Geodesic Deviation and Weak-Field Solutions

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A note on conventions

I here use a spacelike signature $(-,+,+,+)$ and a system of units in which $c=1$. Greek indices are taken to run from 0 to 3, Latin indices from 1 to 3, and we adopt the standard convention for the summation over repeated indices. Furthermore we will indicate three-vectors with an arrow (*e.g.* $\vec{U}$), four-vectors with bold symbols or Greek letters (*e.g.* $\mathbf{U}$ or $\boldsymbol{\xi}$), and tensors with slanted bold symbols (*e.g.* $\boldsymbol{T}$).

1 Geodesic deviation equation

1.1 Geodesic Equation

Given a metric tensor $\boldsymbol{g}$, the *mathematical* definition of a geodesic is that of a curve that parallel transports its tangent vector, *i.e.* $\nabla_{\mathbf{v}}\mathbf{V}=0$, where

$$\nabla_{\mathbf{v}}\mathbf{V}=0=V^{\alpha}_{\ ;\beta}V^{\beta}=V^{\alpha}_{\ ,\beta}V^{\beta}+\Gamma^{\alpha}_{\beta\mu}V^{\beta}V^{\mu}\ , \tag{1}$$

and where ∇ is the covariant derivative with respect to $\boldsymbol{g}$ (see Fig. 1).

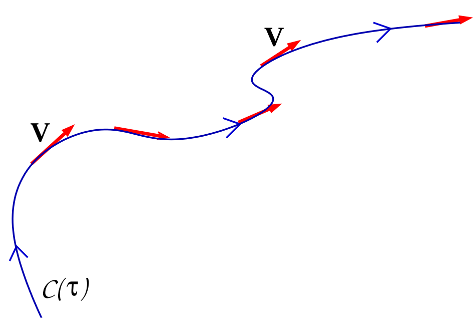

Figure 1: A generic curve $\mathcal{C}(\tau)$ in a 4-dimensional spacetime and its tangent 4-vector $\mathbf{V}$ which is parallel transported along $\mathcal{C}(\tau)$, *i.e.* $\nabla_{\mathbf{v}}\mathbf{V}=0$.

If τ is a parameter along the curve $\mathcal{C}$ having $\mathbf{V}$ as tangent vector, then $V^{\mu}=dx^{\mu}/d\tau$ and

$$V^{\mu}{}_{,\beta}=\frac{\partial}{\partial x^{\beta}}\left(\frac{dx^{\mu}}{d\tau}\right)\ ;\quad V^{\mu}{}_{,\beta}V^{\beta}\equiv\frac{dx^{\beta}}{d\tau}\frac{\partial}{\partial x^{\beta}}\left(\frac{dx^{\mu}}{d\tau}\right)=\frac{d^{2}x^{\mu}}{d\tau^{2}}\ . \tag{2}$$

so that the *geodesic equation* (1) can be written as a second-order differential equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \,, \tag{3}$$

having $x^\mu(P_0)$ and $dx^\mu/d\tau(P_0)$ as initial conditions. If the particle has a nonzero mass, τ can then be associated to the proper time measured by the particle along $\mathcal{C}$.

On the other hand, a *physical* definition of a geodesic is that of a curve along which a freely falling particle moves. Note that the sense of "geodetic-deviation" is *not* that of a deviation from a purely geodetic motion resulting from the application of a force. Rather, the concept of geodetic deviation follows from the comparison between two adjacent geodesics and measures how and when their separation varies. In this Section in particular, we will show how the deviation from a geodesic motion (meant as the evolution of the separation measured between two adjacent geodesics) can indeed be related to a nonzero curvature of the spacetime, or, to use a Newtonian language, to the presence of tidal force.

However, before we consider the concept of geodesic deviation in full General Relativity, it is instructive to first look at this problem in Newtonian gravity, where the basic features are also present.

1.2 Geodesic deviation in Newtonian gravity

For this, let us consider the Newtonian gravitational potential

$$\Phi = \Phi(x^a) \,, \tag{4}$$

and two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ of parameter t (*e.g.* the coordinate time) describing the motion of two test particles P and Q in the gravitational potential Φ. At any given time t the two particles will have positions x^a and z^a which will clearly vary in time. Furthermore, let $\eta^a(t)$ be the components of the 3-vector separating the two particles at any time, *i.e.* (see Fig. 2)

$$z^a = z^a(t) = x^a(t) + \eta^a(t) \,. \tag{5}$$

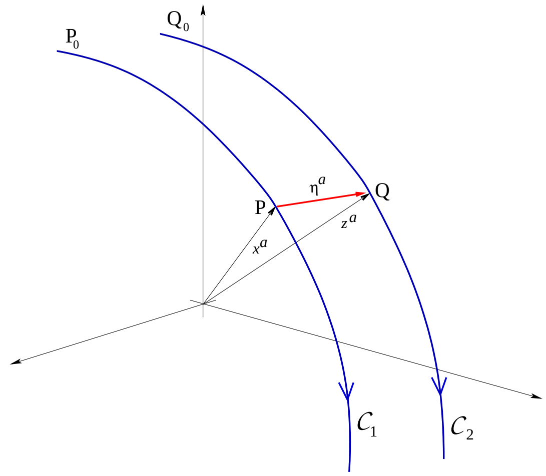

Figure 2: Trajectories of two freely falling particles in a gravitational potential Φ. The 3-vector $\vec{\eta}$ measures the distance between the two particles at any time.

Because there is only one force applied (the gravitational one), the equations of motion will be particularly simple and have the form

$$\ddot{x}^a(t) = -(\partial^a \Phi)_P \ , \tag{6}$$

$$\ddot{z}^a(t) = -(\partial^a \Phi)_Q = \ddot{x}^a(t) + \ddot{\eta}^a(t) \ , \tag{7}$$

where $\ddot{x}^a \equiv d^2x^a/dt^2$ and $(\partial^a \Phi)_{P,Q}$ are the gravitational forces at time t acting on P and Q, respectively. We can now expand in a Taylor series the gravitational field Φ in Q in terms of the separating 3-vector η^a, *i.e.*

$$(\Phi)_Q = (\Phi)_P + \eta^b(\partial_b \Phi)_P + \mathcal{O}(\eta^a \eta_a) \ , \tag{8}$$

so that equations (6) and (7) can be written as

$$\ddot{z}^a = \ddot{x}^a + \ddot{\eta}^a = -(\partial^a \Phi)_Q = -\partial^a[(\Phi)_P + \eta^b(\partial_b \Phi)_P] = -(\partial^a \Phi)_P - \eta^b(\partial^a \partial_b \Phi)_P \ , \tag{9}$$

or equivalently

$$\ddot{\eta}^a = -(\partial^a \partial_b \Phi)\eta^b = -K^a_{\ b}\eta^b \ . \tag{10}$$

where $K^a_{\ b} \equiv \partial^a \partial_b \Phi$.

Equation (10) represents the *geodesic deviation* equation in Newtonian physics and expresses the fact that the distance between the two freely falling particles will vary if they move in a gravitational field that is nonuniform. Furthermore, since $\ddot{\eta}^a = 0$ at all times if $K^a_{\ b} = 0$ at all times, equation (10) underlines how, in principle, the relative acceleration between the two particles cannot distinguish between a uniform gravitational field (*i.e.* $\partial_a \Phi = 0$) and a zero one (*i.e.* $\Phi = 0$). Of course, we know from astronomical observations that in Newtonian gravity $\Phi \sim 1/r$ and so $K^a_{\ b} \sim 0$ only at large distances from the source of Φ, where $\Phi \sim 0$.

