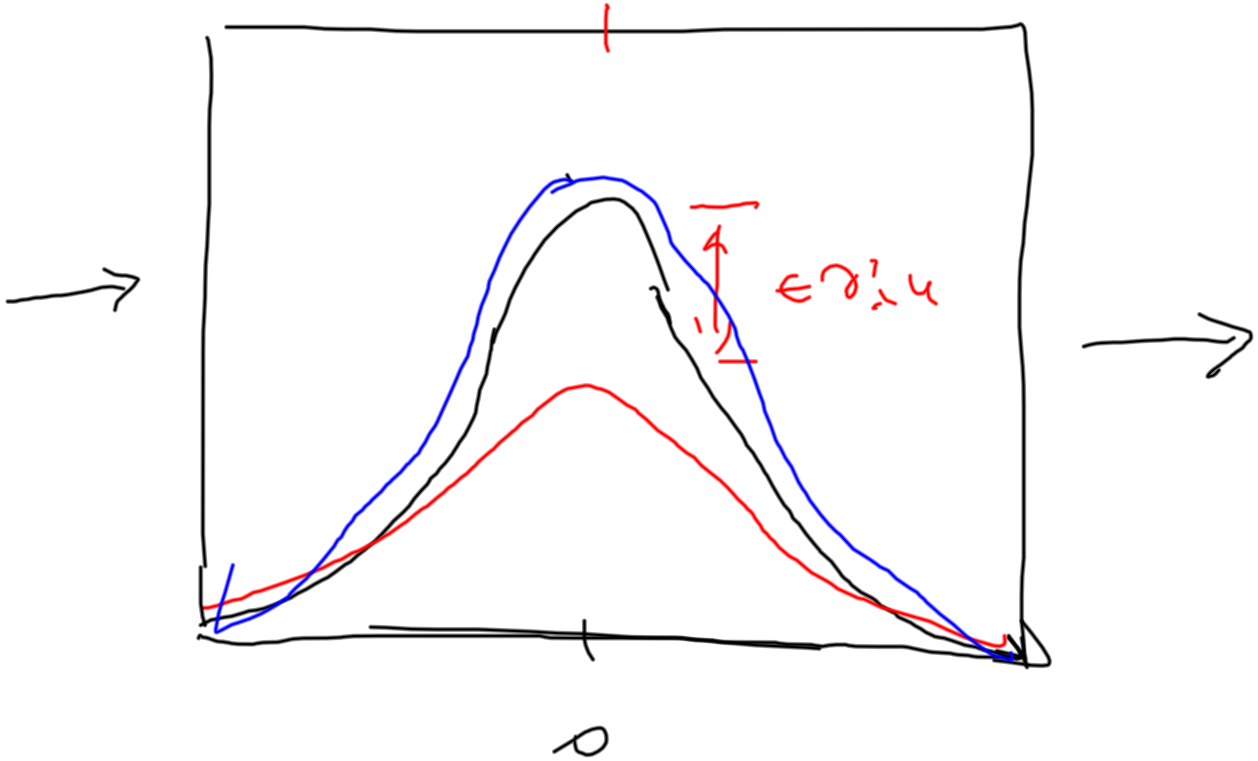
(*) is a 1st order approx. to

$$\partial_t u = v \partial_x \qquad (1)$$

but a 2nd order approx to

$$\boxed{\partial_t u = v \partial_x + \epsilon \partial_x^2 u} \qquad (2)$$

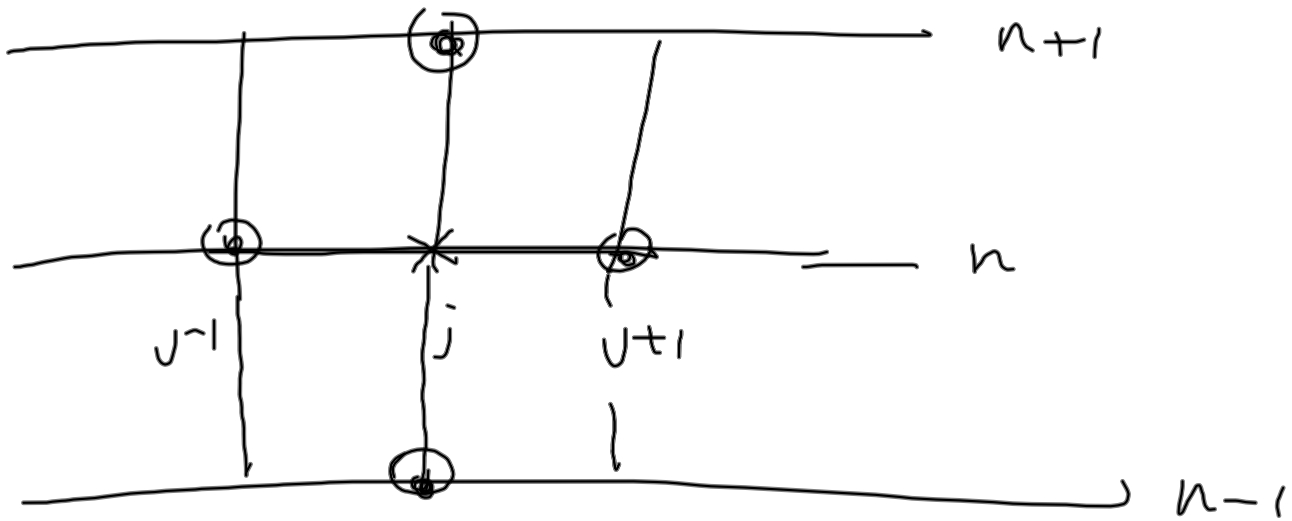
↑ diffusive



$$\epsilon = \epsilon(\Delta x^3)$$

$$\lim_{\Delta x \to 0} (1) = (2)$$

2nd order in space/time



Leap frog

$$\partial_t u \Big|_{n,j} = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}$$

$$u^{n+1}_j = u^{n-1}_j + \alpha \left(u^n_{j+1} - u^n_{j-1} \right) + O(\Delta t^2, \Delta x^2)$$

$$|\xi|^2 = 1$$: has no dissipation

$$\partial_t u = v\, \partial_x u \qquad (1)$$

$$\boxed{\partial_t^2 u = v^2\, \partial_x^2 u} \qquad (4)$$

$$s = \partial_t u$$

$$r = v\, \partial_x u$$

$$\partial_t^2 u = \partial_t (\partial_t u) = \underline{\partial_t s}$$

$$\partial_x^2 u = \partial_x (\partial_x u) = \underline{\partial_x r}$$

$(4) \Leftrightarrow$

$$\begin{cases} \boxed{\partial_t r = v\, \partial_x s} \\ \partial_t s = v\, \partial_x r \\ \partial_t u = s \end{cases}$$

Even a wave equation, ie
a 2nd order PDE can be
easily recast into a 1st order
form

Einstein eqs ~ wave eqs ~
advection eq : we know how
to solve advection eqs

$$\partial_t^2 u = v^2 \partial_x^2 u$$

$$s \equiv \partial_t u$$

$$r \equiv v \partial_x u \implies \partial_t r \stackrel{?}{=} v \partial_t (\partial_x u) = v \partial_x (\partial_t u) = v \partial_x s$$

$$\text{RHS} \iff \partial_t (\partial_t u) = \partial_t s$$

$$\text{LHS} \iff v^2 \partial_x^2 u = v \partial_x (v \partial_x u) = v \partial_x r$$

$v = \text{const}$

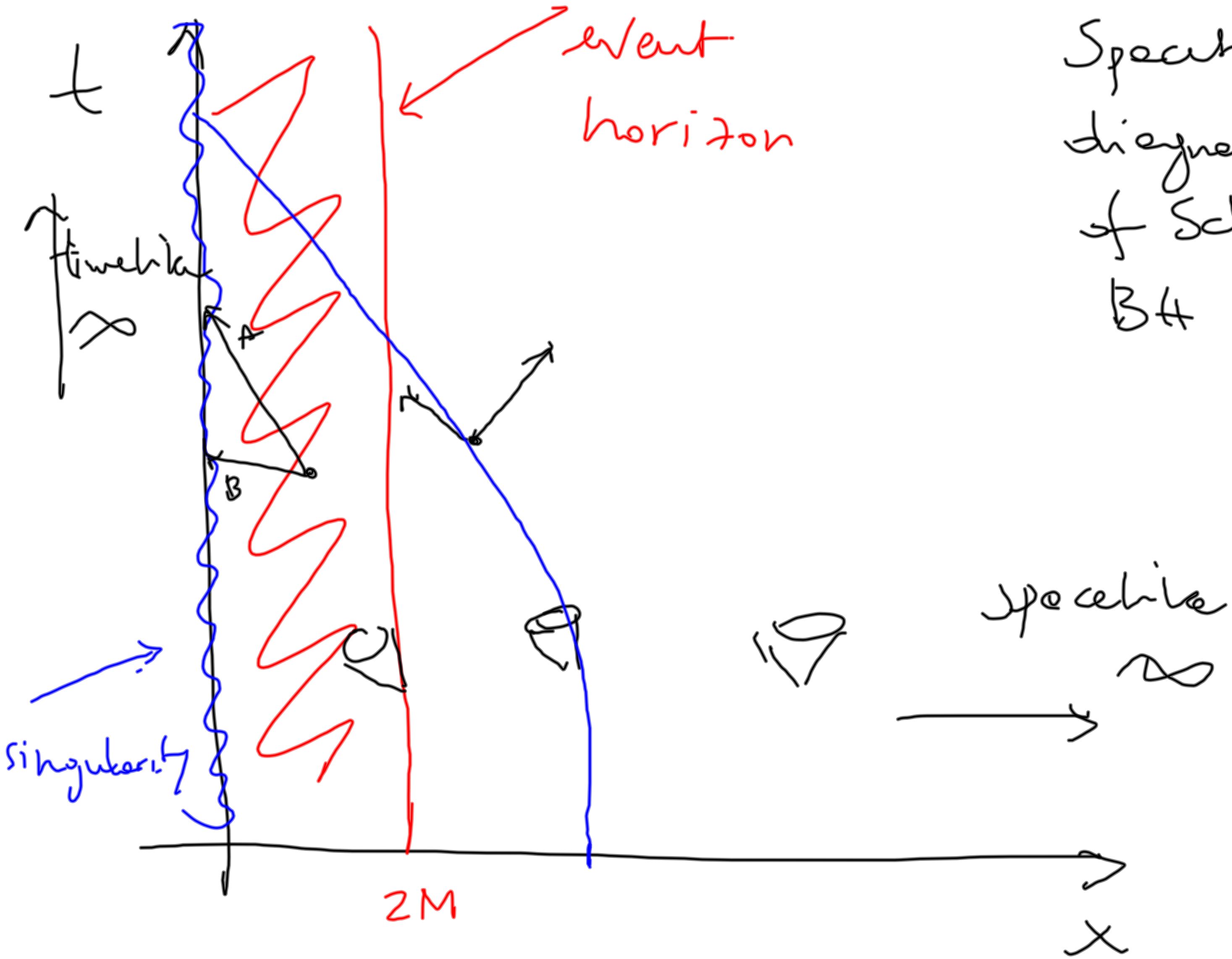
$$\begin{cases} \partial_t s = v \partial_x r \\ \partial_t r = v \partial_x s \\ \partial_t u = s \end{cases}$$

Behind cool movies there is a lot of maths.

That's what we will see these days.

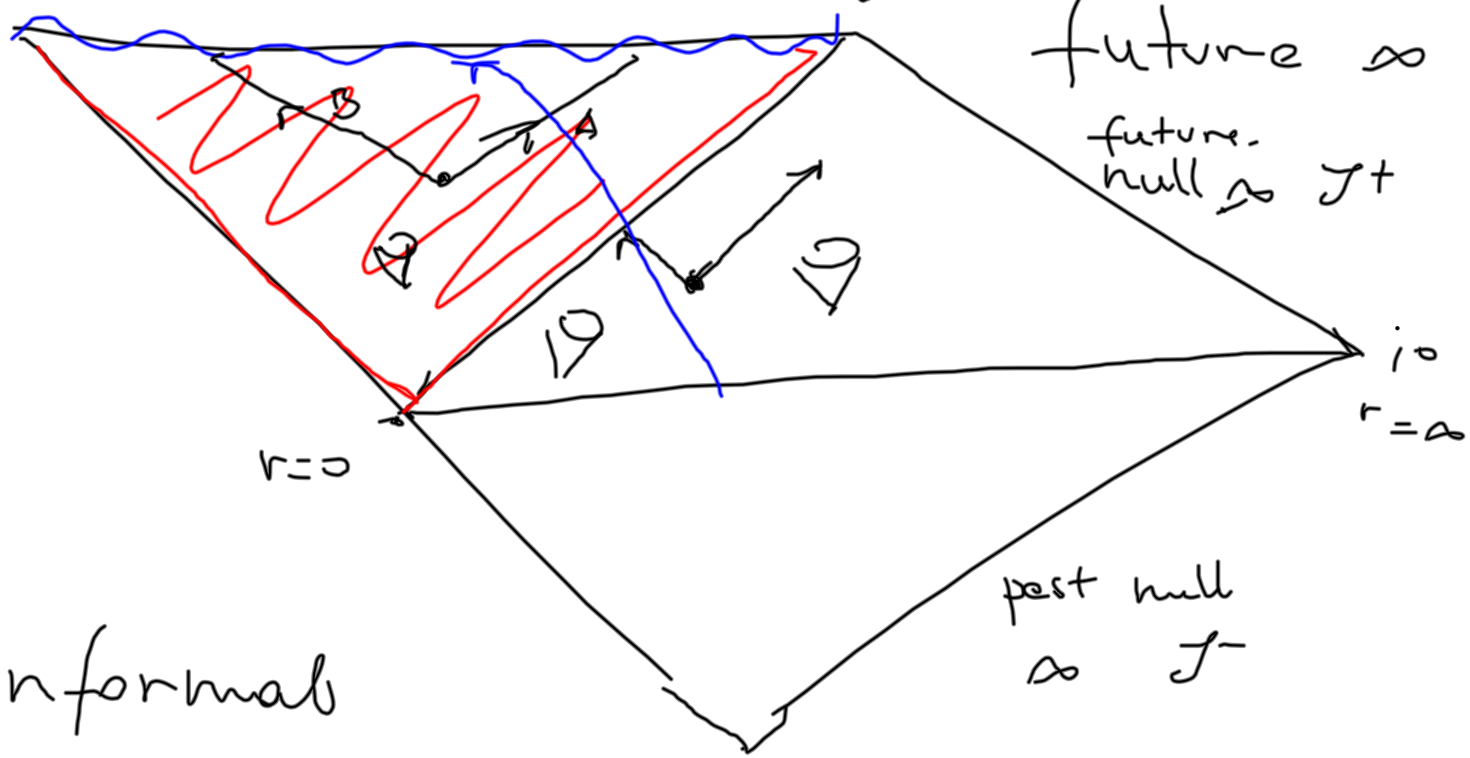
- covering the spacetime
- 3+1 split
- Einstein eqs