In the following Section we will see that many of the considerations made here continue to hold true also in General Relativity.

1.3 Geodesic deviation in General Relativity

With the insight gained in Newtonian gravity, we can now consider the problem of geodetic deviation in General Relativity. It has been shown that the differences that emerge when a 4-vector $\mathbf{V}$ is parallel transported along a loop along two different routes is directly related to the *curvature tensor* of the spacetime in which the loop lies. Using an index notation, this is expressed by the equation

$$2\nabla_{[\alpha}\nabla_{\beta]}V^{\mu} = R^{\mu}_{\ \nu\alpha\beta}V^{\nu} \ , \tag{11}$$

where

$$2\nabla_{[\alpha}\nabla_{\beta]}V^{\mu} = \nabla_{\alpha\beta}V^{\mu} - \nabla_{\beta\alpha}V^{\mu} = V^{\mu}_{\ ;\alpha\beta} - V^{\mu}_{\ ;\beta\alpha} \ , \tag{12}$$

and $\boldsymbol{R}$ is the Riemann (or curvature) tensor defined as

$$R^{\mu}_{\ \nu\beta\alpha} \equiv \Gamma^{\mu}_{\ \nu\alpha,\beta} - \Gamma^{\mu}_{\ \nu\beta,\alpha} + \Gamma^{\mu}_{\ \sigma\beta}\Gamma^{\sigma}_{\ \nu\alpha} - \Gamma^{\mu}_{\ \sigma\alpha}\Gamma^{\sigma}_{\ \nu\beta} \ . \tag{13}$$

Note that because the Christoffel symbols are linear combinations of the first-order partial derivatives of the metric, the Riemann tensor $\boldsymbol{R}$ (13) is effectively a combination of first and second-order partial derivatives of the metric.

Equation (11) is important for a number of different reasons. Firstly it shows that, in general, in a curved spacetime the covariant derivatives do not commute. Secondly it expresses a relation between a "failed parallel transport" (*i.e.* $\nabla_{[\alpha}\nabla_{\beta]}V^{\mu} \neq 0$) and the local curvature, *i.e.*

$$(\text{deviation from parallel transport}) \qquad \leftrightsquigarrow \qquad (\text{curvature}) \ . \tag{14}$$

On the other hand, we have also seen the relation between parallel transport and geodesic motion, *i.e.*

$$(\text{parallel transport}) \qquad \leftrightsquigarrow \qquad (\text{geodesic motion}) \ , \tag{15}$$

and it would not be surprising, therefore, to find out that there is a close logical connection between the deviation from a geodesic in the presence of curvature, *i.e.*

$$(\text{deviation from parallelism in geodesic motion}) \qquad \leftrightsquigarrow \qquad (\text{curvature}) \ . \tag{16}$$

This is what we will show in a rigorous way.

Consider therefore a congruence of geodesic curves of parameter τ and tangent vector $\mathbf{V}$, where

$$V^{\alpha} \equiv \frac{dx^{\alpha}}{d\tau} , \qquad \text{and} \qquad \nabla_{\mathbf{V}} \mathbf{V} = 0 . \tag{17}$$

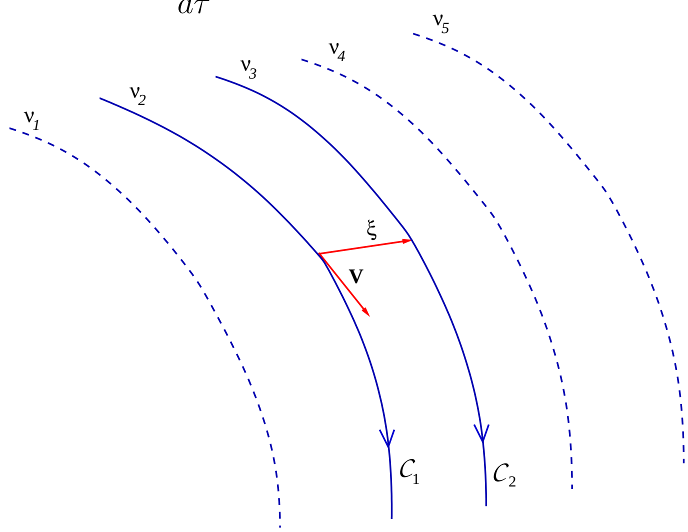

Figure 3: Congruence of geodesic curves of parameter τ and tangent vector $\mathbf{V}$, where each geodesic is distinguished by the value of a parameter ν so that the 4-vector $\boldsymbol{\xi}$ represents the separation vector.

Each geodesic is distinguished by the value of a parameter ν so that the separation vector $\boldsymbol{\xi}$ between two adjacent geodesics is given by

$$\xi^{\alpha} \equiv \frac{dx^{\alpha}}{d\nu} . \tag{18}$$

It is now not difficult to show that the commutator between $\mathbf{V}$ and $\boldsymbol{\xi}$ is given by

$$\begin{aligned} [\mathbf{V}, \boldsymbol{\xi}]^{\alpha} &\equiv V^{\beta}\partial_{\beta}\xi^{\alpha} - \xi^{\beta}\partial_{\beta}V^{\alpha} = \\ &= \left(\frac{dx^{\beta}}{d\tau}\frac{\partial}{\partial x^{\beta}}\right)\xi^{\alpha} - \left(\frac{dx^{\beta}}{d\nu}\frac{\partial}{\partial x^{\beta}}\right)V^{\alpha} = \\ &= \frac{d^2x^{\alpha}}{d\tau d\nu} - \frac{d^2x^{\alpha}}{d\nu d\tau} = 0 , \end{aligned} \tag{19}$$

from which we can deduce that the Lie derivative of $\boldsymbol{\xi}$ along $\mathbf{V}$ is also zero, *i.e.*

$$\mathcal{L}_{\mathbf{V}}\boldsymbol{\xi} \equiv [\mathbf{V}, \boldsymbol{\xi}] = 0 . \tag{20}$$

A more extended discussion of the Lie derivatives and of its properties is presented in the Appendix 1.4.

We can now use the result (20) to show that

$$\begin{aligned} \mathcal{L}_{\mathbf{V}}\xi^{\beta} &= V^{\alpha}\partial_{\alpha}\xi^{\beta} - \xi^{\alpha}\partial_{\alpha}V^{\beta} = \\ &= \xi^{\beta}{}_{,\alpha}V^{\alpha} - V^{\beta}{}_{,\alpha}\xi^{\alpha} = \\ &= \xi^{\beta}{}_{;\alpha}V^{\alpha} - V^{\beta}{}_{;\alpha}\xi^{\alpha} = \\ &= \nabla_{\mathbf{V}}\xi^{\beta} - \nabla_{\boldsymbol{\xi}}V^{\beta} = 0 \,, \end{aligned} \tag{21}$$

where we have used the property that in taking the Lie derivative of a contravariant vector the partial derivatives can be replaced by the covariant derivatives[1].

Stated it differently, expression (21) shows that the parallel transport of $\boldsymbol{\xi}$ along $\mathbf{V}$ is equivalent to the parallel transport of $\mathbf{V}$ along $\boldsymbol{\xi}$, *i.e.*

$$\nabla_{\mathbf{V}}\boldsymbol{\xi} = \nabla_{\boldsymbol{\xi}}\mathbf{V} \,.$$

Recalling that we are interested in finding out about the geodesic deviation through the second-order covariant derivative, we can now use (1.3) to calculate the double covariant derivative of $\boldsymbol{\xi}$ along $\mathbf{V}$, which turns out to be

$$\nabla_{\mathbf{V}}(\nabla_{\mathbf{V}}\boldsymbol{\xi}) = \nabla_{\mathbf{V}}(\nabla_{\boldsymbol{\xi}}\mathbf{V}) \,. \tag{23}$$

At this point it is quite easy to appreciate the similarities between the left-hand-side of expression (23) and the corresponding lef-thand-side of the Newtonian equation (10). In particular, the following association is quite natural

$$\eta^{a} : (\text{separation of particles}' \text{ trajectories}) \quad \rightsquigarrow \quad \xi^{\alpha} : (\text{separation of geodesics}) \,, \tag{24}$$

[1] This property is rooted in the symmetry of the lower indices of the Christoffel symbols $\Gamma^{\alpha}{}_{\beta\mu}$, *i.e.*

$$\mathcal{L}_{\mathbf{V}}\xi^{\alpha} = \xi^{\alpha}{}_{;\beta}V^{\beta} - V^{\alpha}{}_{;\beta}\xi^{\beta} = \xi^{\alpha}{}_{,\beta}V^{\beta} - \xi^{\alpha}{}_{,\beta}V^{\beta} + \Gamma^{\alpha}{}_{\beta\mu}\left[V^{\beta}\xi^{\mu} - V^{\mu}\xi^{\beta}\right] = \xi^{\alpha}{}_{,\beta}V^{\beta} - \xi^{\alpha}{}_{,\beta}V^{\beta} \,. \tag{22}$$

as well as

$$t : (\text{parameter along the particles}' \text{ trajectories}) \quad \leftrightsquigarrow \quad \tau : (\text{parameter along the geodesics}) \,, \tag{25}$$

from which it follows that

$$\ddot{\eta}^a \qquad \leftrightsquigarrow \qquad \nabla_{\mathbf{V}} \nabla_{\mathbf{V}} \xi^\alpha = \frac{D^2}{D\tau^2} \xi^\alpha \,, \tag{26}$$

where we have introduced the symbol D/Dx^α to indicate the covariant derivative along x^α and $D/D\tau$ as a shorthand for the directional covariant derivative along the tangent vector **V**, *i.e.*

$$_{;\alpha} = \frac{D}{Dx^\alpha} \,, \tag{27}$$

$$\nabla_{\mathbf{V}} = V^\alpha \frac{D}{Dx^\alpha} = \frac{dx^\alpha}{d\tau} \frac{D}{Dx^\alpha} = \frac{D}{D\tau} \,, \tag{28}$$

so that

$$\frac{D^2 \xi^\alpha}{D\tau^2} \equiv \frac{d^2 \xi^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{d\xi^\mu}{d\tau} \frac{d\xi^\nu}{d\tau} \,. \tag{29}$$

At this point we need to quantify $\nabla_{\mathbf{V}} \nabla_{\mathbf{V}} \xi^\alpha$ and fortunately this is rather easy if one uses the following identity (prove it!)[2]

$$2\nabla_{[\mathbf{X}} \nabla_{\mathbf{Y}]} Z^\alpha - \nabla_{[\mathbf{X},\mathbf{Y}]} Z^\alpha = \nabla_{\mathbf{X}} (\nabla_{\mathbf{Y}} Z^\alpha) - \nabla_{\mathbf{Y}} (\nabla_{\mathbf{X}} Z^\alpha) - \nabla_{[\mathbf{X},\mathbf{Y}]} Z^\alpha = R^\alpha_{\ \beta\mu\nu} Z^\beta X^\mu Y^\nu \,. \tag{32}$$

where **X**, **Y** are generic 4-vectors.

If we now take $\mathbf{X} = \mathbf{Z} = \mathbf{V}$ and $\mathbf{Y} = \boldsymbol{\xi}$, then $\nabla_{[\mathbf{V},\boldsymbol{\xi}]} V^\alpha = 0$ (*i.e.* $\mathbf{V}$ and $\boldsymbol{\xi}$ commute) and expression (32) simplifies into

$$2\nabla_{[\mathbf{V}} \nabla_{\boldsymbol{\xi}]} V^\alpha = R^\alpha_{\ \beta\mu\nu} V^\beta V^\mu \xi^\nu \,, \tag{33}$$

[2] In your proof bear in mind that

$$2\nabla_{[\mathbf{X}} \nabla_{\mathbf{Y}]} Z^\alpha = \nabla_{\mathbf{X}} (\nabla_{\mathbf{Y}} Z^\alpha) - \nabla_{\mathbf{Y}} (\nabla_{\mathbf{X}} Z^\alpha) \,, \tag{30}$$

and that

$$\nabla_{[\mathbf{X},\mathbf{Y}]} Z^\alpha = X^\mu (\nabla_\mu Y^\nu)(\nabla_\nu Z^\alpha) - Y^\mu (\nabla_\mu X^\nu)(\nabla_\nu Z^\alpha) \,. \tag{31}$$

Exploiting now the fact that $\nabla_{\rm v} V^{\alpha} = 0$ ($\mathbf{V}$ is the tangent vector of a geodesic curve) and $\nabla_{\rm v}\nabla_{\boldsymbol{\xi}}\mathbf{V} = \nabla_{\rm v}\nabla_{\rm v}\boldsymbol{\xi}$ [*i.e.* equation (23)], the left-hand-side of equation (33) can be written explicitely as

$$\nabla_{\rm v}\nabla_{\boldsymbol{\xi}}V^{\alpha} - \nabla_{\boldsymbol{\xi}}\nabla_{\rm v}V^{\alpha} = \nabla_{\rm v}\nabla_{\boldsymbol{\xi}}V^{\alpha} = \nabla_{\rm v}\nabla_{\rm v}\xi^{\alpha}. \tag{34}$$

thus finally leading to the expression for the equation of *geodesic deviation*

$$\nabla_{\rm v}\nabla_{\rm v}\xi^{\alpha} = R^{\alpha}_{\ \beta\mu\nu}V^{\beta}V^{\mu}\xi^{\nu}\ . \tag{35}$$

This tells us about the "forces" acting on an object, *i.e.* how geodesics change their distance as they whirl in spacetime.