Spacetime diagram of Schw. BH



i^+ timelike future ∞

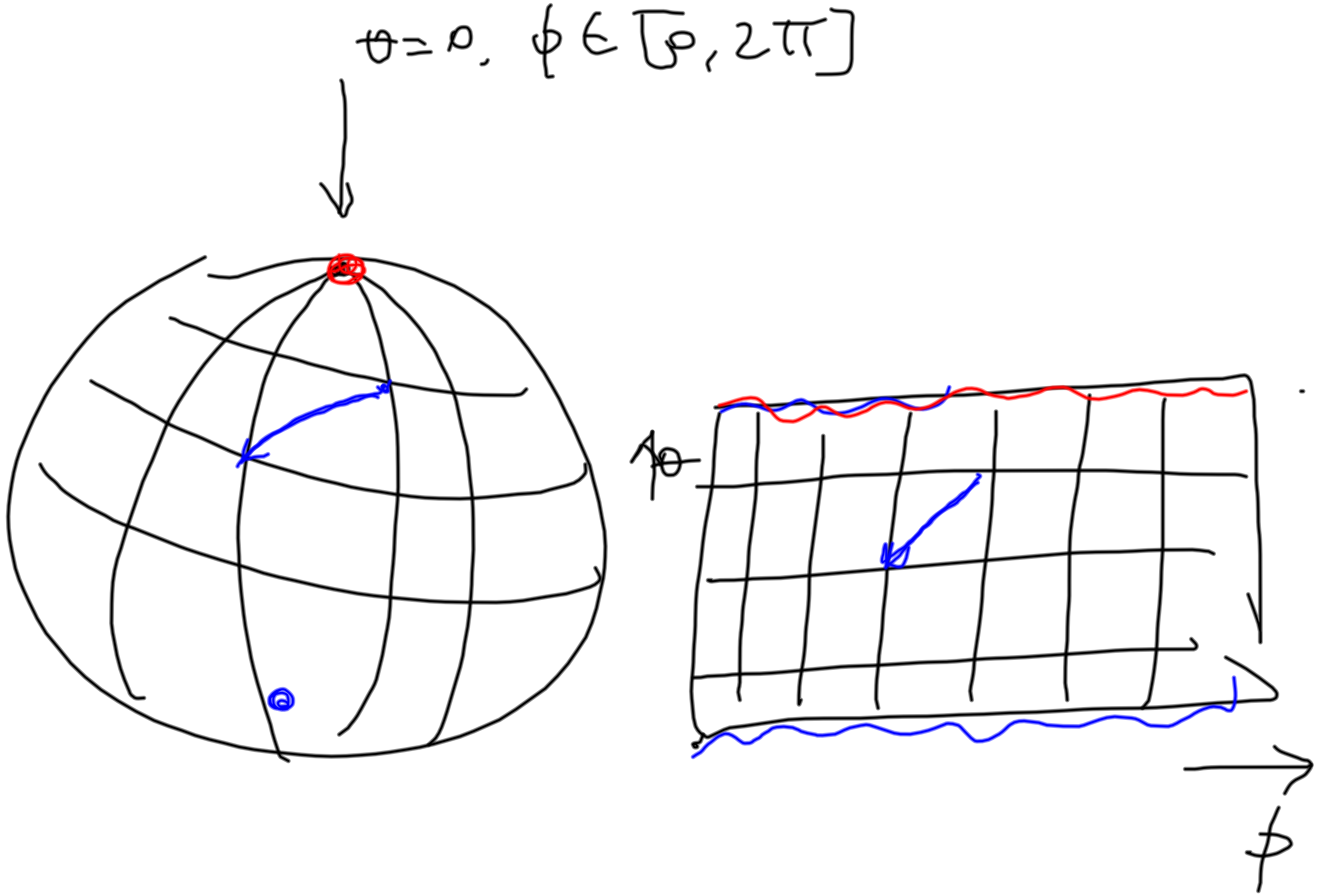
future null ∞ $\mathcal{J}^+$



past null ∞ $\mathcal{J}^-$

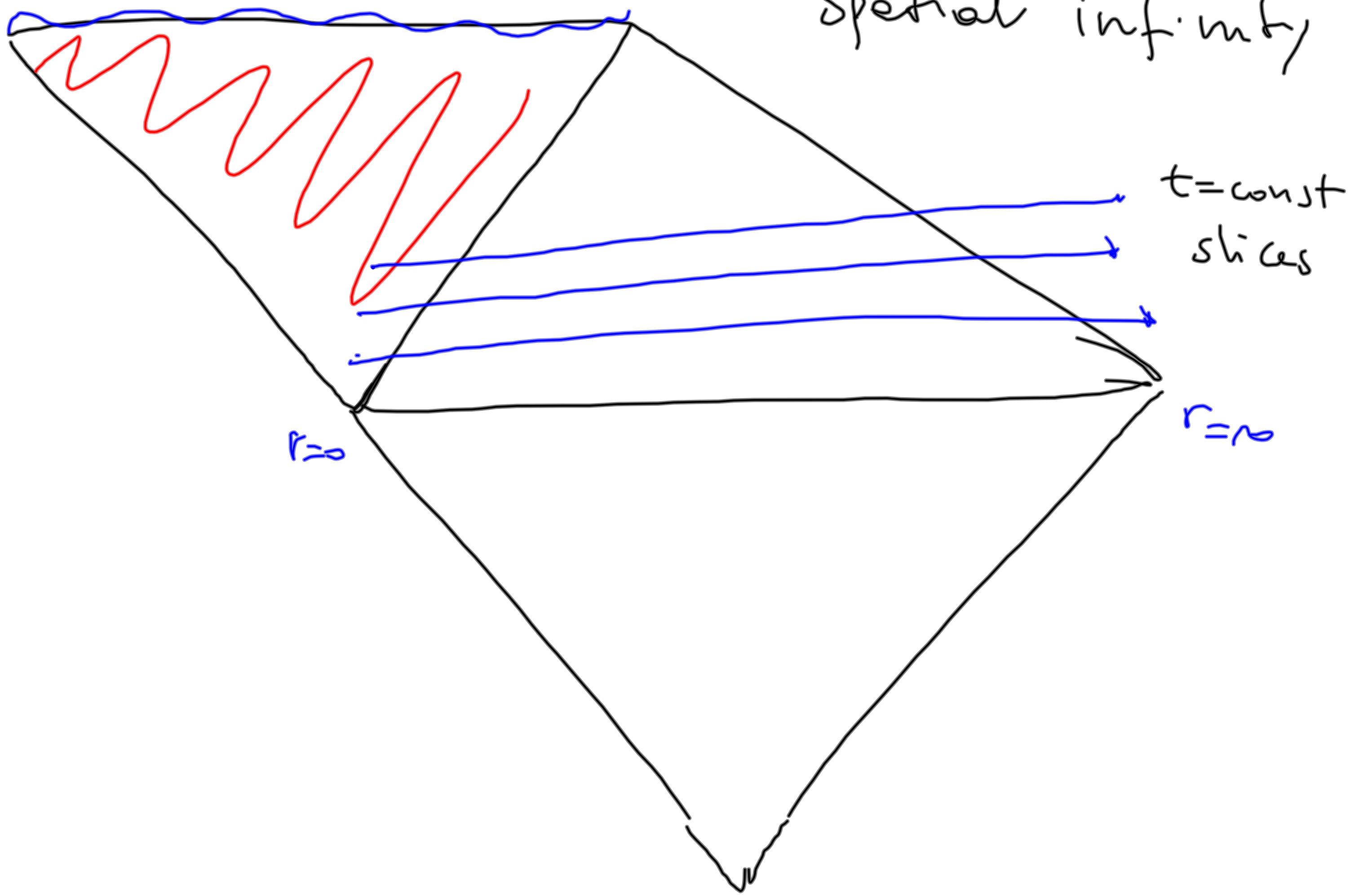
Conformal diagram (Carter-Penrose)

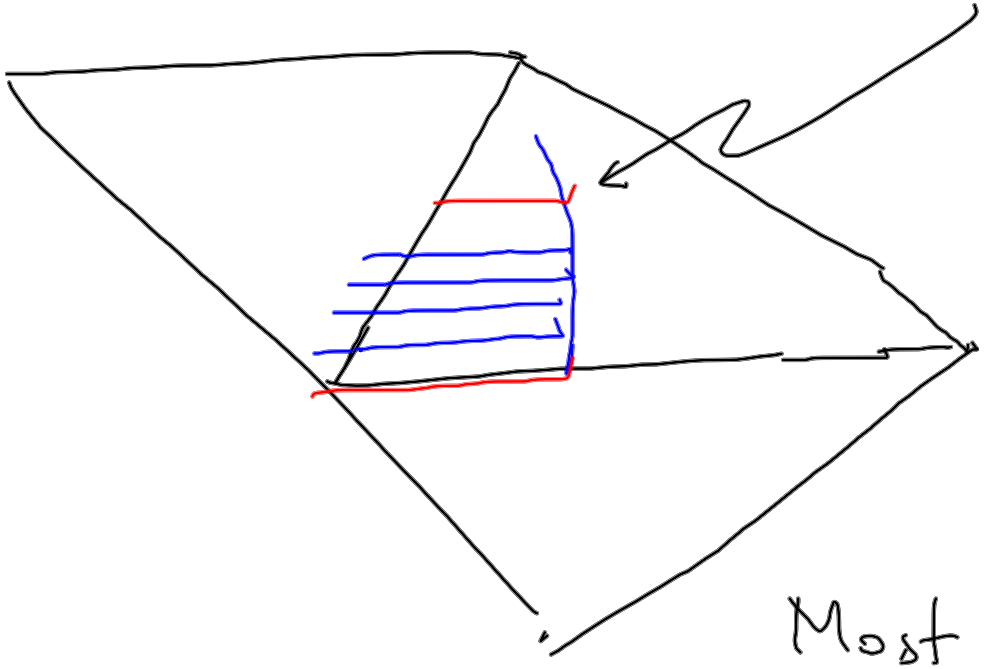
i^- timelike past infinity



θ = 0, φ ∈ [0, 2π]
θ
φ

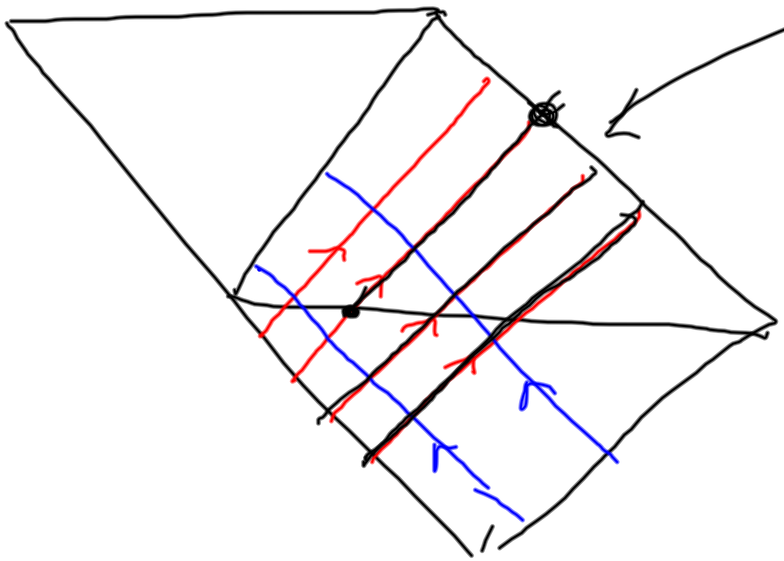
All slices reach spatial infinity



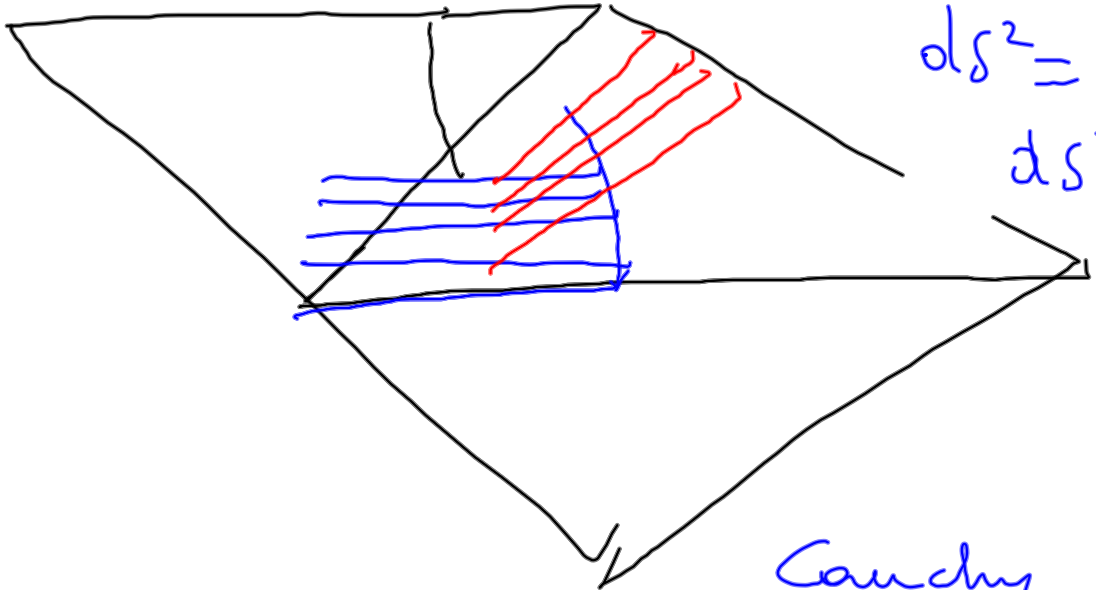


outer boundary is at finite distance

Most standard choice



Null foliation are very powerful to study GWs. However there have many complications (initial data, caustics)



$ds^2 < 0$: time

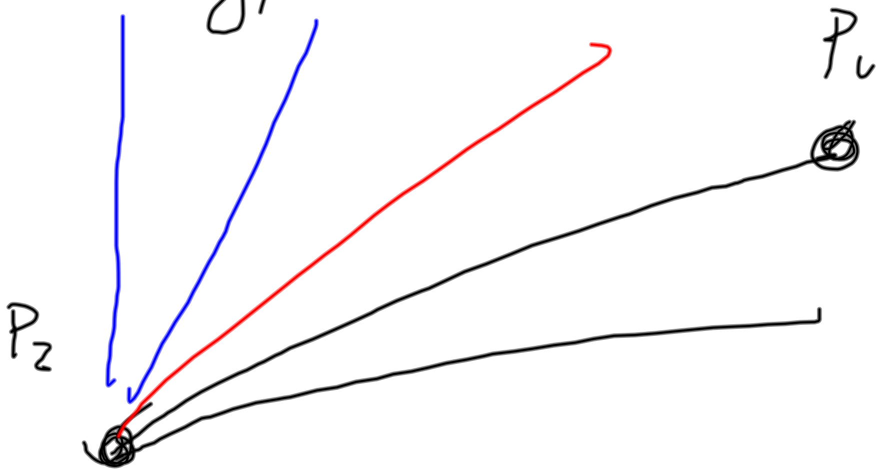
$ds^2 = 0$: null

$ds^2 > 0$: space

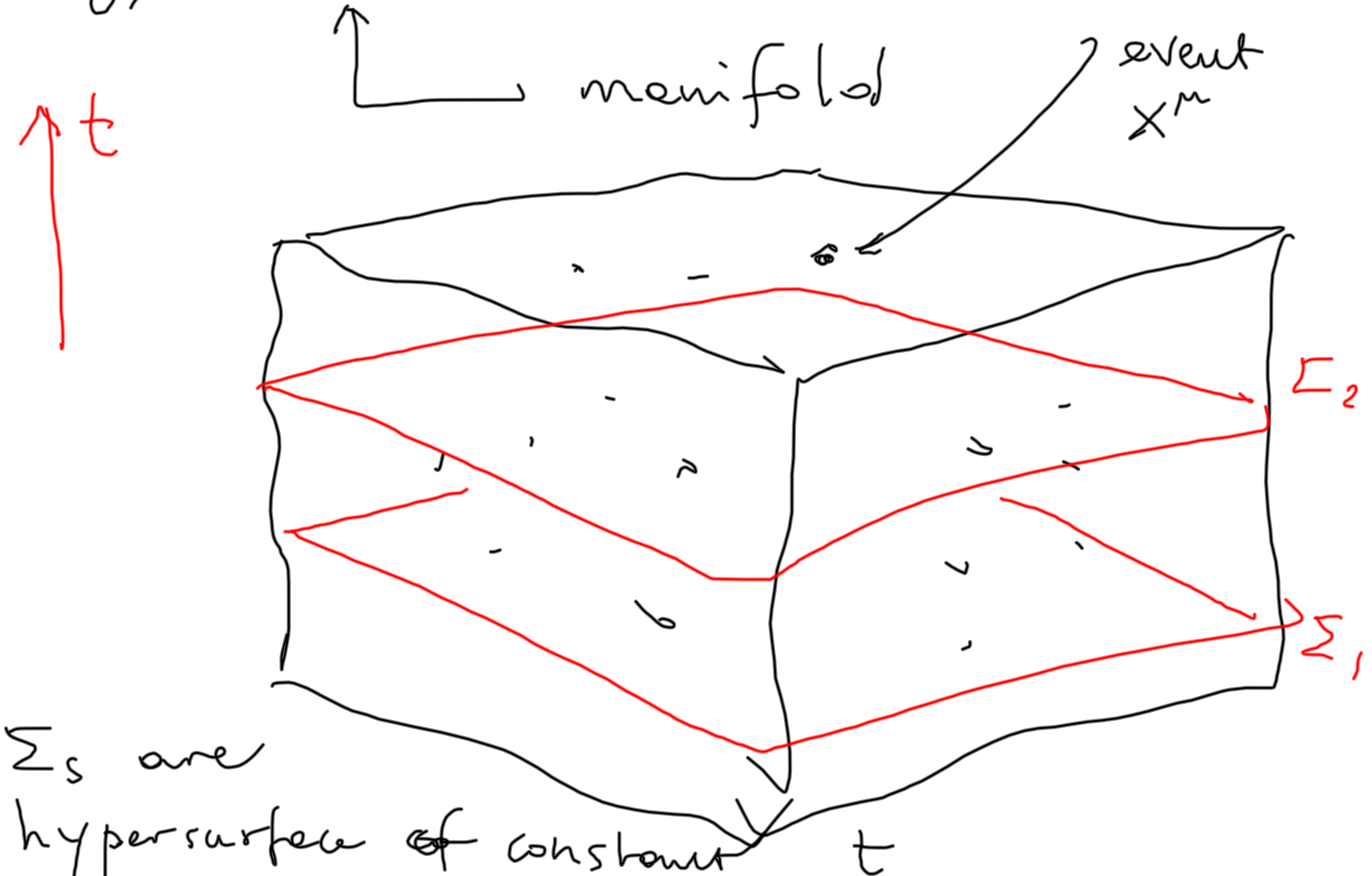
Cauchy characteristic extraction:

combine finite spacelike grid with a null foliation

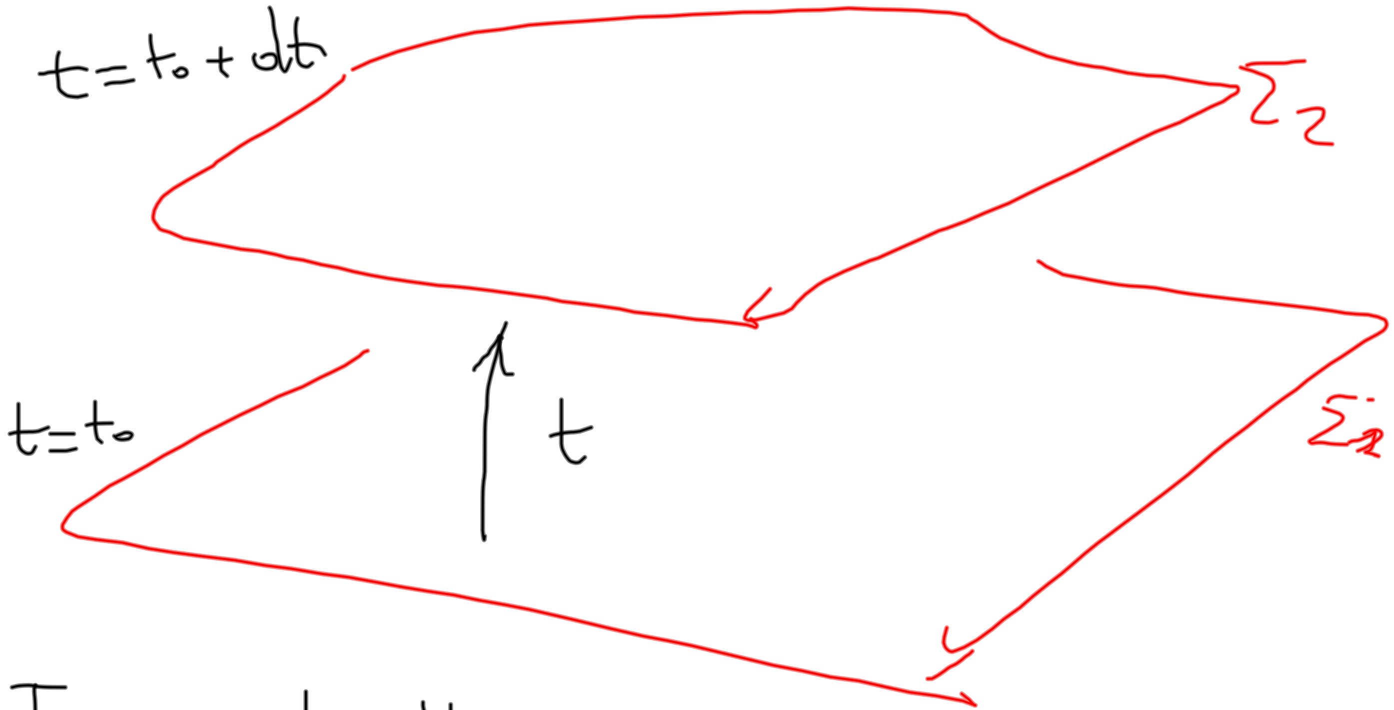
$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$



$(g_{\mu\nu}, \mathcal{M})$



Σs are hypersurface of constant t



I need the normal to Σ

$$\Omega_\mu = \nabla_\mu t$$

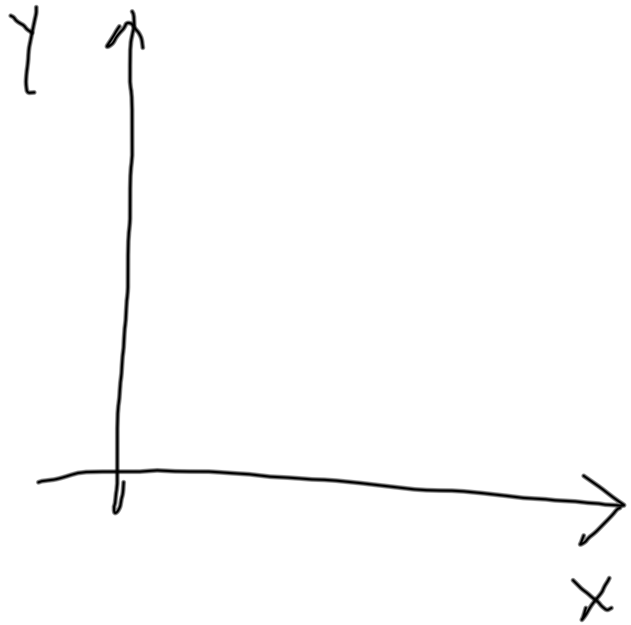
$$|\Omega|^2 = \Omega^\mu \Omega_\mu = g^{\mu\nu} \nabla_\nu \nabla_\mu t \equiv -\alpha^2$$