Recalling expression (27) for the directional covariant derivative along the tangent vector $\mathbf{V}$, then equation (35) can be also written as

$$\frac{D^2\xi^{\alpha}}{D\tau^2} = -K^{\alpha}_{\ \mu}\xi^{\mu}\ , \tag{36}$$

where we have exploited the antisymmetry on the last two indices of the Riemann tensor[3] to rewrite (35) as

$$\nabla_{\rm v}\nabla_{\rm v}\xi^{\alpha} = -R^{\alpha}_{\ \beta\nu\mu}V^{\beta}\xi^{\nu}V^{\mu} = -R^{\alpha}_{\ \beta\mu\nu}V^{\beta}\xi^{\mu}V^{\nu}\ , \tag{39}$$

and where we have defined

$$K^{\alpha}_{\ \mu} \equiv R^{\alpha}_{\ \beta\mu\nu}V^{\beta}V^{\nu}\ . \tag{40}$$

The analogies between the Newtonian expression for the variation of the freely falling particles (10) and the general relativistic expression for the geodesic deviation (36) is particularly apparent when we compare side-by-side the two expressions, i.e

$$\underset{\text{(Newtonian)}}{\frac{d^2\eta^{a}}{dt^2} = -K^{a}_{\ b}\eta^{b}\ ,} \qquad\qquad \underset{\text{(general relativistic)}}{\frac{D^2\xi^{\alpha}}{D\tau^2} = -K^{\alpha}_{\ \beta}\xi^{\beta}\ .} \tag{41}$$

[3] I recall that the Riemann tensor satisfies the following identities:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}\ , \qquad\qquad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\nu\mu}\ , \tag{37}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}\ , \qquad\qquad R_{\alpha[\beta\mu\nu]} = 0\ , \tag{38}$$

A number of considerations are worth making at this point. Firstly, as in Newtonian gravity, equation (35) expresses the fact that the separation between two adjacent geodesics will vary if they move in a spacetime with nonzero curvature (*i.e.* $K^{\alpha}_{\ \beta} \neq 0$). Secondly, since $D^2\xi^{\alpha}/D\tau^2 = 0$ if and only if $K^{\alpha}_{\ \beta} = 0$, equation (35) underlines that only in a flat spacetime two geodesics will remain parallel (*i.e.* with constant separation). Thirdly, note how in General Relativity the limits of a uniform gravitational field (*i.e.* $\partial_a \Phi = 0$) and of a zero gravitational field (*i.e.* $\Phi = 0$) merge into the limit of flat spacetime. Finally, it is important to underline that equation (35) does not describe the force acting on a particle moving along a geodesic. Rather, it expresses how the spacetime curvature influences two nearby geodesics, making them either diverge or converge. Because of this, equation (35) is effectively measuring *tidal effects*.

Equation (35) will be used in Section 2.2 to study the effect that a incident gravitational wave will have on a set of freely falling particles. Before that, however, it is interesting to study the tidal effects in what is (in some sense) the simplest example of a curved spacetime: *i.e.*, that of a Schwarzschild black hole.

1.4 Tidal Forces in a Schwarzschild Spacetime

The Schwarzschild solution is one of the best-known exact solutions of the Einstein equations and was derived a few months after the theory was proposed. Consider, therefore, a spherical coordinate system (t, r, θ, ϕ) in vacuum. Impose the constraints that the metric is spherically symmetric and static (*i.e.* none of the functions $g_{\mu\nu}$ depends on t, θ, ϕ) and that the spacetime is asymptotically flat (*i.e.* $g_{\mu\nu} = 1$ for $r \to \infty$). Under these conditions, the solution to the Einstein equations has a line element

$$ds^2 = -dt^2 \left(1 - \frac{2M}{r}\right) + dr^2 \left(1 - \frac{2M}{r}\right)^{-1} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{42}$$

The spacetime described by (42) is that of a Schwarzschild black hole, where M is the "black-hole mass". Note that despite your intuition and the familiar concept of "mass", the Schwarzschild metric is a solution of the Einstein equations in *vacuum*, *i.e.*

$$R_{\mu\nu} = 0 \ . \tag{43}$$

Indeed, the line element (42) is the only spherically symmetric and asymptotically flat solution that the equations admit (this is the thesis of Birkhoff's theorem). Because of this, the spacetime exterior (*i.e.* for $r \geq R_*$, where R_* is the stellar radius) to a relativistic spherical (*i.e.* non rotating) star will also be given by the line element (42) (The interior spacetime, on the other hand, will be different from that of a Schwarzschild black hole and is in general dependent on the stellar structure and equation of state.).

Let us consider what happens to an extended body located outside the black hole *event horizon*, is defined as the position at which the metric function $g_{tt} = 0$, *i.e.* at $r = 2M$. We will also assume that all the particles in the body move along geodesics (*i.e.* the body has zero internal stresses and is infinitely deformable) and monitor how the separation between two nearby geodesics varies in time. In particular, using the same notation defined above, we build the spatial separation vector

$$\eta^\mu \equiv h^\mu_{\ \nu} \xi^\nu \ , \tag{44}$$

where $\boldsymbol{h}$ represents the spatial projection tensor orthogonal to $\boldsymbol{g}$, *i.e.*

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \ , \qquad \text{and} \qquad \boldsymbol{h} \cdot \boldsymbol{g} = 0 \ . \tag{45}$$

Clearly, the spatial part of $\boldsymbol{\eta}$ coincides with the 3-vector $\vec{\eta}$ introduced in Section 1.2.

The solution of the geodesic deviation equation (35) in the spacetime (42) leads to the following expressions for the spatial components of $\boldsymbol{\eta}$

$$\frac{D^2\eta^r}{D\tau^2} = \frac{2M}{r^3}\eta^r \ , \tag{46}$$

$$\frac{D^2\eta^\theta}{D\tau^2} = -\frac{M}{r^3}\eta^\theta \ , \tag{47}$$

$$\frac{D^2\eta^\phi}{D\tau^2} = -\frac{M}{r^3}\eta^\phi \ , \tag{48}$$

where the positive sign indicates a stretching and a negative one a compression in that direction. A schematic view of the geodesic deviation as well as of the deformation produced on a fluid body in the presence of a strong gravitational field, produced for instance by a compact star, are shown in Fig. 4 for a plane at $\theta = \mathrm{const.}$

Two comments are worth making about expressions (46). Firstly the tidal deformation is finite at $r = 2M$ and thus, depending on the black hole mass, the body may well preserve its shape when crossing the event horizon (This ceases to be true for $r = 0$ when the tidal stresses are divergent.). Secondly, the tidal fields at the horizon are larger for smaller black holes. This is simply because

$$\left|\frac{D^2\eta^a}{D\tau^2}\right| \sim \left|\frac{M}{r^3}\eta^a\right| \sim \left|\frac{1}{M^2}\eta^a\right| \qquad \text{at} \qquad r \sim M \ . \tag{49}$$

For this reason, the tidal forces experienced in the vicinity of a supermassive black hole of, say, $10^8 M_\odot$ will be 16 orders of magnitude smaller than the corresponding ones near a stellar-mass black hole.

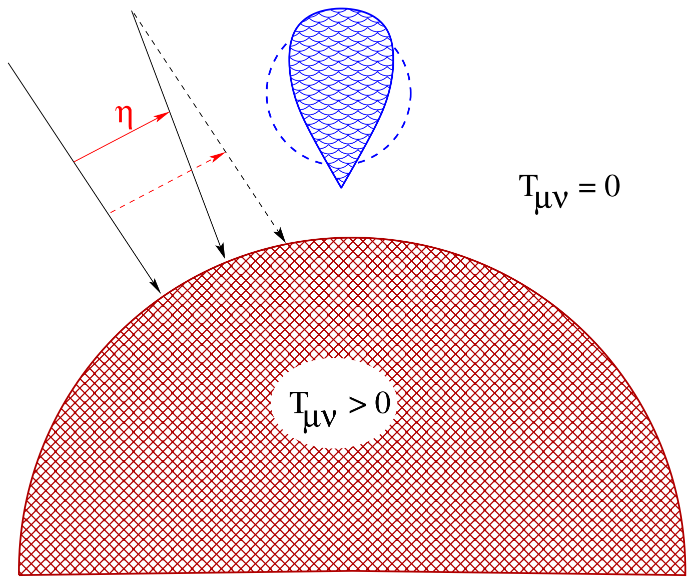

Figure 4: Schematic view of the geodesic deviation as well as of the deformation produced on a fluid body in the presence of a strong gravitational field. In the case considered here the source of the gravitational field is represented by a massive body (*i.e.* $T_{\mu\nu} > 0$) but a qualitative similar scheme would be true also in the case of a black hole.

Appendix: Lie Derivative

That of the Lie derivative is a useful concept which lays the road for the more generic concept of the covariant derivative. In many respects, the Lie derivative can be considered as the extension of the directional derivative (*i.e.* the derivative of along a vector) to the directional derivative in a space where the coordinates are allowed to vary.

Consider therefore a vector field $\mathbf{V}(x^\mu)$ and the family of curves having $\mathbf{V}$ as a tangent vector, *i.e.* (cf. Fig. 1) .

$$V^\mu(x^\mu) \equiv \frac{dx^\mu}{d\tau} \,, \tag{50}$$

where τ is the parameter chosen for the curve (This operation is equivalent to finding the streamlines out of velocity vector field.).

The basic idea behind the Lie derivative is then that of comparing tensors that are *"dragged"* along a certain curve defined by a vector field and taking the limit for infinites-

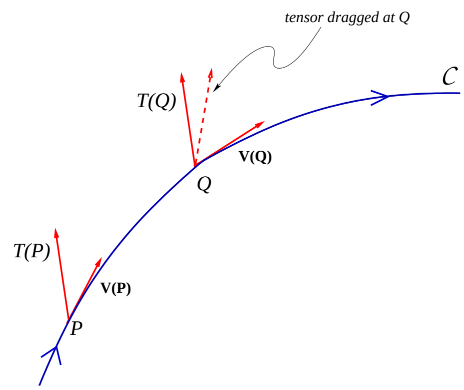

Figure 5: Schematic diagram illustrating how the Lie derivatives compares two tensors at the point Q.

imal displacements, *i.e.*

$$\mathcal{L}_{\mathrm{V}}\,\boldsymbol{T} \equiv \lim_{\delta\tau\to 0} \frac{T^{\nu}(x^{\mu'}) - T^{\nu\,\prime}(x^{\mu'})}{\delta\tau}\ , \tag{51}$$

where $T^{\nu}(x^{\mu'})$ is the tensor $\boldsymbol{T}$ at the point $Q(x^{\mu'})$ and $T^{\nu\,\prime}(x^{\mu'})$ is the tensor $\boldsymbol{T}$ "dragged" at Q. This is illustrate schematically in Fig. 5

The operation of *dragging* is made through a standard coordinate transformation

$$x^{\mu} \longrightarrow x^{\mu'} = x^{\mu} + \delta\tau V^{\mu}, \tag{52}$$

where $\delta\tau V^{\mu}$ is the change of coordinates along the curve. The matrix for the coordinate transformation $\Lambda^{\mu'}{}_{\nu}$ is then expressed as [cf. eq. (52)]

$$\Lambda^{\mu'}{}_{\nu} \equiv \frac{\partial x^{\mu'}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + \delta\tau V^{\mu}{}_{,\nu}\ . \tag{53}$$

so that the tensor $\boldsymbol{T}$ will transform as

$$\begin{aligned} T^{\mu'}(x^{\nu\,\prime}) &\equiv \Lambda^{\mu'}{}_{\nu} T^{\nu} = \\ &= (\delta^{\mu}{}_{\nu} + \delta\tau V^{\mu}{}_{,\nu}) T^{\nu} = \\ &= T^{\mu} + \delta\tau V^{\mu}{}_{,\nu} T^{\nu}\ . \end{aligned} \tag{54}$$

On the other hand, $T^\mu(x')$ can be calculated through a Taylor expansion around the point Q, thus yielding

$$\begin{aligned} T^\mu(x^{\mu'}) &= T^\mu(x^{\nu\,\prime}) = \\ &= T^\mu\left(x^\nu + \delta x^\nu + \mathcal{O}((\delta x^\nu)^2)\right) = \\ &= T^\mu\left(x^\nu + \delta\tau V^\nu + \mathcal{O}(\delta\tau^2)\right) = \\ &= T^\mu(x^\nu) + \delta\tau V^\nu \frac{\partial T^\mu}{\partial x^\nu} + \mathcal{O}(\delta\tau^2)\,. \end{aligned} \tag{55}$$

Collecting now expressions (54) and (55) in the definition (51) and taking the limit for infinitesimal displacements on the curve one obtains that the Lie derivative of $\boldsymbol{T}$ along the vector field $\mathbf{V}$ is

$$(\mathcal{L}_{\mathrm{V}}\,\boldsymbol{T})^\mu = T^\mu{}_{,\nu}V^\nu - V^\mu{}_{,\nu}T^\nu = \mathcal{L}_{\mathrm{V}}T^\mu\,, \tag{56}$$

$$(\mathcal{L}_{\mathrm{V}}\,\boldsymbol{T})_\mu = T_{\mu,\,\nu}V^\nu + V^\nu{}_{,\mu}T_\nu = \mathcal{L}_{\mathrm{V}}T_\mu\,. \tag{57}$$

Note that in addition to the simple derivative along $\mathbf{V}$ (*i.e.* $T^\mu{}_{,\nu}V^\nu$), the Lie derivative (56) also contains a second term (*i.e.* $V^\mu{}_{,\nu}T^\nu$) providing information on how the coordinates themselves change along the curve with tangent vector $\mathbf{V}$.

For a generic mixed tensor with m covariant components and n contravariant ones (*i.e.* $T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}$), the Lie derivative will then be expressed as

$$\begin{aligned} \mathcal{L}_{\mathrm{V}}T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n} &= T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n,\nu}V^\nu - V^{\alpha_1}{}_{,\nu}T^{\nu\alpha_2\ldots\alpha_m}{}_{\beta_1\ldots\beta_n} - \ldots - V^{\alpha_m}{}_{\nu}T^{\alpha_1\ldots\alpha_{m-1}\nu}{}_{\beta_1\ldots\beta_n} - \\ & V_{\beta_1}{}^{,\nu}T^{\alpha_1\ldots\alpha_m}{}_{\nu\beta_2\ldots\beta_n} + \ldots + V_{\beta_n}{}^{,\nu}T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_{n-1}\nu}\,. \end{aligned} \tag{58}$$

1.5 Properties of the Lie derivative

1. *It is a linear operator*

   Given two generic tensors $\boldsymbol{Y}$, and $\boldsymbol{Z}$, the

   $$\mathcal{L}_{\mathrm{V}}(\tau Y^{\alpha\nu} + \sigma Z^{\beta\nu}) = \tau\mathcal{L}_{\mathrm{V}}Y^{\alpha\nu} + \sigma\mathcal{L}_{\mathrm{V}}Z^{\beta\nu}\,. \tag{59}$$

   where τ and σ are two constant real coefficients.