The lapse defines the normal to Σ

α → lapse and $\alpha > 0$

Now I can build the normal vector to Σ

$$n^\mu \equiv -\alpha g^{\mu\nu} \Omega_\nu = -\alpha g^{\mu\nu} \nabla_\nu t$$

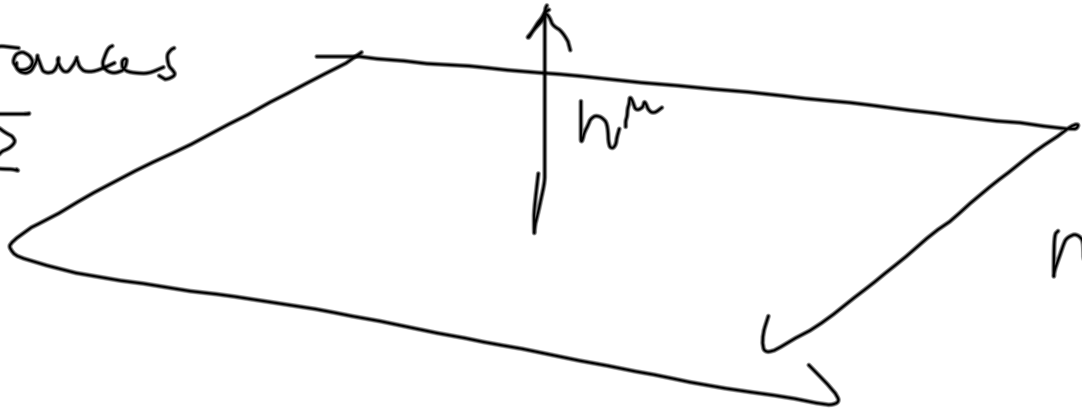


$$\bar{V} = V^x \underline{e}_x + V^y \underline{e}_y$$

$$(e_x)_j = \nabla_j x$$

$$= \{1, 0\}$$

γ measures distances on Σ



$n^\mu n_\mu = -1$

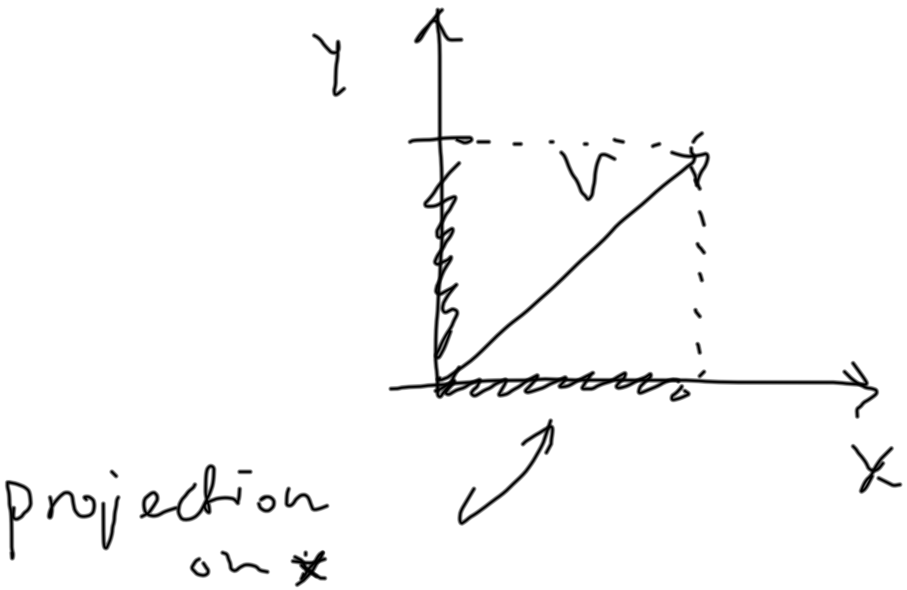
Once I have n^μ I can calculate the spatial metric

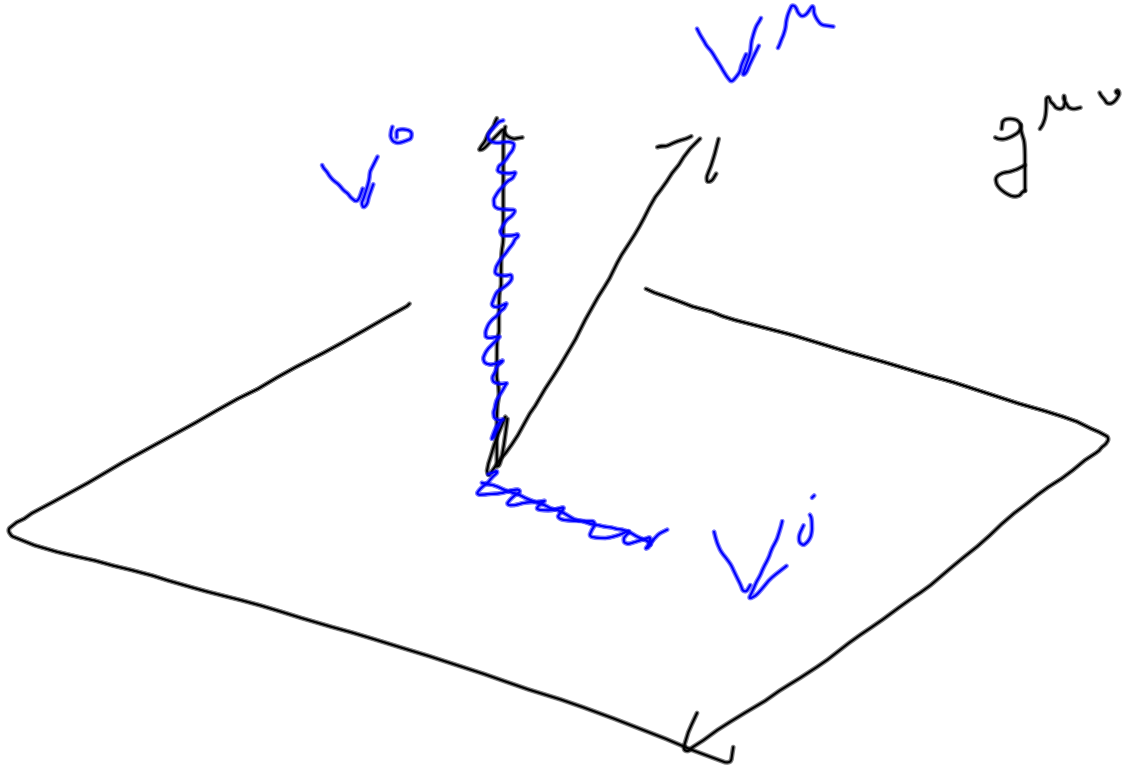
$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$

γ is the spatial metric

With n^μ and $\gamma_{\mu\nu}$ I can build a 3+1 split

I can define a spatial projector and a timelike projector





$$g^{\mu\nu} g_{\mu\alpha} = \delta^\nu_\alpha$$

spatial projector

$$\gamma^\mu{}_\nu = g^{\mu\alpha}\gamma_{\alpha\nu} = g^{\mu\alpha}(g_{\alpha\nu} + n_\alpha n_\nu)$$

$$= \delta^\mu{}_\nu + n^\mu n_\nu$$

Timelike projector

$$N^{\mu}{}_{\nu} = -n^{\mu} n_{\nu} \qquad N^{\mu}{}_{\mu} = 1$$

$$N^{\mu}{}_{\nu} \gamma^{\nu}{}_{\mu} = 0$$

In what we will do we will transform 4-D objects into 3D ones by taking spatial projections

$$\underbrace{D_\gamma T^\beta_{\ \alpha}}_{\text{3D covariant deriv.}} = \gamma^\tau_{\ \alpha}\gamma^\beta_{\ \sigma}\gamma^\rho_{\ \gamma}\underbrace{\nabla_\rho T^\sigma_{\ \tau}}_{\text{4D cov. deriv.}}$$

$$\nabla_\mu g^{\mu\nu} = 0$$

3-metric (spatial) $\quad D_\alpha \gamma^{\alpha\beta} = 0$

is compatible with covariant deriv.

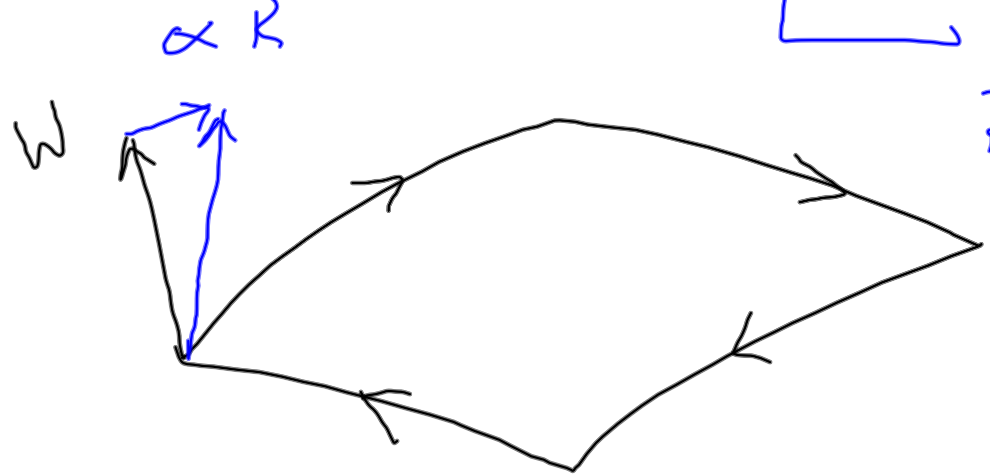
3D Christoffel symbols

$$\Gamma^\alpha_{\beta\delta} = \frac{1}{2} \gamma^{\alpha\mu} \left(\gamma_{\mu\beta,\delta} + \gamma_{\mu\delta,\beta} - \gamma_{\beta\delta,\mu} \right)$$

In the same way you can define the 3D Riemann tensor

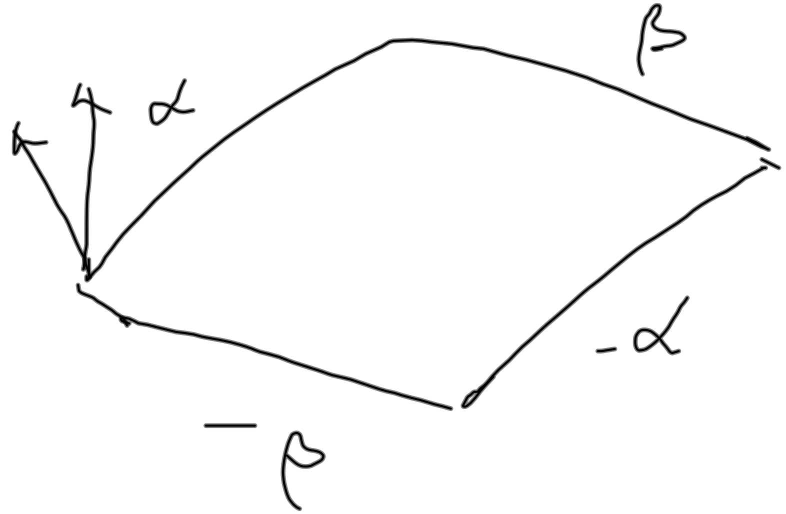
$$2\, D_{[\alpha} D_{\beta]} W_\delta = R^{\mu}{}_{\alpha\beta\delta}\, W_\mu$$

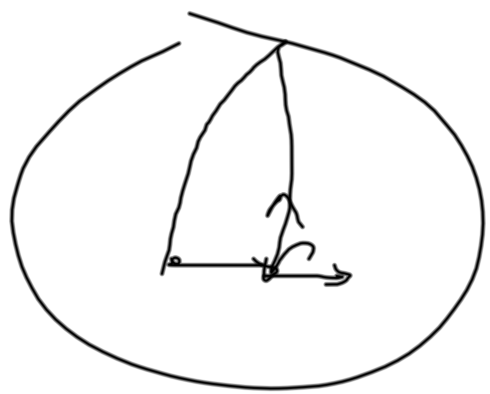
3D Riemann



$$D_\alpha \underbrace{\left(\underbrace{D_\beta W_\mu}\right)} =$$

$$D_{[\alpha} D_{\beta]} W_\mu = D_\alpha D_\beta W_\mu - D_\beta D_\alpha W_\mu$$





$$R^{\mu}{}_{\delta\alpha\beta}\, n_{\mu} = 0$$

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\alpha}{}_{\mu\gamma}$$

$$- \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\mu\delta}$$

$$R_{\alpha\beta} = R^{\delta}{}_{\alpha\delta\beta} \quad ; \quad R = R^{\alpha}{}_{\alpha}$$

$R^{\mu}{}_{\alpha\beta\gamma}$: 3D Riem.

$^{(4)}R^{\mu}{}_{\alpha\beta\gamma}$: 4D Riem.

intrinsic curvature on Σ

curvature in 4D

The missing information is in the

extrinsic curvature $K_{\mu\nu}$

There are a no. of different ways to compute $K_{\mu\nu}$

$$K_{\alpha\beta} = - \gamma_{\alpha}{}^{\mu} \gamma_{\beta}{}^{\nu} \nabla_{(\mu} n_{\nu)} \qquad (1)$$

$$2 T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}$$

$$2 T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$$

in terms of the acceleration of normal observers

$$\tilde{a}^\mu = u^\nu \nabla_\nu u^\mu$$: fluid acceler.

(u fluid 4-vel)

$$a^\mu = n^\nu \nabla_\nu n^\mu$$

$$K_{\alpha\beta} = - \nabla_\alpha n_\beta - n_\alpha a_\beta$$

$$K_{\alpha\beta} = -\frac{1}{2} \mathcal{L}_n \gamma_{\alpha\beta}$$

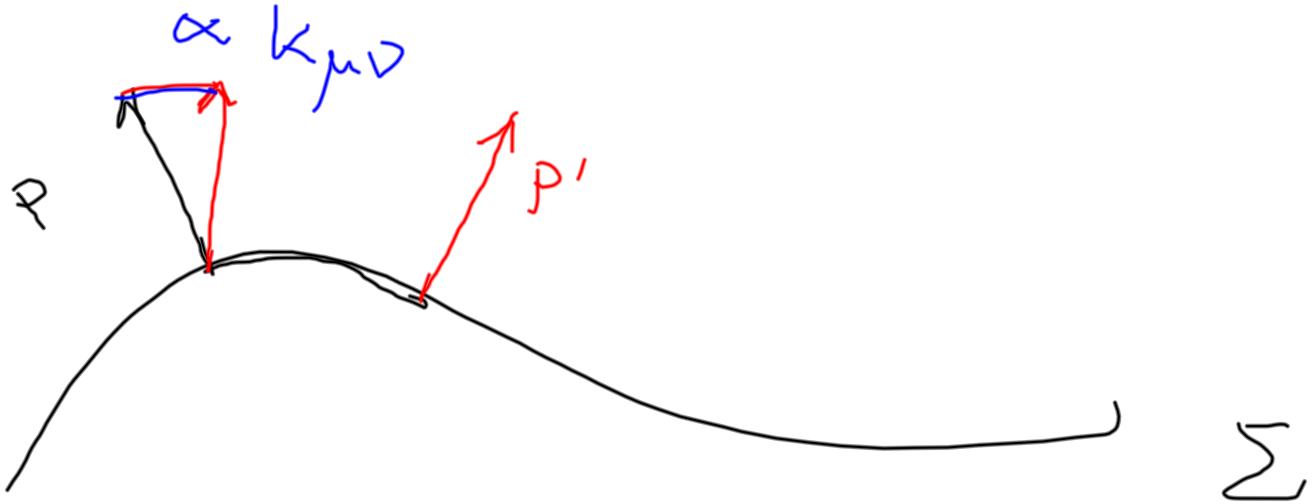
$\mathcal{L}_n$: Lie derivative along n

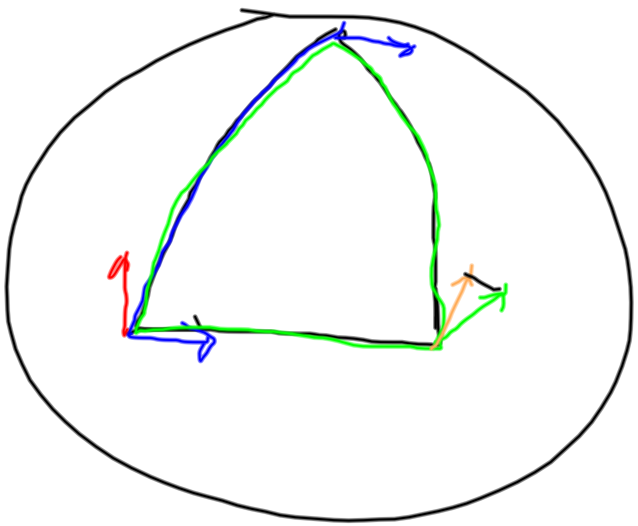
V^μ directional derivative along W^μ

$$Z^\mu = W^\nu \nabla_\nu V^\mu$$

$$\mathcal{L}_X \phi = X^\mu \nabla_\mu \phi = X^\mu \partial_\mu \phi$$

$$\mathcal{L}_X V^\nu = X^\mu \nabla_\mu V^\nu - V^\mu \nabla_\mu X^\nu$$





Intrinsic curvature tells me the difference between red/blue

Extrinsic curveture tells me the difference between orange/green

RECAP

- Take 4D spacetime $g_{\mu\nu}$
- introduce time coord t
- " normal to $t=$const surf.

$$n^\mu = -\alpha g^{\mu\nu} \nabla_\nu t \qquad n^\mu n_\mu = -1$$

- Define spatial metric

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

construct 3D objects by projecting with $\gamma^{\mu\nu}$ and $N^{\mu}{}_{\nu} = -n^{\mu} n_{\nu}$

Eg

$$\underbrace{D_\alpha V^\beta}_{\text{3D spatial covariant deriv.}} = \gamma^{\beta}_{\nu} \gamma^{\mu}_{\alpha} \underbrace{\nabla_\mu V^\nu}_{\text{cov. 4-Deriv.}}$$

$$R^{\alpha}{}_{\mu\nu\beta} \longleftarrow {}^{(4)}R^{\alpha}{}_{\mu\nu\beta}$$

$$K_{\mu\nu} = -\gamma^{\alpha}{}_{\mu}\,\gamma^{\beta}{}_{\nu}\,\nabla_{(\alpha}\, n_{\beta)}$$

Extrinsic curvature

Properties

1.) Spatial tensor $K_{\mu\nu} =$ 

2.) sym. tensor $K_{\mu\nu} = K_{\nu\mu}$

3) $K = K^{\mu}{}_{\mu} = -\nabla^{\mu} n_{\mu}$

$$\nabla_{\mu} G^{\mu\nu} = 0 = \nabla_{\mu} T^{\mu\nu}$$

$$^{(4)}G^{\mu\nu} \rightleftharpoons 8\pi \, ^{(4)}T^{\mu\nu}$$

$$\underbrace{^{(4)}R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \, ^{(4)}R}_{\text{curvature}} = \underbrace{8\pi \, ^{(4)}T^{\mu\nu}}_{\text{energy}}$$

$$\propto \Gamma^{\alpha}_{\mu\nu,\beta} \propto \partial_x^2 g_{01}$$

$$= \overbrace{\gamma^{\mu}_{\alpha}\, \gamma^{\nu}_{\beta}\, \gamma^{\rho}_{\delta}\, \gamma^{\sigma}_{\gamma}}^{S}\; {}^{(4)}R_{\mu\nu\rho\sigma}$$

$$R_{\alpha\beta\delta\gamma} + K_{\alpha\beta} K_{\delta\gamma} - K_{\alpha\delta} K_{\beta\gamma}$$

<u>Gauss eqs</u>

$$= \overbrace{\gamma^{\rho}{}_{\beta}\,\gamma^{\mu}{}_{\alpha}\,\gamma^{\nu}{}_{\gamma}}^{s}\,\overbrace{n^{\sigma}}^{t}\;{}^{(4)}R_{\rho\mu\nu\sigma}$$

$$D_\alpha K_{\beta\gamma} - D_\beta K_{\alpha\gamma}$$

Codazzi eqs

Ricci eqs

$$\overbrace{\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}}^{s}\overbrace{n^{\delta}n^{\gamma}}^{t}\;{}^{(4)}R_{\gamma\delta\mu\nu}$$

$$= \mathcal{L}_n K_{\alpha\beta} - \frac{1}{\alpha} D_\alpha D_\beta \alpha - K^{\gamma}_{\beta} K_{\alpha\gamma}$$

(α ← lapse function)

$$n^\mu = -\alpha g^{\mu\nu} \nabla_\nu t$$

$U^\mu n_\mu = 0$: U is spatial $\quad D_\mu U^\nu = \gamma^{\rho}_{\mu} \nabla_\rho U^\nu +$

$K_{\mu e} \, v^{e} n^{\nu}$ $\square$

Gauss (A), Codazzi (B) and Ricci (C) eqs give 3 different splitting of the Riemann tenso

A: $\gamma \gamma \gamma \gamma \, {}^{(4)}R$

B: $\gamma \gamma \gamma n \, {}^{(4)}R$

C: $\gamma \gamma n n \, {}^{(4)}R$

$$T^{\mu\nu} \equiv (e+p)\, u^\mu u^\nu + p\, g^{\mu\nu}$$
$$= h\rho\, u^\mu u^\nu + p\, g^{\mu\nu}$$

u^μ : fluid four-velocity

$u^\mu u_\mu = -1$: timelike unit vector

$u^\mu n_\mu = +1$: // 4-vect

e : energy density

p : pressure

h : specific enthalpy $\quad h \equiv \dfrac{e+p}{\rho}$

ρ: rest mass density

$$m_B\, n = \rho$$

(n ← no density)

$$e = \rho(1+\epsilon) = \text{rest mass + inter. eng.}$$

(ϵ ← specific internal energy)

$$e = n^\mu n^\nu T_{\mu\nu}$$

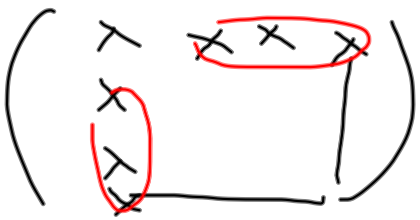
$$j_\alpha = -\gamma^\mu_\alpha n^\nu T_{\mu\nu}$$

↑ momentum density

We have derived all the relevant elements to proceed to a 3+1 splitting of the Einstein eqs.

Anticipate the results

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$



10 eqs

10 eqs : 2nd order PDEs

12 evolution eqs 1st order

$\Longleftrightarrow$ 6 evolution eqs 2nd order

$$\partial_t^2 u = v^2 \partial_x^2 u \longrightarrow \begin{aligned} \partial_t r &= v\, \partial_x s \\ \partial_t s &= v\, \partial_x r \end{aligned}$$

4 constraint eqs are not evolution

$$\partial_t \bar{E} = \cdots$$

$$\partial_t \bar{B} =$$

} evolution eqs

$$\nabla \cdot \bar{E} = 4\pi \rho_e$$

$$\nabla \cdot \bar{B} = 0$$

} constraint eqs

$$\Box E = 0$$

$$2\, n^\mu n^\nu\, {}^{(4)}G_{\mu\nu} = R + K^2 + K_{\mu\nu} K^{\mu\nu}$$

Hamiltonian constraint (no time derivs)

$$= 2e \cdot 8\pi = 16\pi e$$

$$\boxed{R + K^2 + K_{\mu\nu} K^{\mu\nu} = 16\pi e} \quad 1\text{ eq}$$

$$\nabla^2 \phi = 4\pi \rho$$

$$-\gamma^{\mu}_{\alpha} n^{\nu} \, {}^{(4)}G_{\mu\nu} = -R_{\alpha\nu} n^{\nu} + \frac{1}{2} n_{\alpha} R$$

$$= -D_{\mu} K^{\mu}_{\alpha} + D_{\alpha} K$$

$$= -8\pi j_{\alpha}$$

$$\boxed{D_{\mu} K^{\mu}{}_{\alpha} - D_{\alpha} K = 8\pi j_{\alpha}}$$

Momentum constraint eq 3 eqs

$$\Omega_\mu = \nabla_\mu t$$

$$n^\mu = -\alpha\, g^{\mu\nu} \nabla_\mu t$$

So far we have only defined the normal n to Σ but n is not the natural direction of time evolution (n is not dual to Ω)

$$n^\mu \Omega_\mu = \frac{1}{\alpha} \neq 1$$

We need a different 4-vector which is still time like

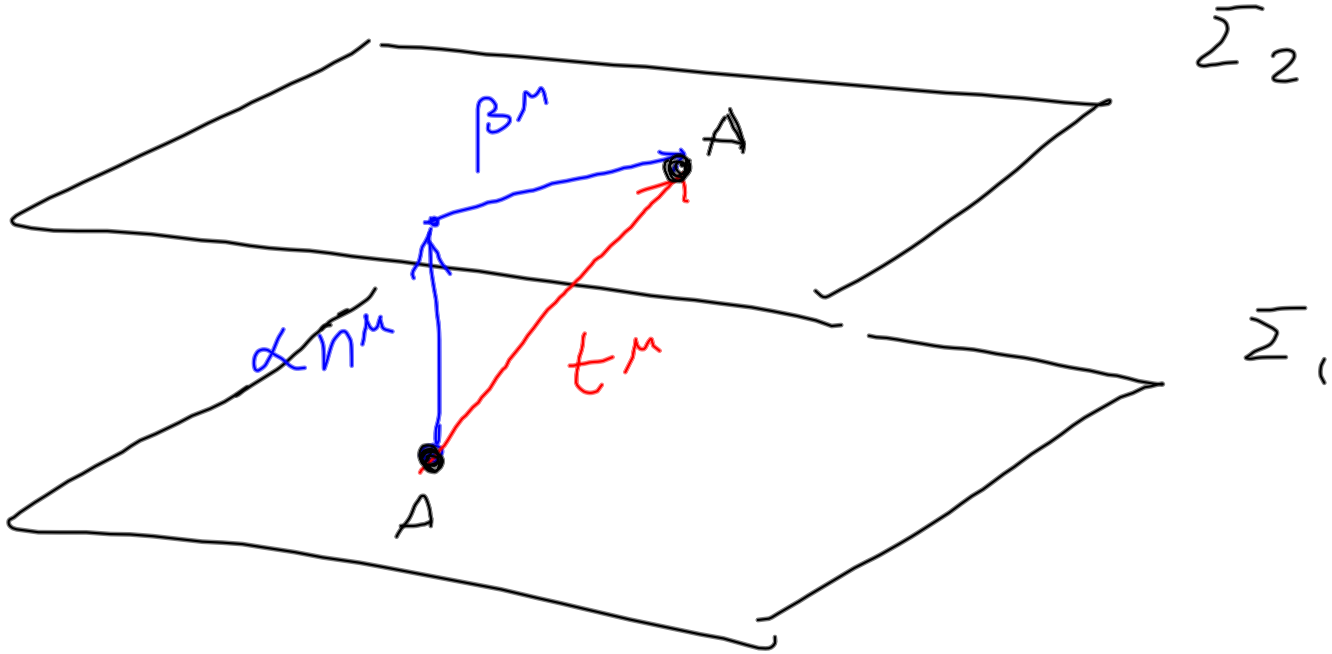
$$t^\mu = \alpha n^\mu + \beta^\mu$$

β^μ : is purely spatial

$$t^\mu \Omega_\mu = (\alpha n^\mu + \beta^\mu) \Omega_\mu$$

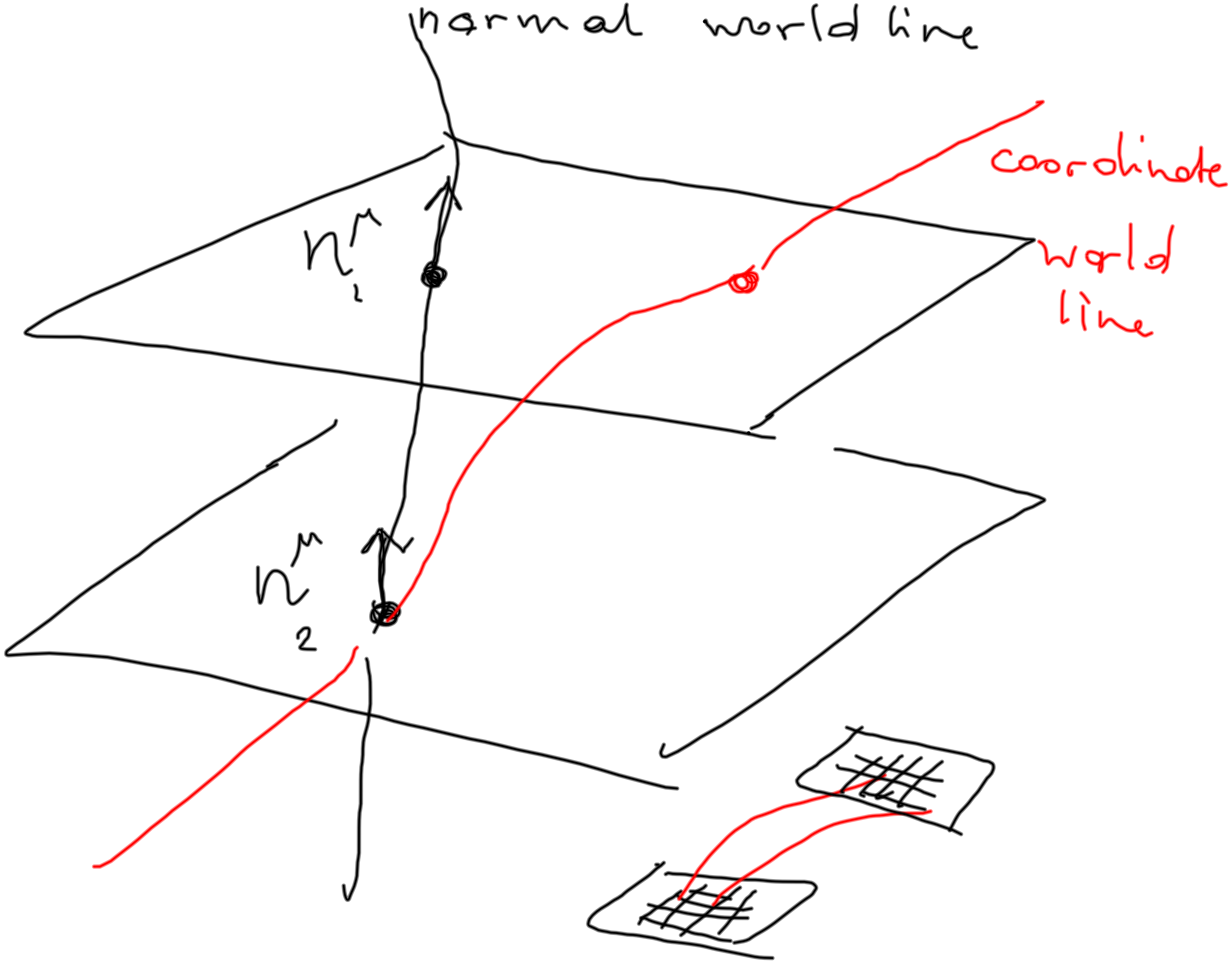
$$= \alpha n^\mu \Omega_\mu + \cancel{\beta^\mu \Omega_\mu}$$

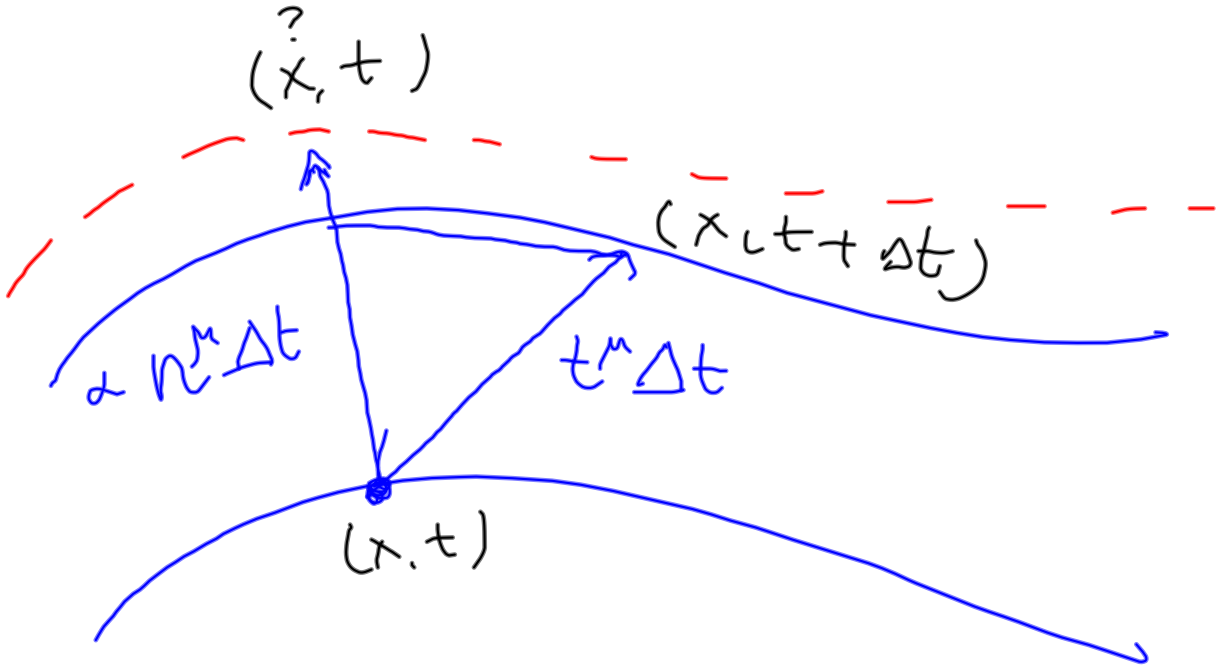
$$= 1$$



t^μ is dual to Ω_μ

t


normal world line
coordinate world line
n_1^μ
n_2^μ



$$n^\mu \Omega_\mu \neq 1$$

$$t^\mu \Omega_\mu = 1$$

Recall one def. of ext. curv.

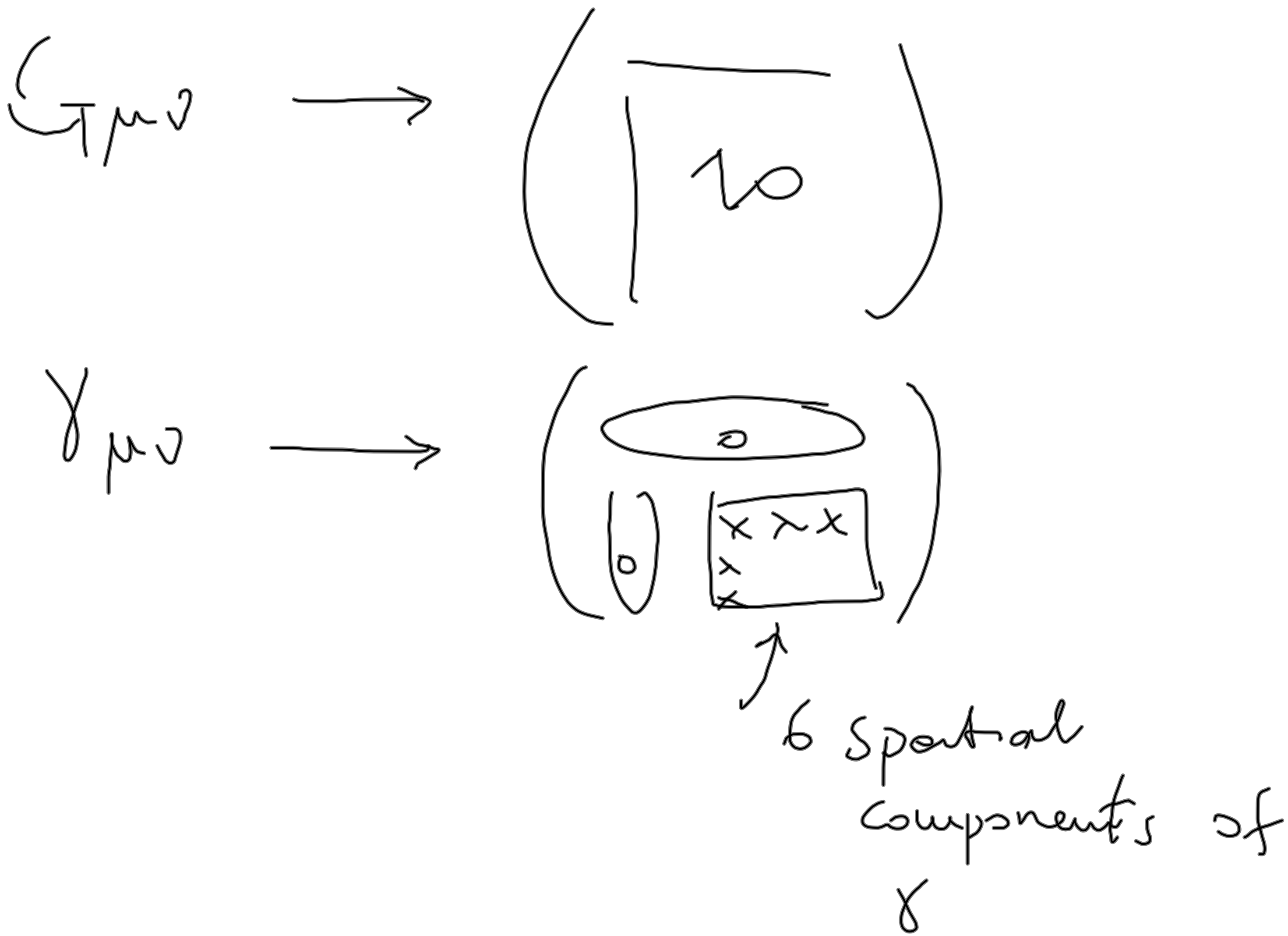
$$K_{\alpha\beta} = -\frac{1}{2} \mathcal{L}_n \gamma_{\alpha\beta}$$

$$t^\mu = \alpha n^\mu + \beta^\mu \Rightarrow \alpha \mathcal{L}_n = \mathcal{L}_t - \mathcal{L}_\beta$$

$$\Rightarrow \boxed{\mathcal{L}_t \gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta \gamma_{\mu\nu}}$$

definition of ext. curvature

but gives evolution eq for $\gamma_{\mu\nu}$


$G_{\mu\nu}$
10
$\gamma_{\mu\nu}$
0
0
x x x
x
x
6 spatial
components of
γ

$$n^\mu n^\nu \, {}^{(4)}G_{\mu\nu} \Longrightarrow \text{elliptic eq}$$

$$\gamma^\mu_\alpha \gamma^\nu_\beta \, {}^{(4)}G_{\mu\nu} = 8\pi S_{\alpha\beta} = 8\pi \, \gamma^\mu_\alpha \gamma^\nu_\beta T_{\mu\nu}$$