2. *It follows the Leibniz rule*

   Given two generic tensors $\boldsymbol{Z}$, and $\boldsymbol{U}$, then

$$\mathcal{L}_{\mathbf{v}}(Z^{\mu\nu}U_{\alpha\beta}) = \mathcal{L}_{\mathbf{v}}(Z^{\mu\nu})U_{\alpha\beta} + Z^{\mu\nu}\mathcal{L}_{\mathbf{v}}U_{\alpha\beta} \ . \tag{60}$$

3. *It is "type-preserving"*

   Given a generic $\begin{pmatrix} m \\ n \end{pmatrix}$-form, its Lie derivative is still a $\begin{pmatrix} m \\ n \end{pmatrix}$-form

$$\mathcal{L}_{\mathbf{v}}\left[\begin{pmatrix} m \\ n \end{pmatrix} \text{form}\right] = \begin{pmatrix} m \\ n \end{pmatrix} \text{form} \ . \tag{61}$$

   .

4. *Directional derivative*

   When acted upon a scalar function $\Phi(x^\alpha)$, the Lie derivative provides a directional derivative

$$\mathcal{L}_{\mathbf{v}}\Phi = \Phi_{,\nu}V^\nu \ . \tag{62}$$

5. *It commutes with contraction*

   Given a generic tensor $\boldsymbol{T}$ then

$$\delta^\nu{}_\mu(\mathcal{L}_{\mathbf{v}}T^\mu{}_\nu) = \mathcal{L}_{\mathbf{v}}(\delta^\nu{}_\mu T^\mu{}_\nu) = \mathcal{L}_{\mathbf{v}}T^\nu{}_\nu \ . \tag{63}$$

6. *It is equivalent to the commutator operator*

   Given two generic tensors $\boldsymbol{Y}$, and $\boldsymbol{X}$ (hereafter just two 4-vectors), then

$$\mathcal{L}_{\mathbf{X}}Y^\mu = Y^\mu{}_{,\nu}X^\nu - X^\mu{}_{,\nu}Y^\nu = [\mathbf{X}, \mathbf{Y}]^\mu \ . \tag{64}$$

2 Linearized Einstein Equations

The starting point in discussing gravitational waves cannot but come from the Einstein field equations, expressing the close equivalence between matter-energy and curvature

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}\ . \tag{65}$$

In the 10 linearly independent equations (65), $R_{\mu\nu}$ and R are the Ricci tensor and scalar, respectively, $g_{\mu\nu}$ and $G_{\mu\nu}$ are the metric and Einstein tensors, respectively, while $T_{\mu\nu}$ is the stress-energy tensor of the matter in the spacetime considered.

Looking at the Einstein equations (65) as a set of second-order partial differential equations it is not easy to predict that there exist solutions behaving as waves. Indeed, and as it will become more apparent in this Section, the concept of gravitational waves as solutions of Einstein equations is valid only under some rather idealized assumptions such as: a vacuum and asymptotically flat spacetime and a linearized regime for the gravitational fields. If these assumptions are removed, the definition of gravitational waves becomes much more difficult. In these cases, in fact, the full nonlinearity of the Einstein equations complicates the treatment considerably and solutions can be found only numerically. It should be noted, however, that in this respect gravitational waves are not peculiar. Any wave-like phenomenon, in fact, can be described in terms of homogenous wave equations only under very simplified assumptions such as those requiring an uniform "background" for the fields propagating as waves.

These considerations suggest that the search for wave-like solutions to Einstein equations should be made in a spacetime with very modest curvature and with a metric line element which is that of flat spacetime but for small deviations of nonzero curvature, *i.e.*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}([h_{\mu\nu}]^2)\ , \tag{66}$$

where

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)\ , \tag{67}$$

and the linearized regime is guaranteed by the fact that

$$|h_{\mu\nu}| \ll 1 \ . \tag{68}$$

Is this condition verified in Cascina???

Fortunately, the conditions expressed by equations (66) and (68) are, at least in our Solar system, rather easy to reproduce and, in fact, the deviation away from flat spacetime that could be measured, for instance, on the surface of the Sun are

$$|h_{\mu\nu}| \sim |h_{00}| \simeq \frac{M_\odot}{R_\odot} \sim 10^{-6} \,, \tag{69}$$

and which should be compared with the equivalent value as measured on the surface of the Earth $M_\oplus/R_\oplus \sim 10^{-9}$.

Before writing the linearized version of the Einstein equations (65) it is necessary to derive the linearized expression for the Christoffel symbols. In a coordinate basis (as the one will will assume hereafter), the general expression for the affine connection is

$$\Gamma^{\mu}_{\ \alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \,, \tag{70}$$

where the partial derivatives are readily calculated as

$$g_{\nu\alpha,\beta} = \eta_{\nu\alpha,\beta} + h_{\nu\alpha,\beta} = h_{\nu\alpha,\beta} \,, \tag{71}$$

so that the linearized Christoffel symbols become

$$\begin{aligned} \Gamma^{\mu}_{\ \alpha\beta} &= \frac{1}{2}\eta^{\mu\nu}(h_{\nu\alpha,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) = \\ &= \frac{1}{2}(h_{\alpha\ \ ,\beta}^{\ \mu} + h_{\beta\ \ ,\alpha}^{\ \mu} - h_{\alpha\beta}^{\ \ \ ;\mu}) \,. \end{aligned} \tag{72}$$

Note that the operation of lowering and raising the indices in expression (72) is not made through the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ but, rather, through the spacetime metric tensors $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$. This is just the consequence of linearized approximation and, despite this, the spacetime is really curved!

Once the linearized Christoffel symbols have been computed, it is possible to derive the linearized expression for the Ricci tensor which takes the form

$$\begin{aligned} R_{\mu\nu} &= \Gamma^{\alpha}_{\ \mu\nu,\alpha} - \Gamma^{\alpha}_{\ \mu\alpha,\nu} \\ &= \frac{1}{2}(h_{\mu\ \ ,\nu\alpha}^{\ \alpha} + h_{\nu\ \ ,\mu\alpha}^{\ \alpha} - h_{\mu\nu,\ \ \alpha}^{\ \ \ \ \alpha} - h_{,\mu\nu}) \,, \end{aligned} \tag{73}$$

where

$$h \equiv h^{\alpha}_{\ \ \alpha} = \eta^{\mu\alpha} h_{\mu\alpha} \ , \tag{74}$$

is the trace of the metric perturbations. The resulting Ricci scalar is then given by

$$R \equiv g^{\mu\nu} R_{\mu\nu} \simeq \eta^{\mu\nu} R_{\mu\nu} \ . \tag{75}$$

Making now use of (73) and (75) it is possible to rewrite the Einstein equations (65) in a linearized form as

$$h_{\mu\alpha,\ \nu}^{\ \ \ \ \alpha} + h_{\nu\alpha,\ \mu}^{\ \ \ \ \alpha} - h_{\mu\nu,\ \alpha}^{\ \ \ \ \alpha} - h_{,\mu\nu} - \eta_{\mu\nu}(h_{\alpha\beta,}^{\ \ \ \ \alpha\beta} - h_{,\alpha}^{\ \ \alpha}) = 16\pi T_{\mu\nu} \ . \tag{76}$$