Using Ricci eq

$$\mathcal{L}_t K_{\mu\nu} = D_\mu D_\nu \alpha + \alpha (\cdots)$$

$$\cdots + \mathcal{L}_\beta K_{\mu\nu}$$

6 evol. eqs for $K_{\mu\nu}$

$n n \, {}^{(4)}G \longrightarrow$ Ham. eq

$\gamma n \, {}^{(4)}G \longrightarrow$ mom. const

4

$\gamma \gamma \, {}^{(4)}G \longrightarrow$

$$\mathcal{L}_t K_{\mu\nu} = D_\mu D_\nu \alpha + \cdots$$

$$\mathcal{L}_t \gamma_{\mu\nu} = -\alpha K_{\mu\nu} + \cdots$$

$6 + 6 = 12$

Arnowitt

Deser

Misner eqs

Chosing a proper coordinate basis to make the eqs more transparent

$$\mathcal{L}_{\underline{t}} \longrightarrow \partial_t$$

We want a set of unit vectors $(e_\nu)^\mu$ which highlight the spatial nature of $\gamma_{\mu\nu}$, $K_{\mu\nu}$

Requirements:

$(e_j)^\mu$: have to be spatial

$$\{e_1\}^\mu = \{0, 1, 0, 0\}$$

$$\vdots$$

$$\{e_3\}^\mu = \{0, 0, 0, 1\}$$

$$\boxed{n_\mu (e_j)^\mu = 0}$$

$$(e_0)^\mu = t^\mu = (1, 0, 0, 0)$$

$$\Longrightarrow \quad \mathcal{L}_t \longrightarrow \partial_t$$

$$n^\mu = \frac{1}{\alpha}\left(1, -\beta^i\right) \quad ; \quad n_\mu = (-\alpha, 0, 0, 0)$$

$$\beta^\mu = \left(0, \beta^i\right) \quad ; \quad \beta_\mu = (0, \beta_i)$$

$\gamma^{\alpha 0} = 0$ for γ spatial

$$g^{\mu\nu} = \gamma^{\mu\nu} - n^\mu n^\nu$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

$\gamma^{ij}\gamma_{ij} = \delta^i_j$ $\qquad$ γ^{ij}, γ_{ij} are inverse

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$dl^2 = \gamma_{ij}\,dx^i dx^j$ $\qquad$ spatial 3 metric

γ can be used to lower and raise the indices

$$V_i = \gamma_{ij} V^j$$

$$V^i = \gamma^{ij} V_j$$

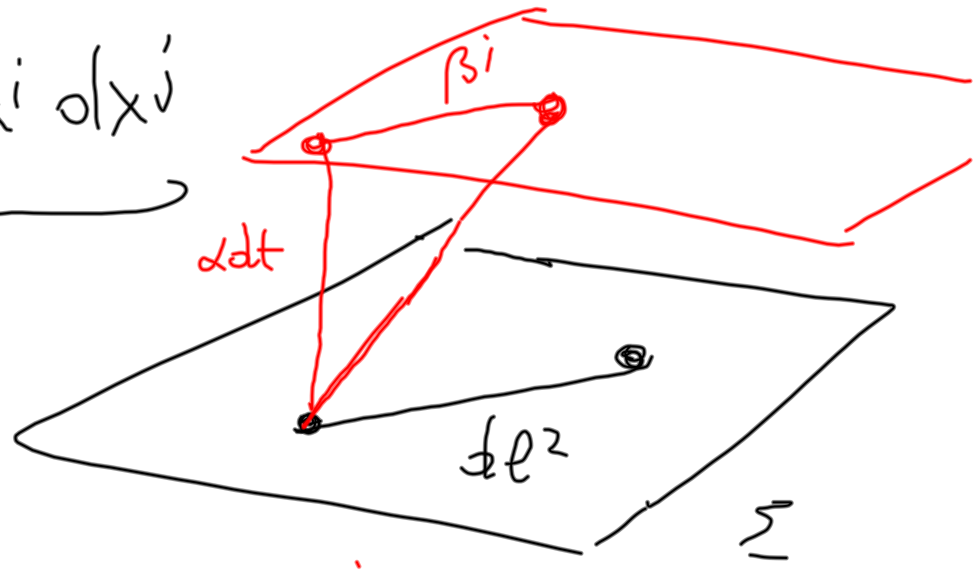
3 vector

$- + + +$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt$$

$$+ \underbrace{\gamma_{ij} dx^i dx^j}$$



$$dl^2 = \gamma_{ij} dx^i dx^j$$

$$d\tau^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 \overset{\beta^i = 0}{=} -\alpha^2 dt^2$$

The lapse function measures (increment) how time changes from one slice to the next

Let's compare Einstein with Maxwell

$$\partial_t \bar{E} = \bar{\nabla} \times \bar{B} - 4\pi \bar{J}$$

$$\partial_t \bar{B} = -\bar{\nabla} \times \bar{E}$$

$$\bar{\nabla} \cdot \bar{E} = 4\pi \rho_e$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\partial_t E_i = \epsilon_{ijk} \partial^j B^k - 4\pi J_i$$

3+3 evol.

$$\partial_t B_i = -\epsilon_{ijk} \partial^j E^k$$

$$\partial_i E^i = 4\pi \rho_e$$

$$\partial_i B^i = 0$$

1+1 constr.

4D : $G_{\mu\nu} = 8\pi\, T_{\mu\nu}$

10, 2nd-order PDEs

—— · ——

Introduce 3+1 split and obtain

$$\underbrace{(3+1)}_{4} + \underbrace{(6 + 6)}_{12,\ \text{1st order eqs}}$$

constraint eqs

evolution eqs

$n \cdot n \cdot G \Longrightarrow$ $R + K^2 - K_{ij} K^{ij} = 16\pi e$

$\gamma \cdot n \cdot G \Longrightarrow$ $D_j K^j{}_i - D_i K = 8\pi J_i$

$\gamma \cdot \gamma \cdot G \Longrightarrow$ $\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} - 2K_{ik} K_j{}^k + K K_{ij} \right) - 8\pi\alpha \left(R_{ij} - \frac{1}{2} \gamma_{ij} (S - e) \right) + \mathcal{L}_\beta K_{ij}$$

$$\partial_t \bar{E} = \bar{\nabla} \times \bar{B} - 4\pi \bar{J}$$

$$\partial_t \bar{B} = -\bar{\nabla} \times \bar{E}$$

$$\bar{\nabla} \cdot \bar{E} = 4\pi \rho_e$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\partial_t E_i = \epsilon_{ijk} \partial^j B^k - 4\pi J_i$$

$$\partial_t B_i = - \epsilon_{ijk} \partial^j E^k$$

$$\partial_i E^i = 0$$

$$\partial_i B^i = 0$$

$$A_\mu = (-\phi, A_i)$$

$$\phi \longleftrightarrow \beta^i$$
$$A_i \longleftrightarrow \gamma_{ij}$$
$$E_i \longleftrightarrow K_{ij}$$

$$\partial_t A_i = - E_i - D_i \phi$$

$$\partial_t E_i = - D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$\partial_t K_{ij} = - D_i D_j \alpha + \alpha (R_{ij} + \dots)$$

ADM

The ADM eqs, as the Maxwell eqs we have written are <u>weakly</u> hyperbolic and hence <u>ill-posed</u>: the solution <u>can</u> grow unbounded

In contrast, a <u>strongly</u> hyperb. formulation is <u>well posed</u>

Wave eq is a typical example of a strongly hyperbolic eq.

Let's try to write Maxwell eq as wave eq

$$\partial_t A_i = -E_i - D_i \phi$$

take time derivative

$$-\partial_t^2 A_i + D^j D_j A_i - \underline{D_i D^j A_j} = D_i \partial_t \phi - 4\pi J_i$$

mixed derivatives

$$\Box u = 0 \qquad \partial_t^2 u - \partial_i \partial^i u = 0$$

The system is weakly hyperb.

but we can introduce a new variable

$$\Gamma \equiv D^i A_i$$

$$\partial_t E_i = -D^j D_j A_i + D_i \Gamma - 4\pi J_i$$

$$\underbrace{\Box A_i}_{\text{wave eq. principal part}} = - D_i \Gamma - D_i \partial_t \phi + 4\pi J_i$$

The price to pay is an additional eq. for Γ

$$\partial_t \Gamma = \partial_t D^i A_i = D^i \partial_t A_i$$

$$= -4\pi \rho_e - D_i D^i \Phi$$

$$\partial_t^2 u = \partial_x^2 u \implies \begin{cases} \partial_t r = v \partial_x s \\ \partial_t s = v \partial_x r \end{cases}$$

Adding a new eq. is not a problem

$$\partial_t K_{ij} = - D_i D_j \alpha + \alpha \left(R_{ij} + \cdots \right.$$

contains
mixed derivs
and spoils wave struc.

We do the same done for Maxwell and introduce a number of new quantities;
conformal, trace-free formulation
(BSSNOK)

$$\phi = \frac{1}{12} \ln \left(\det (\gamma_{ij}) \right) = \frac{1}{12} \ln (\gamma)$$

: conformal factor

$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu$$

$$ds^2 = \tilde{g}_{\mu\nu} \, dx^\mu dx^\nu$$

} conformally related

$$\tilde{g}_{\mu\nu} = \Omega \, g_{\mu\nu}$$

↑ conformal factor

$$\tilde{\gamma}_{ij} = e^{-4\phi} \gamma_{ij}$$

$\tilde{\gamma}_{ij}$ conformal 3-metric

└ this can be very simple, eg flat

$$K = \gamma^{ij} K_{ij}$$

$$\tilde{A}_{ij} = e^{-4\phi} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right)$$

└ trace free conf. extrinsic curv

$$\Gamma = D^i A_i$$

$$\Gamma^i = \gamma^{jk} \Gamma^i_{jk}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk}$$

After doing all this we obtain the following set of eqs.

$$\rightarrow D_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} \qquad D_t = \partial_t - \mathcal{L}_\beta$$

$$D_t \phi = -\frac{1}{6}\alpha K$$

$$\rightarrow \underline{D_t \tilde{A}_{ij}} = e^{-4\phi}\left[-\nabla_i \nabla_j \alpha + \overline{D^i D_i \tilde{\gamma}_{ij}} + \cdots\right]$$

$$D_t K = -2\tilde{A}^{ij} \nabla_i \nabla_j \alpha + \cdots + \epsilon \nabla^6 \tilde{A}_{ij}$$

$$D_t \tilde{\Gamma}^i = -2\tilde{A}^{ij} \partial_j \alpha + \cdots .$$

BSSNOK : is hyperbolic and therefore well-posed

$$|\xi|^2 \leq 1$$

$$\partial_t \tilde{\gamma}_{ij}$$

$$\partial_t \tilde{A}_{ij}$$

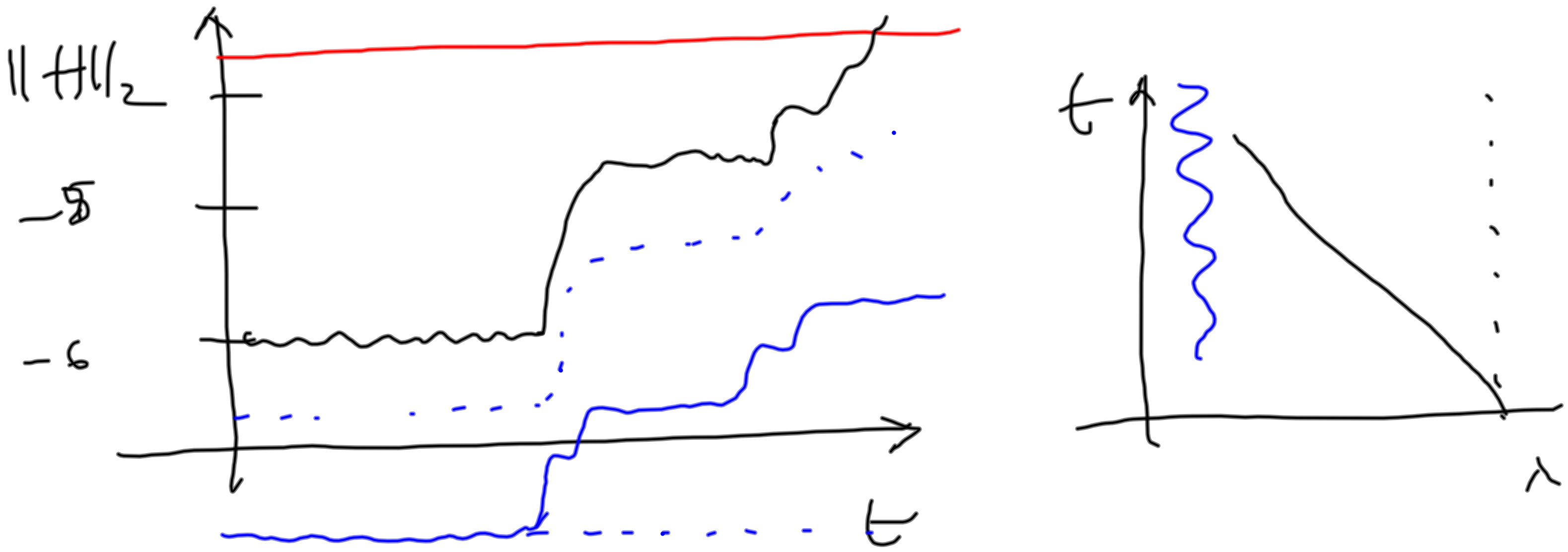
$$H = R + K^2 - K_{ij} K^{ij} = 0 \qquad (*)$$

$$M_j = D_j \left(K^{ij} - g^{ij} K \right) = 0$$

At each timelevel we compute the LHS of (*) and monitor them

$$\| H \|_2 = \frac{1}{N} \Sigma_{ijk} \left(H(x,y,z) \right)^2$$

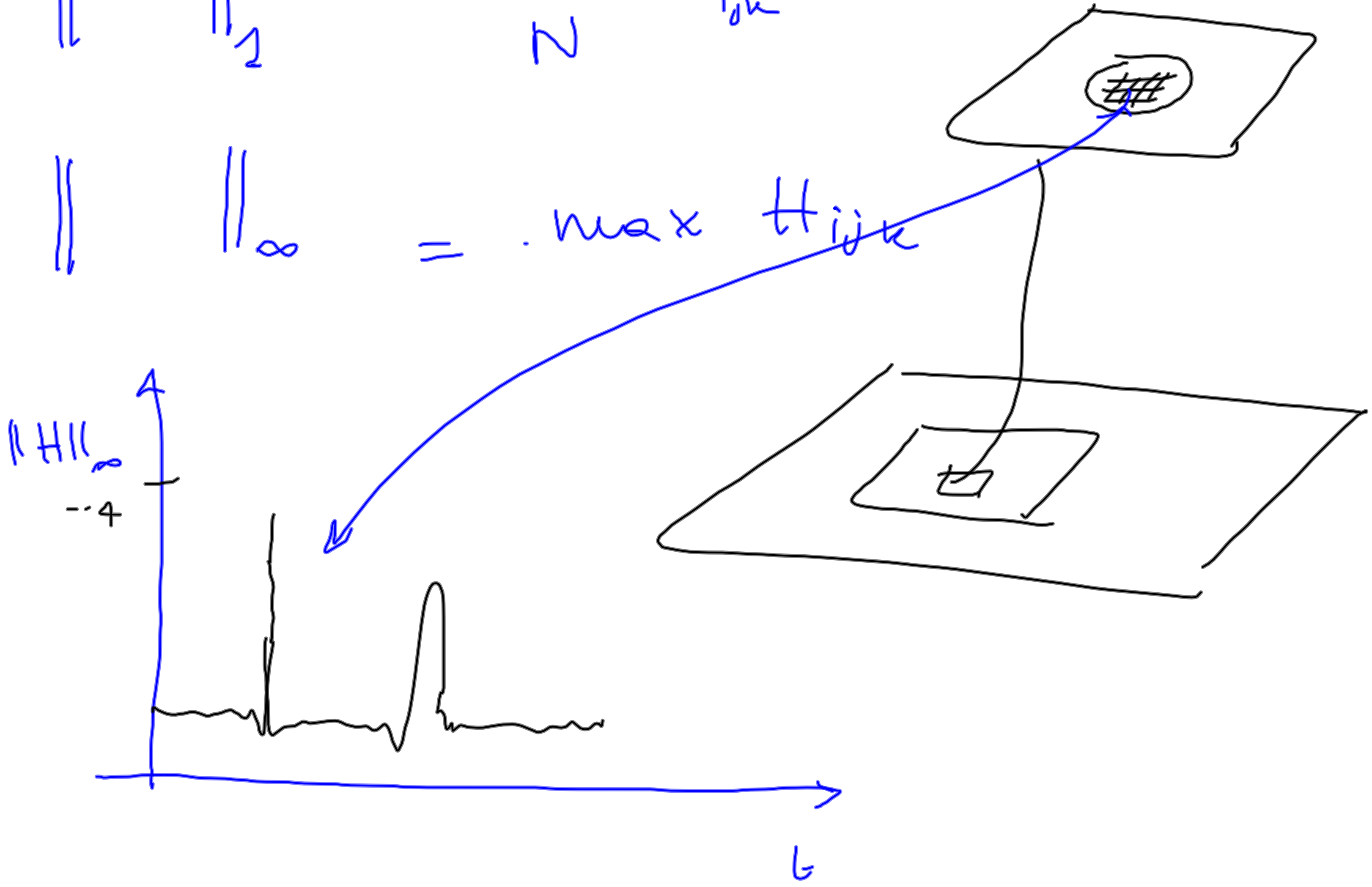
We check that $\|H\|_2 \le \epsilon$



The constraints are not solved but monitored

$$\| \quad \|_1 = \frac{1}{N} \sum_{ijk} H$$

$$\| \quad \|_\infty = \max H_{ijk}$$



There are alternative formulations to BSSNOK and these are called generalized harmonic (GH)

This formulation imposes that the coordinates follow a wave eq.

$$0 = \Box X^\mu = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\, g^{\nu\beta} \partial_\beta X^\mu \right)$$

If this is imposed the Einstein eqs have a wave principal part

$$\Box g_{\mu\nu} = C(g_{\mu\nu}, \partial_\mu g_{\mu\nu}, \cdots)$$

However $\Box x^\mu = 0$ is problematic because of caustics

$$\Box x^\mu = H^\mu$$

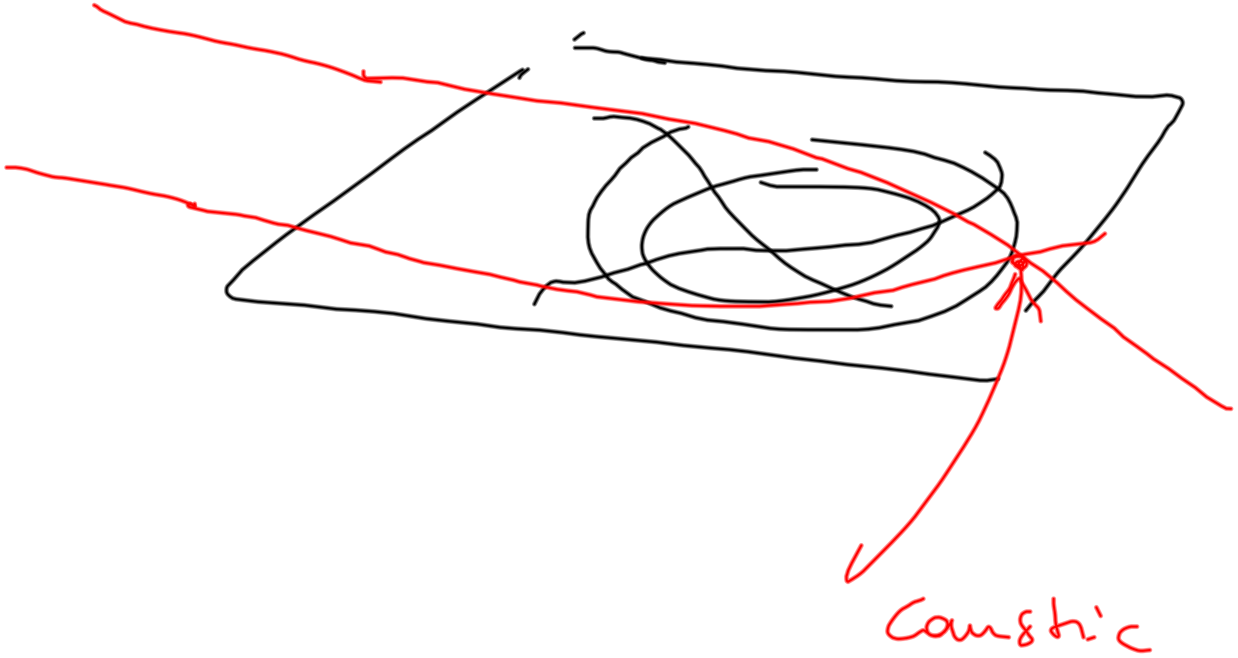
generalized harmonic

The specification of H^μ amounts to selecting a gauge condition

3+1 metric

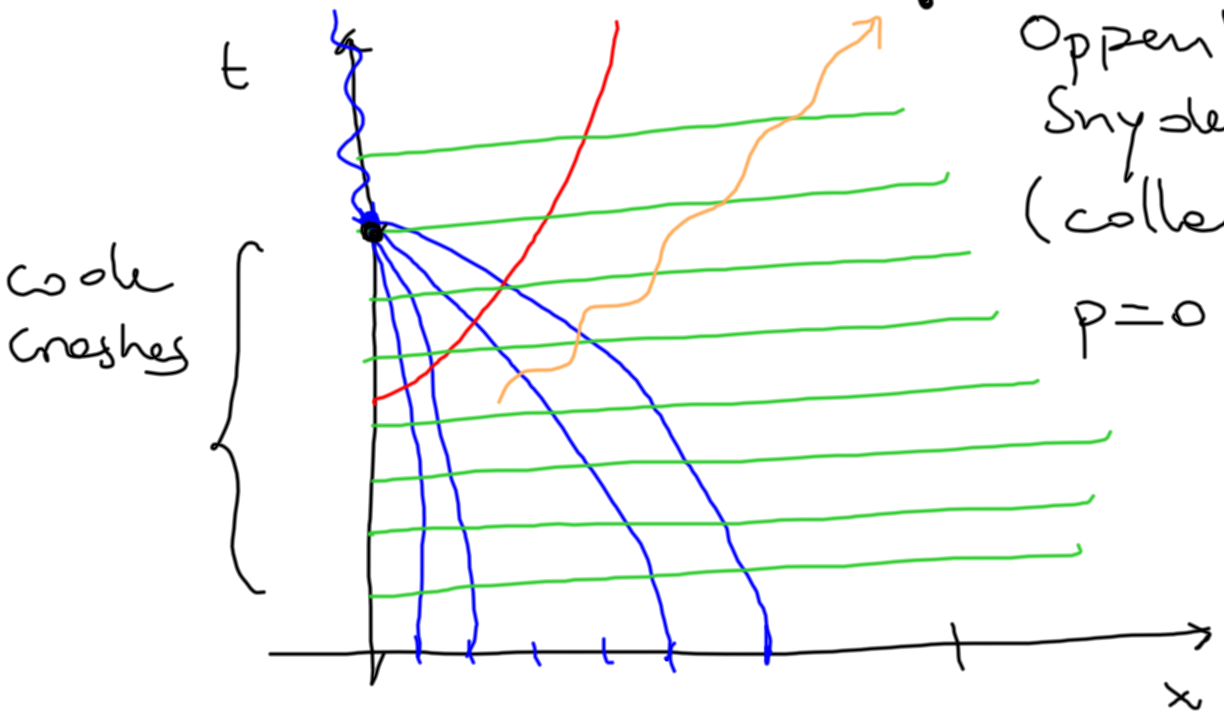
$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -(\alpha^2 - \beta^i \beta_i)\, dt^2 + 2\beta_i\, dx^i dt$$
$$+ \gamma_{ij}\, dx^i dx^j$$

α, β (lapse and shift) are arbitrary and reflect our arbitrariness in laying coords.


Caustic

Defining "good" gauge conditions is an art in NR.

What is "good"?

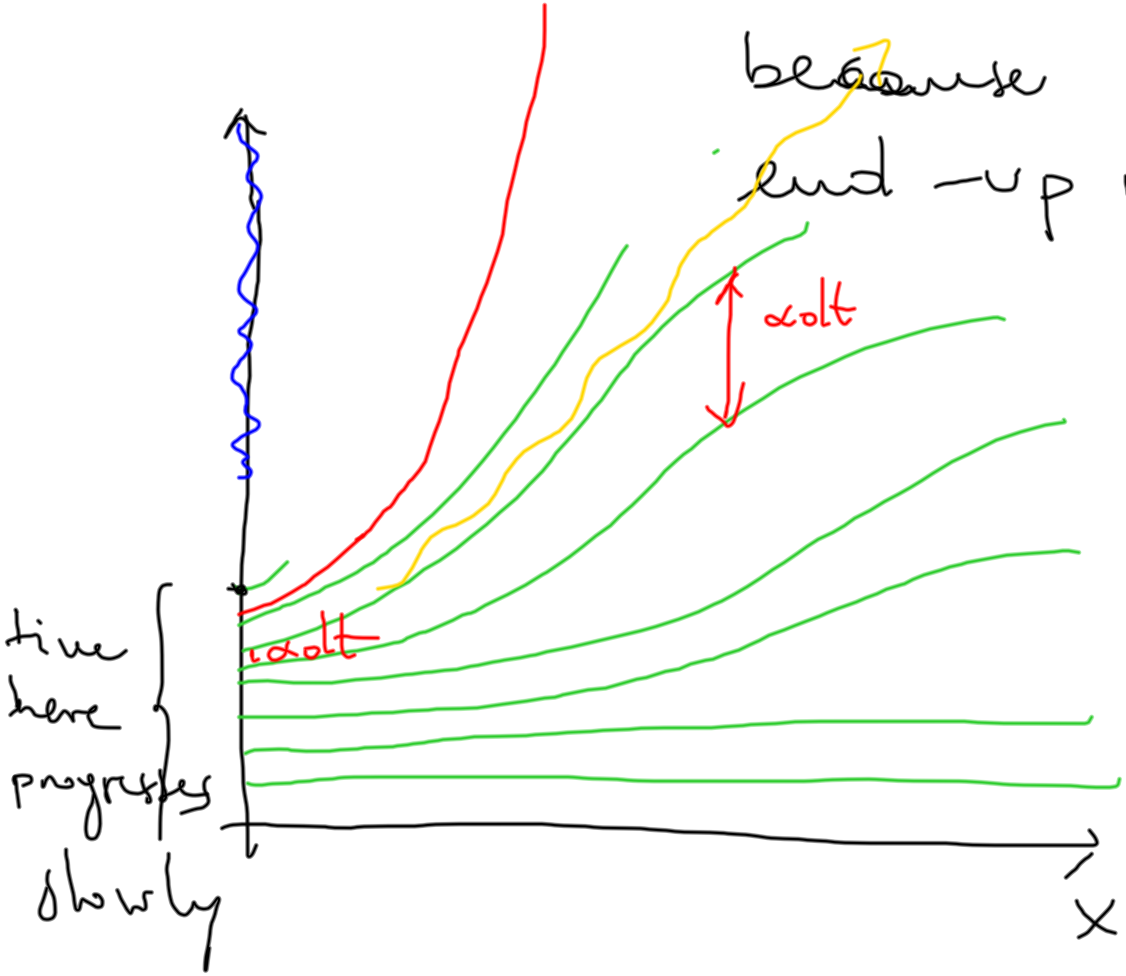


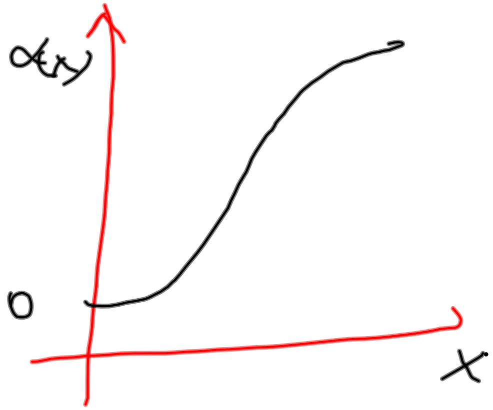
Oppenheimer Snyder collapse (collapse of dust $p = 0$)

$\alpha = 1, \beta^i = 0$

$\alpha = 1, \; \beta^i = 0$

geodesic slicing is not a good idea because grid points end-up in the singul.





The trick is to slice the spacetime avoiding the singularity and this is called sing. avoiding slicing.

You do the trick by "slowing" proper time in regions where curvature is large choice of $\alpha \Longleftrightarrow$ slicing condition

" " $\beta \Longleftrightarrow$ spatial gauge conditions,

It is possible to define perfect gauge conditions but these are always the result of elliptic eqs and hence very expensive

Eg

$\partial_t u \pm v \partial_x u = 0$

Maximal slicing: guarantees singularity avoidance but is elliptic eq.

$$\partial_t K = 0$$

$\Longleftrightarrow$

$$D^i D_i \alpha = \alpha \left[K_{ij} K^{ij} + 4\pi (e + S) \right]$$

$$d_t \equiv \partial_t - \beta^i \partial_i$$

We rather use

$$\underbrace{\partial_t \alpha - \beta^i \partial_i \alpha}_{d_t \alpha} = - \alpha^2 f(\alpha) (K - K_0)$$

K_0 ↑ initial

$$\partial_t \alpha = -\alpha^2 f(\alpha)\,(K - K_0)$$

$$f(\alpha) = \begin{cases} 0 & : \text{geodesic slicing } (\alpha = 1 \quad t = 0) \\ 1 & : \text{harmonic} \\ 2/\alpha & : \text{"1 + log"} \end{cases}$$

$y' = -y$

$f \to \infty$: maximal slicing

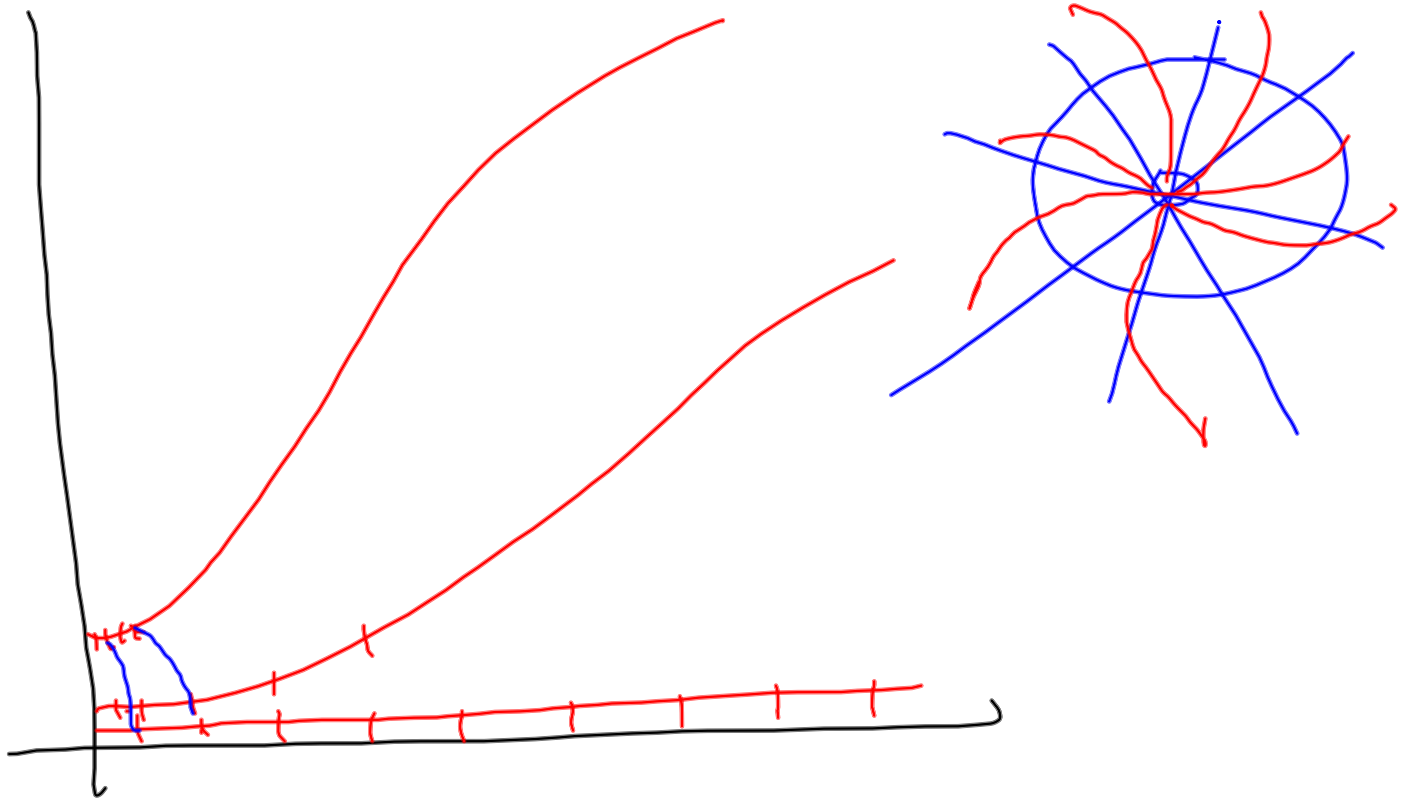
$$\boxed{\partial_t \alpha = -2\alpha}\,(K - K_0)$$

$$\alpha = \alpha_0 e^{-t/\tau}$$

1+log suppresses exponentially fast the lapse in the presence of large curvature.

a

Spatial gauge condition must avoid distortions



Good gauge spatial conditions need to avoid distortions (usually via elliptic eqs)

Alternatives are

$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \frac{3}{4} \alpha B^i$$

$$d_t \beta^i = \frac{3}{4} \alpha B^i$$

$$\partial_t B^i - \beta^j \partial_j B^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i$$

$$d_t B^i = d_t \tilde{\Gamma}^i$$

This enforces that $d_t \tilde{F}_i = 0$

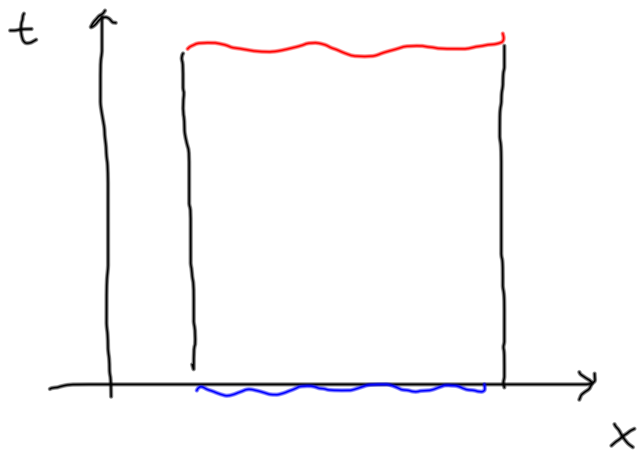
"gamma-driver" condition

Initial Data

GW extraction

Relativistic hydrodynamics

□



Simplest way is to assume there is an equilibrium $\Longrightarrow$ remove time derivatives $\Longrightarrow$ constraint eqs.

$$R + K^2 - K_{ij} K^{ij} = e$$

$$D_j K^j_i - D_i K = 8\pi j_i$$

elliptic eqs

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} = \psi^4 \delta_{ij}$$

CONFORMAL FLAT APPROX.