Although linearized, the Einstein equations (76) do not seem yet to suggest a wave-like behaviour. A good step in the direction of unveiling this behaviour can be made if a more compact notation is introduced and which makes use of *"trace-free" tensors* defined as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \ , \tag{77}$$

where the "bar-operator" can be applied to any symmetric tensor so that, for instance, $\bar{R}_{\mu\nu} = G_{\mu\nu}$ and $\bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$[4]. Using this notation, the linearized Einstein equations (76) take the more compact form

$$-\bar{h}_{\mu\nu,\alpha}^{\ \ \ \ \ \alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}^{\ \ \ \ \alpha\beta} + \bar{h}_{\nu\alpha,\ \mu}^{\ \ \ \ \alpha} = 16\pi T_{\mu\nu} \ . \tag{78}$$

It is now straightforward to recognize that the first term on the right-hand-side of equation (78) is simply the Dalambertian (or wave) operator

$$\bar{h}_{\mu\nu,\alpha}^{\ \ \ \ \ \alpha} = \Box \bar{h}_{\mu\nu} = -(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\bar{h}_{\mu\nu} \ , \tag{79}$$

where the last equality is valid for a Cartesian (t, x, y, z) coordinate system only. At this stage the gauge freedom inherent to General Relativity can (and should) be exploited to recast equations (79) in a more convenient form. A good way of exploiting this gauge freedom is by choosing the metric perturbations $h_{\mu\nu}$ so as to eliminate the terms in (78)

[4]Note that the "bar" operator can in principle be applied also to the trace so that $\bar{h} = -h$

that spoil the wave-like structure. Most notably, the metric perturbations can be selected so that

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0 \,. \tag{80}$$

Making use of the gauge (80), which is also known as *"Lorentz"* (or Hilbert) gauge, the linearized field equations take the form

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \,. \tag{81}$$

Despite they are treated in a linearized regime and with a proper choice of variables and gauges, Einstein equations (81) do not *yet* represent wave-like equations if matter is present (*i.e.* if $T_{\mu\nu} \neq 0$). A further and final step needs therefore to be taken and this amounts to consider a spacetime devoid of matter, in which the Einstein equations can finally be written as

$$\Box\bar{h}_{\mu\nu} = 0 \,, \tag{82}$$

indicating that, in the Lorentz gauge, the "gravitational field" propagates in spacetime as a wave perturbing flat spacetime.

Having recast the Einstein field equations in a wave-like form has brought us just half-way towards analysing the properties of these objects. More will be needed in order to discuss the nature and features of gravitational waves and this is what is presented in the following Section.

2.1 A Wave Solution to Einstein Equations

The simplest solution to the linearized Einstein equations (82) is that of a plane wave of the type

$$\bar{h}_{\mu\nu} = \Re \left\{ A_{\mu\nu} \exp(i\kappa_\alpha x^\alpha) \right\} , \tag{83}$$

where $\Re$ selects the real part, $\boldsymbol{A}$ is the *"amplitude tensor"*, and $\boldsymbol{\kappa}$ is a null four-vector, *i.e.* $\kappa^\alpha \kappa_\alpha = 0$. In such a solution, the plane wave (83) travels in the spatial direction $\vec{k} = (\kappa_x, \kappa_y, \kappa_z)/\kappa^0$ with frequency $\omega \equiv \kappa^0 = (\kappa^j \kappa_j)^{1/2}$.

Note that the amplitude tensor $\boldsymbol{A}$ in the wave solution (83) has in principle $16 - 6 = 10$ independent components. On the other hand, a number of considerations indicate that there are only two dynamical degrees of freedom in General Relativity. This "excess" of independent components can be explained simply. Firstly, $\boldsymbol{A}$ and $\boldsymbol{\kappa}$ cannot be arbitrary if they have to describe a plane wave; as a result, an orthogonality condition between the two quantities will constrain *four* of the *ten* components of $\boldsymbol{A}$ (see condition *(a)* below). Secondly, while a global Lorentz gauge has been chosen [*cf.* equation (80)], this does *not* fix completely the coordinate system of a linearized theory. A residual ambiguity, in fact, is preserved through arbitrary *"gauge changes"*, *i.e.* through infinitesimal coordinate transformations that are not entirely constrained, even if a global gauge has been selected. To better appreciate this, consider an infinitesimal coordinate transformation in terms of a small but otherwise arbitrary displacement four-vector $\boldsymbol{\xi}$

$$x^{\alpha'} = x^\alpha + \xi^\alpha . \tag{84}$$

Applying this transformation to the linearized metric (66) generates a "new" metric tensor that at the lowest order is

$$g^{\text{NEW}}_{\mu'\nu'} = \eta_{\mu\nu} + h^{\text{OLD}}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} , \tag{85}$$

so that the "new" and "old" perturbations are related by the following expression

$$h^{\text{NEW}}_{\mu'\nu'} = h^{\text{OLD}}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} . \tag{86}$$

or, alternatively, by

$$\bar{h}^{\rm NEW}_{\mu'\nu'} = \bar{h}^{\rm OLD}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} \; . \tag{87}$$

Requiring now that the new coordinates satisfy the condition (80) of the Lorentz gauge $\bar{h}^{\rm NEW}_{\mu'\nu'}{}^{,\nu'} = 0$, forces the displacement vector to be solution of the homogeneous wave equation

$$\xi^{\alpha,\beta}{}_{\beta} = 0 \; . \tag{88}$$

As a result, the plane-wave vector with components

$$\xi^{\alpha} \equiv -iC^{\alpha}\exp(i\kappa_{\beta}x^{\beta}) \; , \tag{89}$$

generates, through the *four* arbitrary constants C^{α}, a gauge transformation that changes arbitrarily *four* components of $\boldsymbol{A}$ in addition to those coming from the condition $\boldsymbol{A}\cdot\boldsymbol{\kappa} = 0$. Effectively, therefore, $A_{\mu\nu}$ has only $10 - 4 - 4 = 2$ linearly independent components, corresponding to the number of degrees of freedom in General Relativity [1].

Note that all this is very similar to what happens in classical electrodynamics, where the Maxwell equations are invariant under transformations of the vector potentials of the type $A_{\mu} \to A_{\mu'} = A_{\mu} + \Psi_{,\mu}$, so that the corresponding electromagnetic tensor $F^{\rm NEW}_{\mu'\nu'} = A_{\mu',\nu'} - A_{\nu',\mu'} = F^{\rm OLD}_{\mu'\nu'}$. Similarly, in a linearized theory of General Relativity, the gauge transformation (86) will preserve the components of the Riemann tensor, *i.e.* $R^{\rm NEW}_{\alpha\beta\mu\nu} = R^{\rm OLD}_{\alpha\beta\mu\nu}$.