The consequence of this is that the initial data do not contain waves $\Longrightarrow$ NR simulation all start with a little bit of "junk" radiation

GW extraction

There are different ways of doing this

1) perturbative matching

2) extraction via Weyl scalars

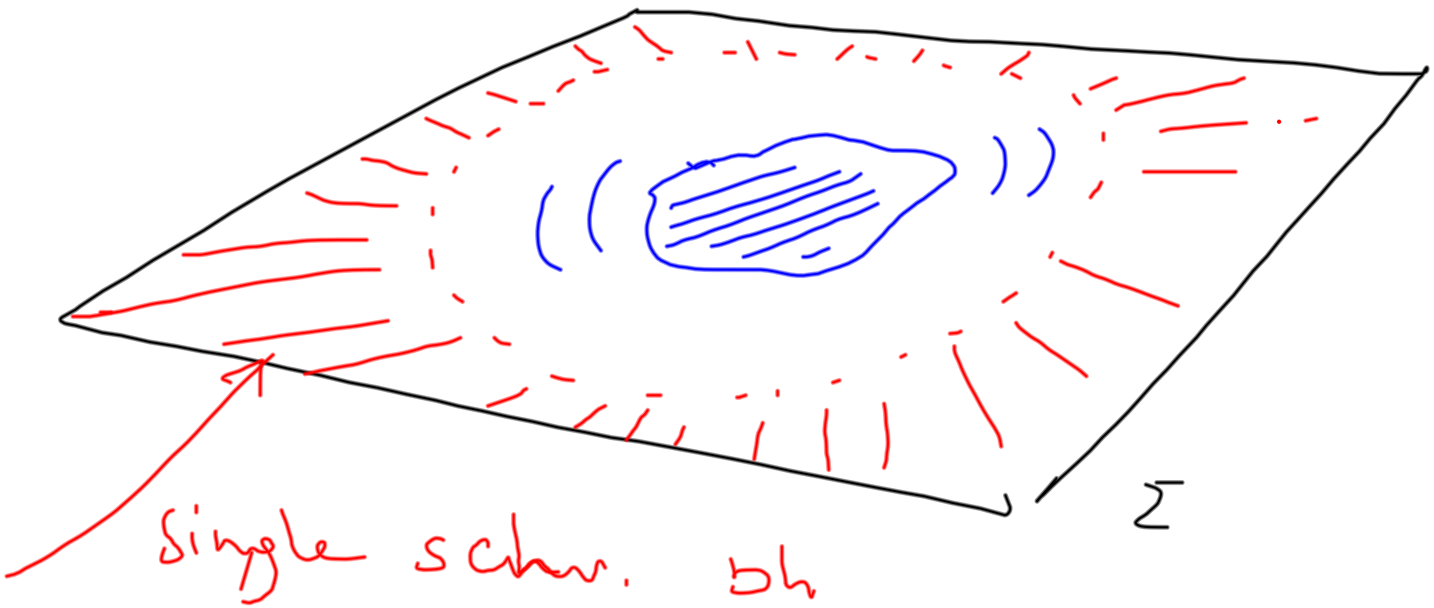
M ; $\quad \dfrac{\Delta E_{GW}}{M} = \underbrace{\text{few} \times 10^{-2}}_{\text{binary bhs}}$

$= 10^{-3}$ binary NS

$= 10^{-6}$

collapse to bh

1) Assume that at certain distance the spacetime is that of Schwarzschild bh



$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}$$

$$\underbrace{h_{\mu\nu}}_{\text{waves}} = \underbrace{g_{\mu\nu}}_{\text{computed}} - \underbrace{\mathring{g}_{\mu\nu}}_{\text{backg.}}$$

In practice $h_{\mu\nu}$ is decomposed into tensor spherical harmonics

2) Weyl

Computes the radiative part of Weyl tensor (ie the trace free part of the Riemann tensor)

$$\Psi_4 \propto \frac{C_{\dot{\alpha}\beta\mu\nu}}{r} \quad : \text{outgoing radiation}$$

$h_+ , h_\times$

1) $h_{\mu\nu} \longrightarrow Q^+_{\ell m} , Q^\times_{\ell m}$

$$h_+ - i h_\times = \frac{1}{\sqrt{2} r} \sum_{\ell m} \left(Q^+_{\ell m} - \right.$$

$$\left. + i \int_{-\infty}^{t} Q^\times_{\ell m} (t') dt' \right)$$

2)

$$h_+ - i h_\times = \int dt \int dt \, \Psi_4$$

$$\ddot{h} \propto \Psi_4$$

Relativistic hydrodymics

$$T_{\mu\nu} = (e+p)\, u_\mu u_\nu + p\, g_{\mu\nu}$$

perfect fluid

$$= h\rho\, u_\mu u_\nu + p\, g_{\mu\nu}$$

$$T_{\mu\nu} = T_{\mu\nu}^{A} + T_{\mu\nu}^{B} + \ldots\,.$$

fluid EM field ν

$$u^\mu u_\mu = -1$$

(3) $\gamma \cdot (\nabla T) = 0$: conservation of mom.

(1) $n \cdot (\nabla T) = 0$: cons. of energy

(1) $(\nabla \rho u) = 0$: cons. of baryon

$$p = p(\rho, \epsilon, \ldots)$$

Euler eqs : describe fluid motion in Newt. gravity

$$\partial_t(\rho v^i) + \underbrace{\partial_i(\rho v^i v^j + p\delta^{ij})}_{\text{gravity}} = -\partial_i\Phi\rho$$

$$\partial_t = 0 = v^i$$

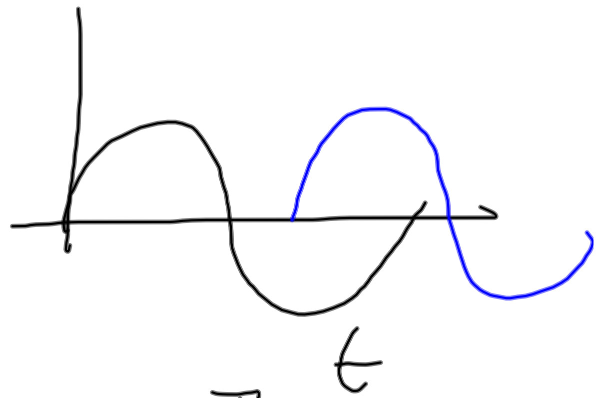
$$\boxed{\partial_r p = -\rho\,\partial_r\phi}$$

$\partial_t u = \underline{v}\, \partial_x u$: hyperb.

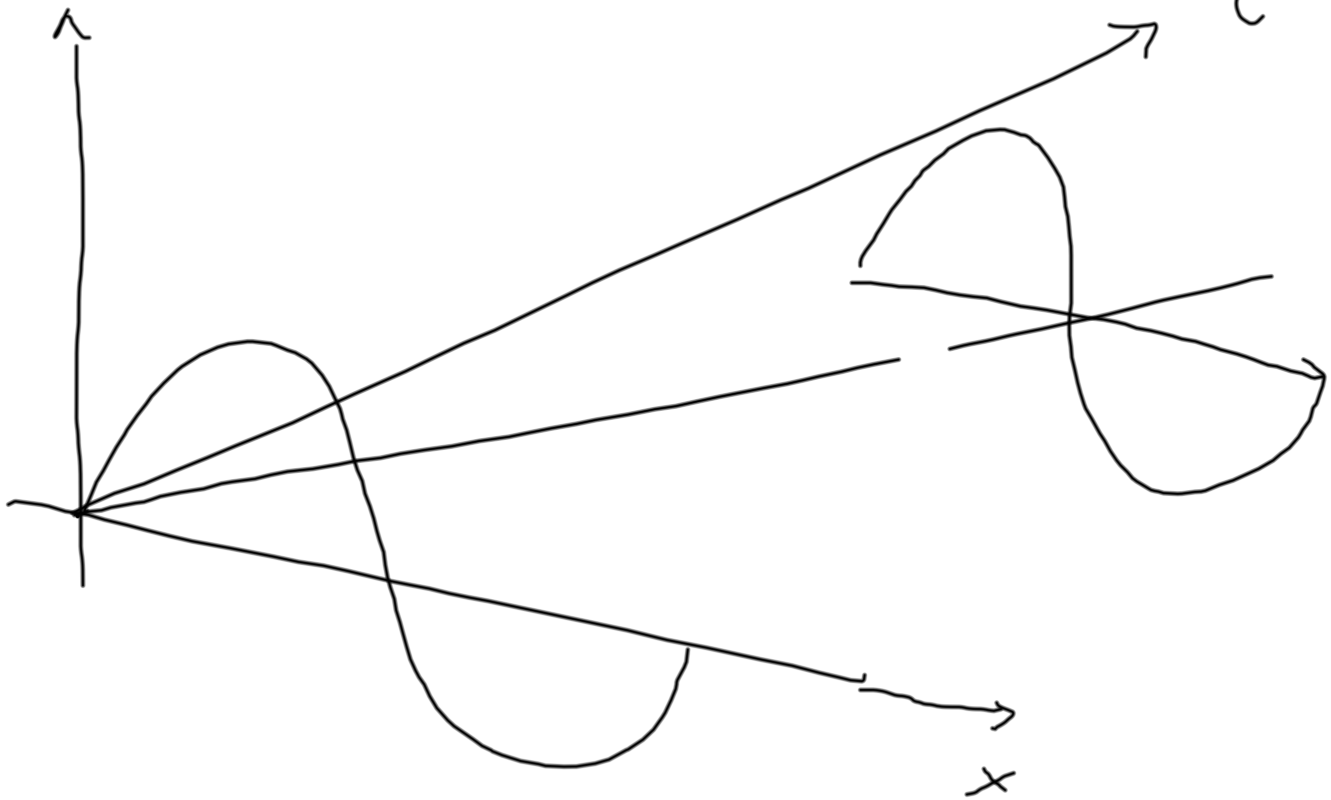
Hydrodynamic eqs. are also hyperbolic but <u>nonlinear</u>

$\partial_t u = \underline{u}\, \partial_x u$: Burgers equation

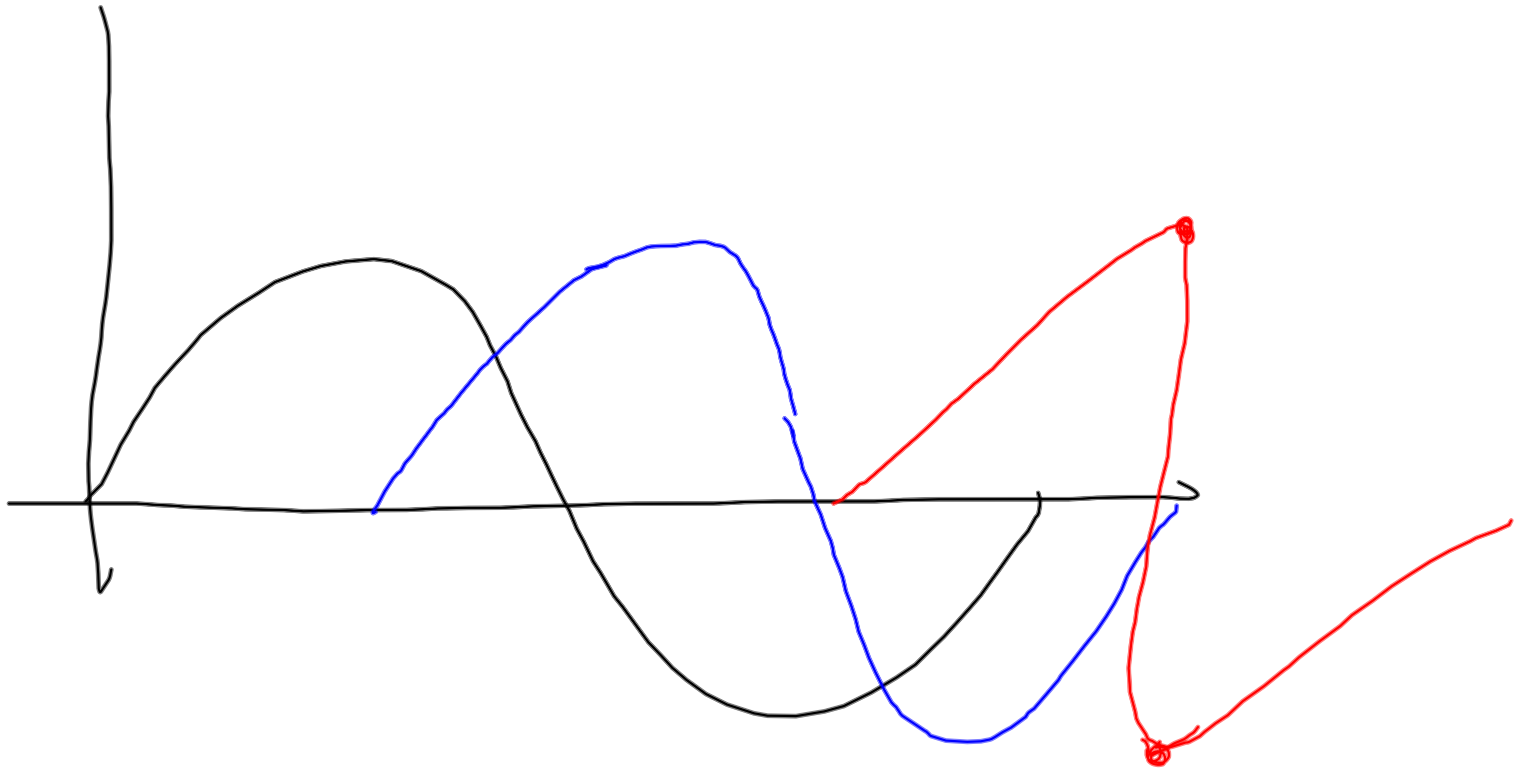
The evolution of a nonlinear hyp. eq leads to development of discontinuities



$\partial_t u = v\, \partial_x u$

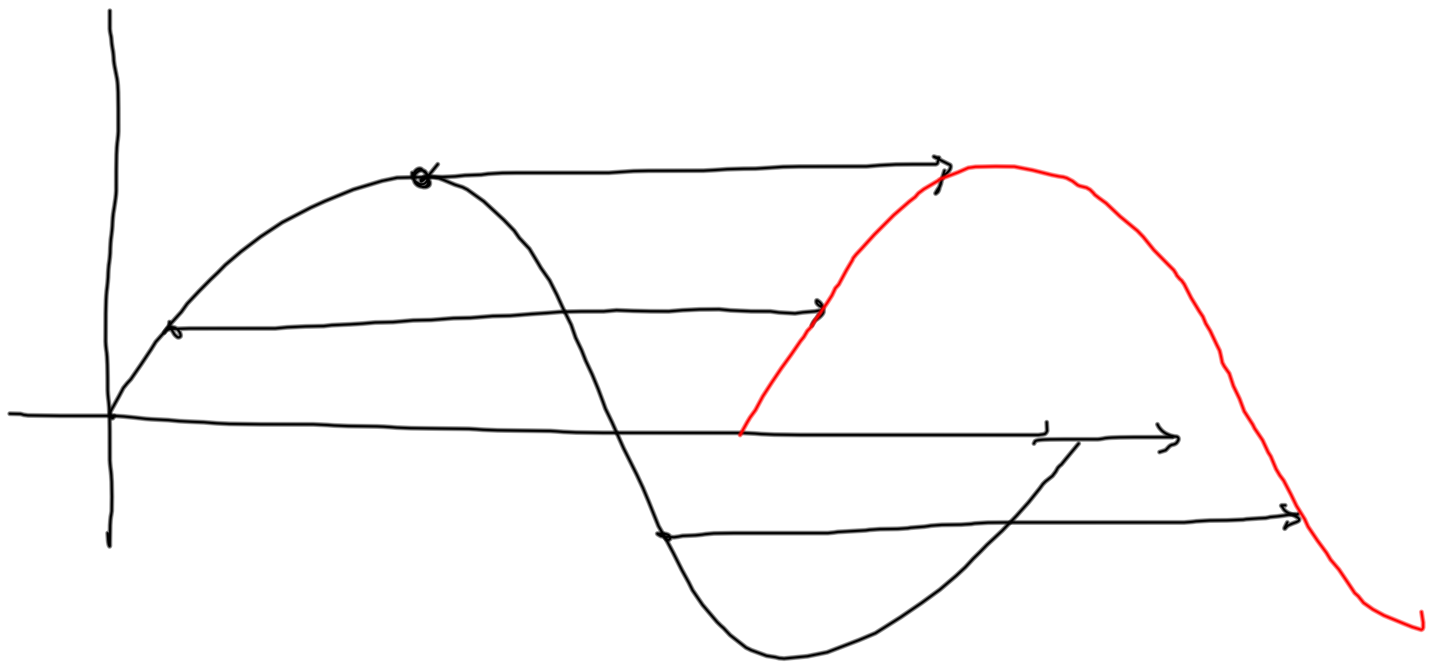


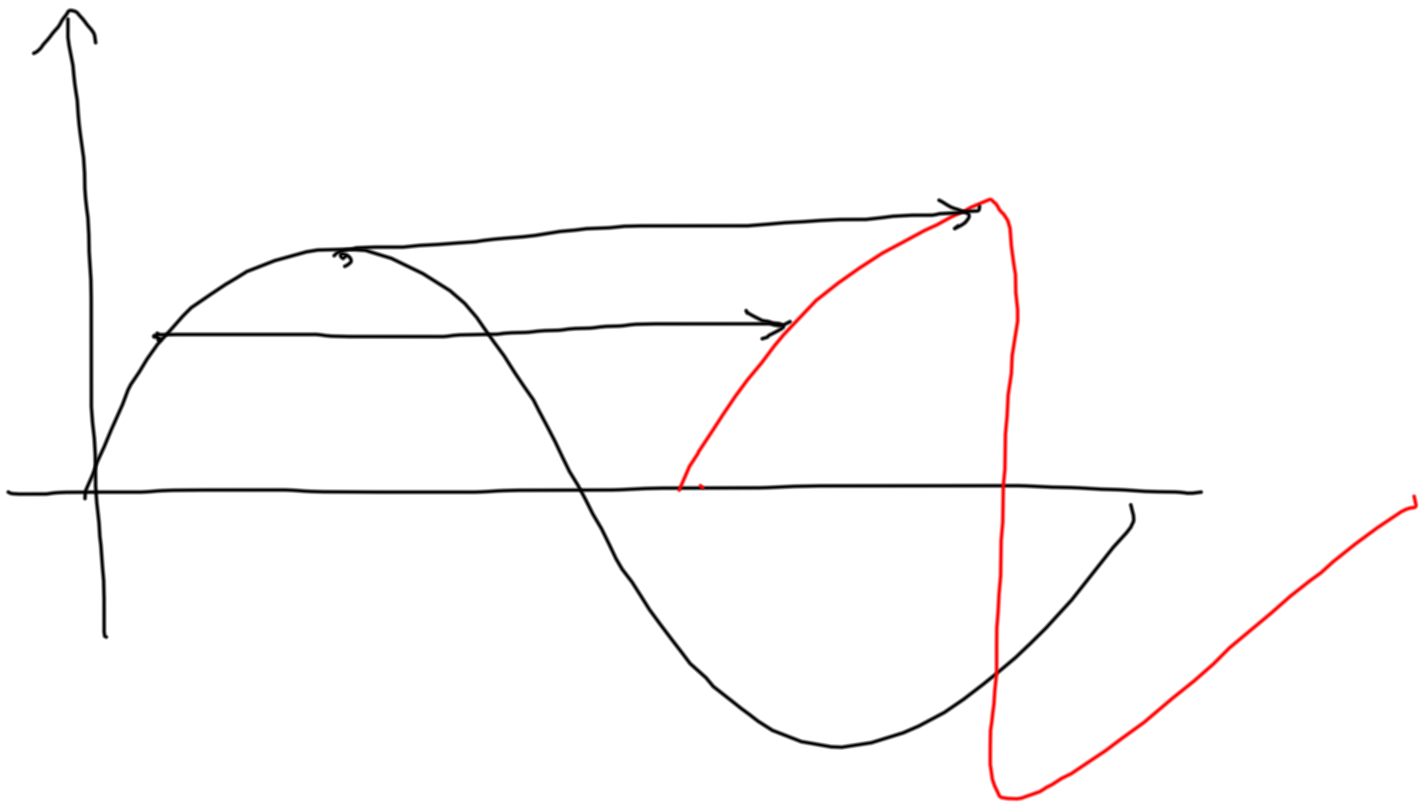
$$\partial_t u = \underline{\underline{u}}\, \partial_x u$$



1) Why does that happen?

2) Why do we care?





We care because the
hydrodynamic eqs are nonlinear

$\Rightarrow$ even smooth initial data will produce disconti-nuities (shocks)

$$\partial_t u = u\,\partial_x u + \epsilon\,\partial_x^2 u$$

Because of this nonlinear feature of hydro eqs they can easily develop shock but we have to be able to follow them properly

In other words we need to write the eqs. in flux-conserv. form

$$\partial_t \underline{u} + \bar{\nabla} \cdot \bar{F}(\underline{u}) = S(\underline{u})$$

1) Only a FC formulation of the eqs leads to correct weak solution

2) A non FC formulation leads to the incorrect weak solution

(1) $\partial_t u = u \, \partial_x u$: Non FC

(2) $\partial_t u = \partial_x \underbrace{\left(\frac{u^2}{2} \right)}_{v}$: FC

$$\partial_t \underline{u} + \partial_x \underline{F}(u) = 0$$

$$A \equiv \partial F / \partial u$$

$$\partial_t \underline{u} + A \, \partial_x \underline{u} = 0$$

$$\Lambda = R^{-1} A R$$: diagonal

R: right eigenvector

$$\Lambda = (\lambda_1, \lambda_2, \lambda_3 \ldots)$$

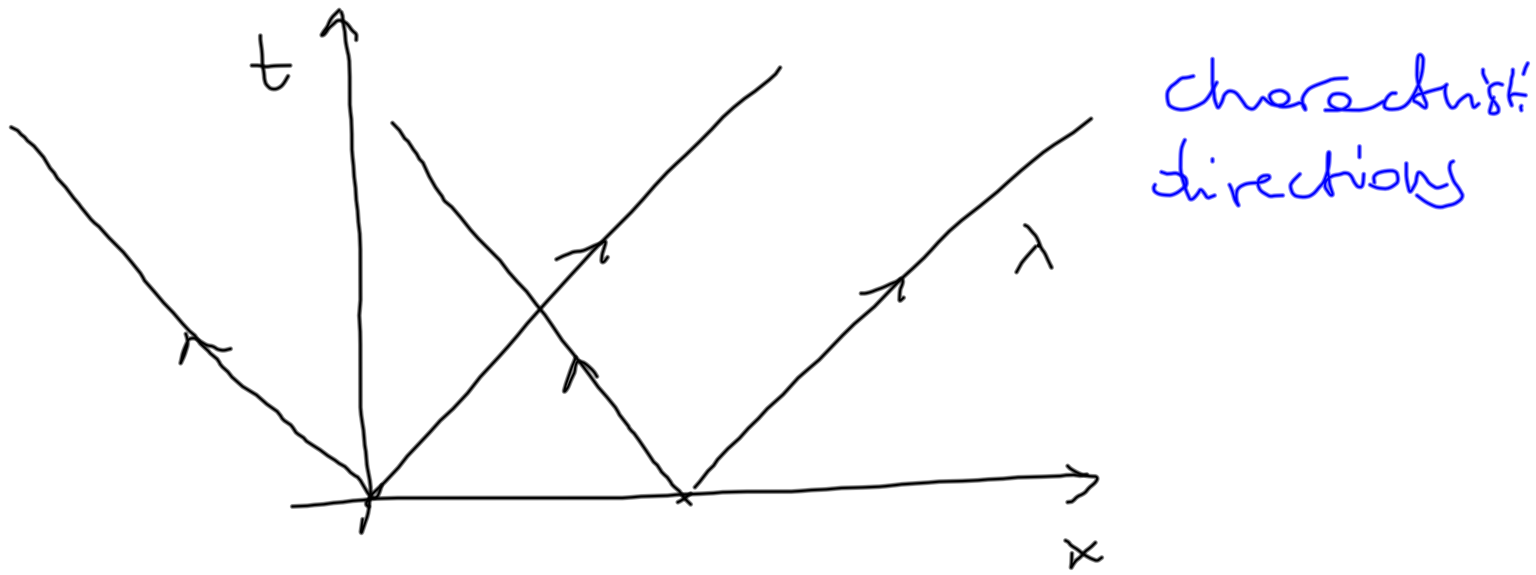
W : characteristic variable

$$\underline{W} \equiv R^{-1} \underline{u}$$

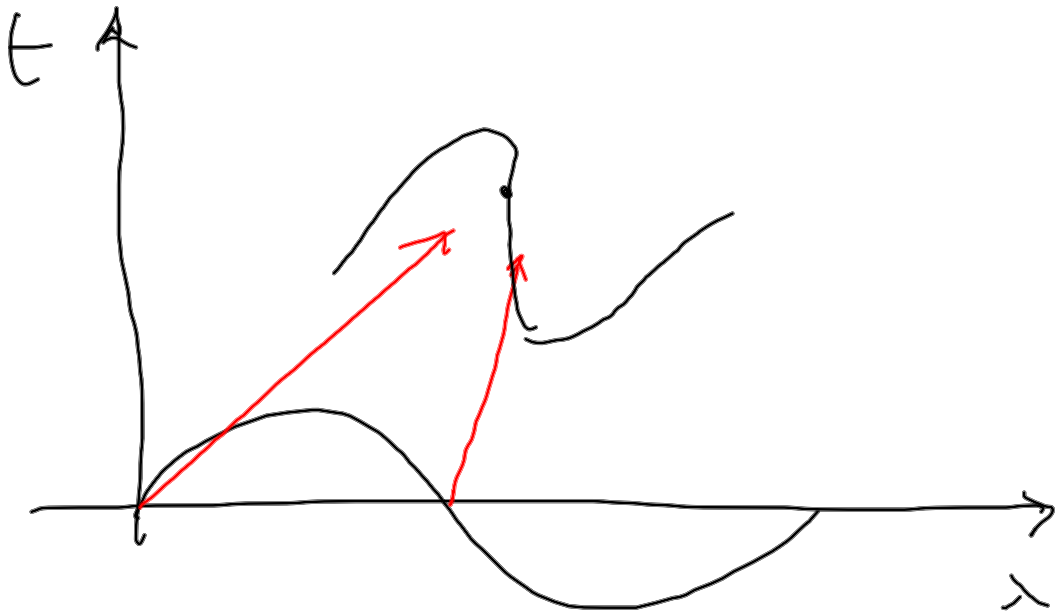
$$\boxed{\partial_t W + \Lambda \partial_x W = 0}$$

this is our advection eq, along the directions given by λ

$$\frac{d}{dt} W = 0 \qquad \text{along } \frac{dx}{dt} = \pm\lambda_i$$



Then what happens in the Burgers eq is that the characteristics merge (caustics)



Recap

- hydro eqs are nonlinear

- hydro eqs develop shock <u>even</u> from smooth I.D.

- eqs with discontinuities "<u>must</u>" be written in a FC form

We have to write the rel. hydro eqs in a FC: Valencia formulation

$$v^i = \frac{\gamma^i_\mu u^\mu}{-n^\mu \cdot u_\mu} = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i\right)$$

$$v_i = \gamma_{ij} v^j$$

$$W = \alpha u^0 = \frac{1}{\sqrt{1 - v^i v_i}}$$: Lorentz factor

$$T_{\mu\nu} = T_{\mu\nu}(\underbrace{u^\mu, p, \epsilon, \rho \ldots}_{\text{these cannot be used}})$$

$D = \rho W$: conserved rest mass density

$S_j = \rho h W^2 v_j$: conserved mom. flux

$\tau = \rho h W^2 - \rho W - p$: cons. energy

$$\frac{1}{\sqrt{-g}}\left\{\partial_t\left(\sqrt{\gamma}\,F^0\right)+\partial_i\left(\sqrt{\gamma}\,F^i\right)\right\} = S$$

$$\sqrt{-g}=\alpha\,\gamma$$

$$F^0=\underline{U}=\left(D,\,S_j,\,\tau\right)^T$$

$$F^i=F^i\left(\,\cdots\cdots\,\right)$$

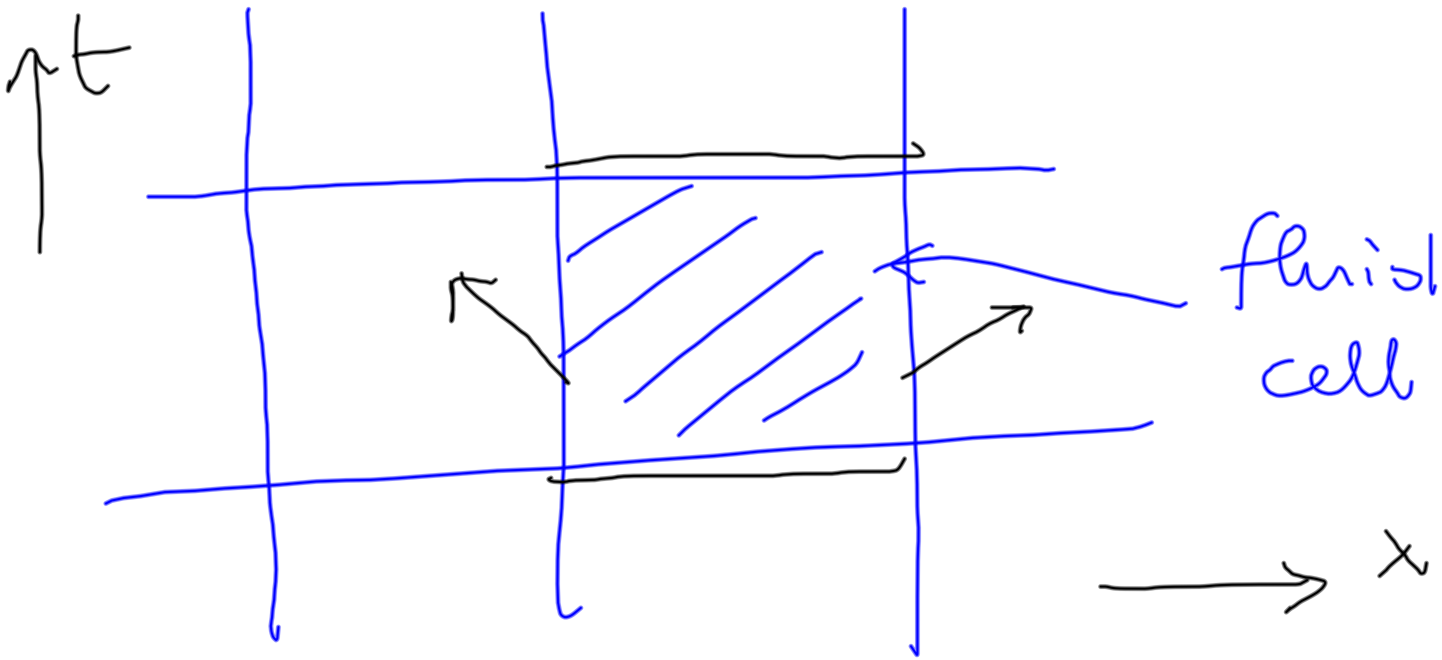
$$S=S\left(g,\,\partial_x g,\,\right)$$

: No deriv. of the fluid variables

Solution of hypho eqs.

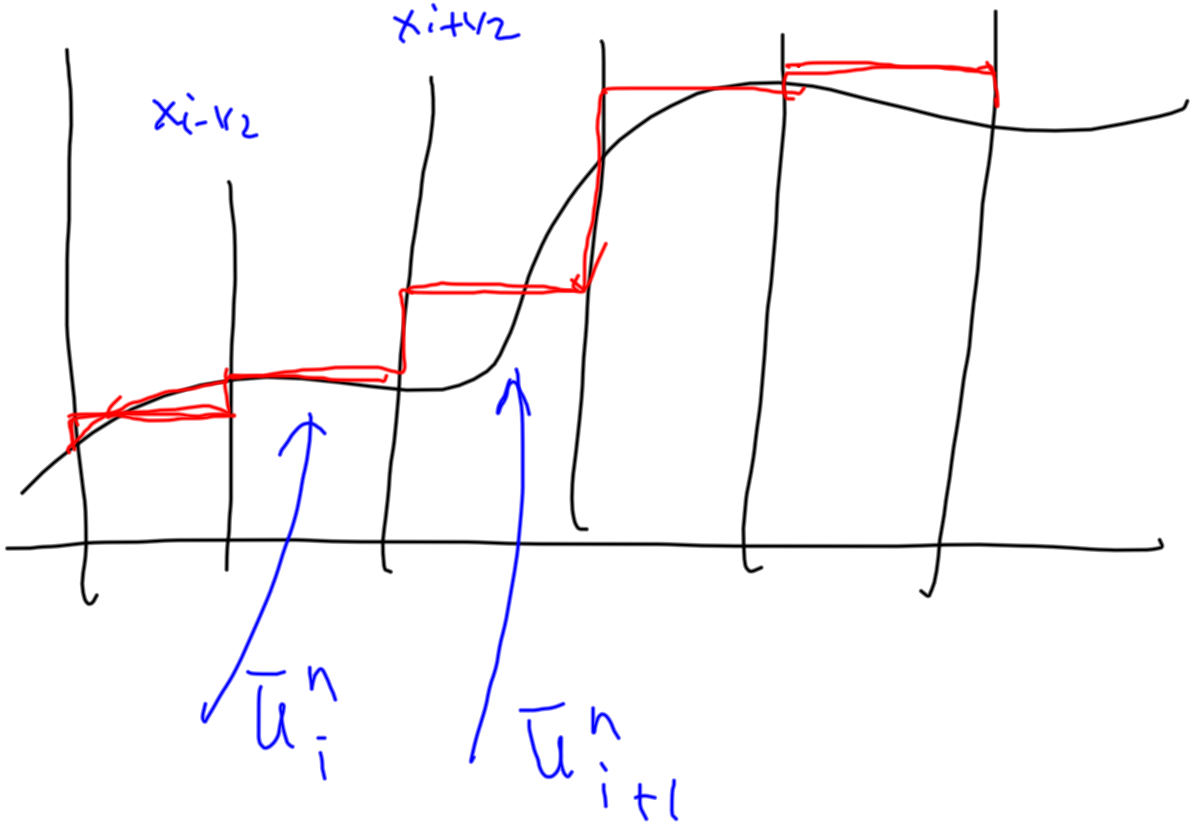
Finite-difference methods are not adequate for the solution of the hypho eq.

Finite volume methods are instead preferable



$$\partial_t u + \nabla F = 0$$

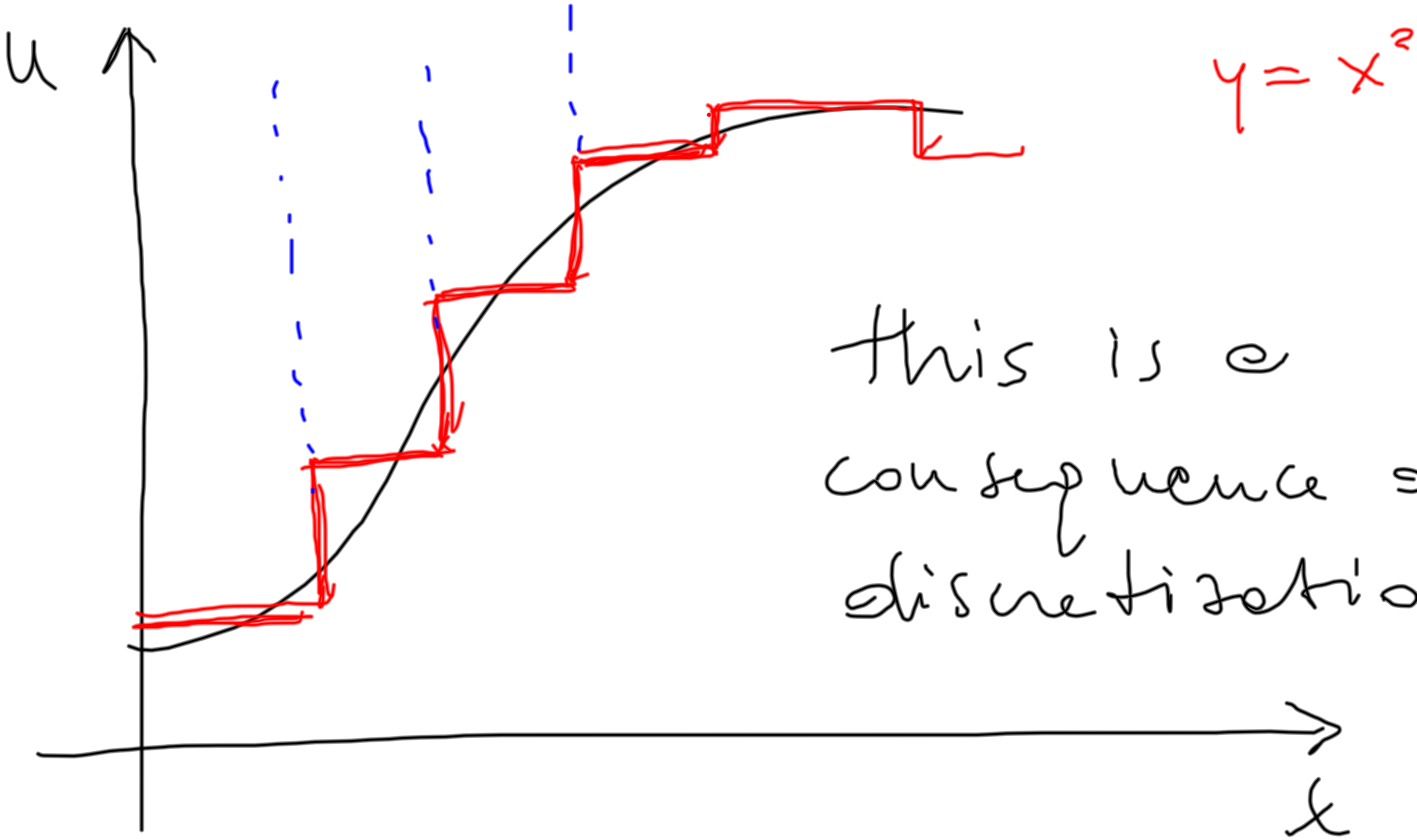
$$\bar{u}_j^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n)\, dx$$



$$(*) \quad \bar{u}_j^{n+1} = \bar{u}_j^{n} - \frac{\Delta t}{\Delta x}\left(\hat{F}_{i+1/2} - \hat{F}_{i-1/2}\right)$$

$$\hat{F}_{i+1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(u(x_{i+1/2}))\,dt$$

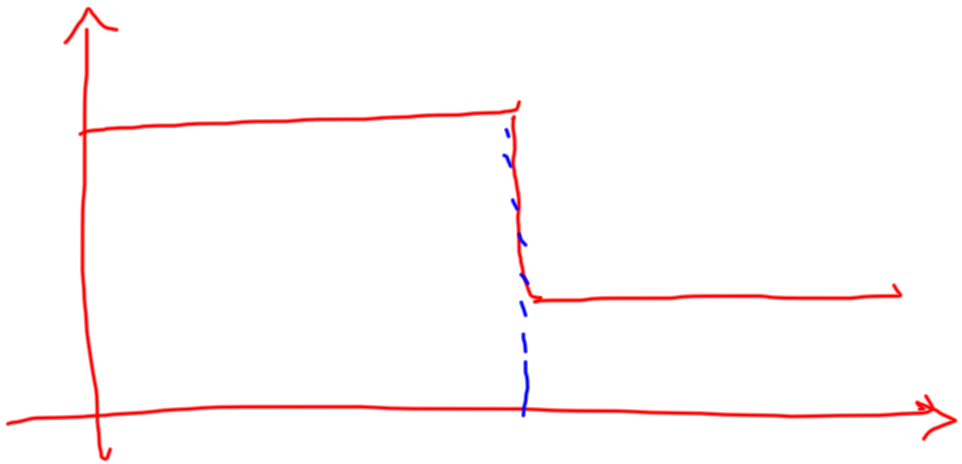
(*) is the standard 1st order (Godunov) representation of a finite volume conservative eq.



this is a consequence of discretization

One can therefore think that a piecewise const representation of hydro var.

is a series of initial conditions of local Riemann problems



evolution of 2 discontinuous initial states

Godunov realized that and produced a series of methods (high resolution shock capturing HRSC) that allow to solve the hydro eq. to great accuracy even with shock.