To summarize, it is convenient to constrain the components of the amplitude tensor through the following conditions:

(a): **Orthogonality Condition**: *four* components of the amplitude tensor can be specified if $\boldsymbol{A}$ and $\boldsymbol{\kappa}$ are chosen to be orthogonal, *i.e.*

$$A_{\mu\nu}\kappa^{\nu} = 0 \; . \tag{90}$$

(b): **Global Lorentz Frame**: just like in Special Relativity, a global Lorentz frame relative

to an observer with four-velocity $\mathbf{u}$ can be defined. In this case, *three*[5] components of the amplitude tensor can be specified after selecting a four-velocity $\mathbf{u}$ orthogonal to $\boldsymbol{A}$, *i.e.*

$$A_{\mu\nu}u^{\nu} = 0 \,. \tag{91}$$

(c): **Infinitesimal Gauge Transformation**: *one* final independent component in the amplitude tensor can be eliminated after selecting the infinitesimal displacement vector $\xi^{\mu} = iC^{\mu}\exp(i\kappa_{\alpha}x^{\alpha})$ so that

$$A^{\mu}{}_{\mu} = 0 \,. \tag{92}$$

Conditions *(a)*, *(b)* and *(c)* define the so called *"Transverse and Traceless" (TT)* gauge and represent the standard gauge for the analysis of gravitational waves.

To appreciate the significance of the constraint conditions (90)–(92), consider them implemented in a reference frame which is globally at rest, *i.e.* $u^{\alpha} = (1, 0, 0, 0)$. In this frame, the components of the wave vector κ^{μ} do not appear directly, and the above conditions for the amplitude tensor can be written as

(a):

$$A_{\mu\nu}\kappa^{\nu} = 0 \qquad \Longleftrightarrow \qquad h_{ij,j} = 0 \,, \tag{93}$$

i.e. the spatial components of $h_{\mu\nu}$ are *divergence-free*.

(b):

$$A_{\mu\nu}u^{\nu} = 0 \qquad \Longleftrightarrow \qquad h_{\mu 0} = 0 \,, \tag{94}$$

i.e. only the spatial components of $h_{\mu\nu}$ are *nonzero* (hence the *transverse* character of the TT-gauge).

[5] Note that the conditions (90) fix three and not four components because one further constraint needs to be satisfied, *i.e.* $\kappa^{\mu}A_{\mu\nu}u^{\nu} = 0$.

(c):

$$A^{\mu}{}_{\mu} = 0 \qquad \Longleftrightarrow \qquad h = h^{j}{}_{j} = 0 \,, \tag{95}$$

i.e. the spatial components of $h_{\mu\nu}$ are ***trace-free*** hence the *trace-free* character of the TT-gauge). Because of this, and only in this gauge, $\bar{h}_{\mu\nu} = h_{\mu\nu}$

An obvious question that might emerge at this point is about the generality of the TT gauge. A simple answer to this question can be provided by reminding that any linear gravitational wave can, just like any electromagnetic wave, be decomposed in the linear superposition of planar waves. Because all of the conditions (94)–(95) are linear in $h_{\mu\nu}$, any of the composing planar waves can be chosen to satisfy (94)–(95), which, as a result, are satisfied also by the original gravitational wave. Indeed, all of what just stated is contained in a theorem establishing that: once a global Lorentz frame has been chosen in which $u^{\alpha} = \delta^{\alpha}{}_{0}$, it is then always possible to find a gauge in which the conditions (94)–(95) are satisfied.

2.2 Making Sense of the TT Gauge

As introduced so far, the TT gauge might appear rather abstract and not particularly interesting. Quite the opposite, the TT gauge introduces a number of important advantages and simplifications in the study of gravitational waves. The most important of these is that, in this gauge, the only nonzero components of the Riemann tensor are

$$R_{j0k0} = R_{0j0k} = -R_{j00k} = -R_{0jk0} \ . \tag{96}$$

Since, however,

$$R_{j0k0} = -\frac{1}{2} h^{\mathrm{TT}}_{jk,00} \ , \tag{97}$$

the use of the TT gauge indicates that a travelling gravitational wave with periodic time behaviour $h^{\mathrm{TT}}_{jk} \propto \exp(i\omega t)$ can be associated to a local oscillation of the spacetime curvature, *i.e.*

$$h^{\mathrm{TT}}_{jk,00} \sim -\omega^2 \exp(i\omega t) \sim R_{j0k0} \ , \tag{98}$$

and thus

$$R_{j0k0} = \frac{1}{2}\omega^2 h^{\mathrm{TT}}_{jk} \ . \tag{99}$$

To better appreciate the effects of the propagation of a gravitational wave, it is useful to consider the separation between two neighbouring particles A and B on a geodesic motion and how this separation changes in the presence of an incident gravitational wave (see Fig. 6). Note that considering a single particle would not be sufficient to establish the effect of an incident gravitational wave. Its coordinate position does not change at the passage of the wave and it is, indeed, the relative displacement between two adjacent particles that allows for the detection of the measurement.

For this purpose, let us introduce a coordinate system $x^{\hat{\alpha}}$ in the neighbourhood of particle A so that along the worldline of the particle A the line element will have the form

$$ds^2 = -d\tau^2 + \delta_{\hat{i}\hat{j}} dx^{\hat{i}} dx^{\hat{j}} + \mathcal{O}(|x^{\hat{j}}|^2) dx^{\hat{\alpha}} dx^{\hat{\beta}} \ , \tag{100}$$

where, of course, we are interested exactly in quantifying the $\mathcal{O}(|x^{\hat{j}}|^2)$ terms.

The arrival of a gravitational wave will perturb the geodesic motion of the two particles and produce a nonzero contribution to the geodesic deviation equation. I remind that the changes in the separation four-vector $\boldsymbol{\xi}$ between two geodesic trajectories with tangent four-vector $\mathbf{V}$ are expressed through the geodesic deviation equation [*cf.* equation (35)]

$$V^{\gamma}V^{\beta}\xi^{\alpha}{}_{;\beta\gamma} = -R^{\alpha}_{\ \beta\gamma\delta}V^{\beta}\xi^{\gamma}V^{\delta}\,, \tag{101}$$

or, equivalently, as

$$V^{\gamma}V^{\beta}\left(\frac{D^2\xi^{\alpha}}{D\tau^2}\right) \equiv V^{\gamma}V^{\beta}\left(\frac{d^2\xi^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{d\xi^{\alpha}}{d\tau}\frac{d\xi^{\beta}}{d\tau}\right) = -R^{\alpha}_{\ \beta\gamma\delta}V^{\beta}\xi^{\gamma}V^{\delta}\,. \tag{102}$$

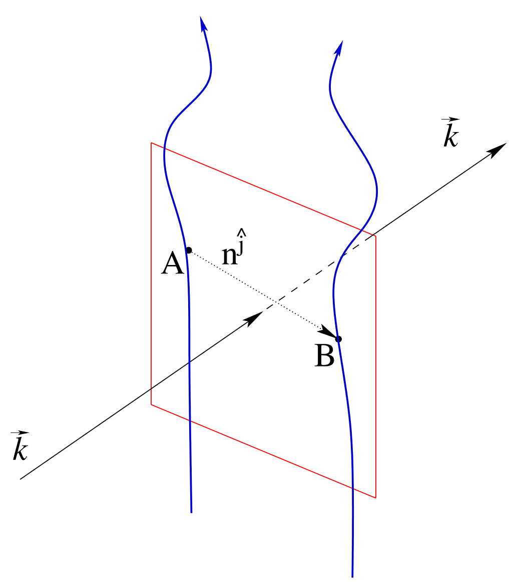

Figure 6: Schematic diagram for the changes in the separation vector between two particles A and B moving along geodesic trajectories produced by the interaction with a gravitational wave propagating along the direction $\vec{k}$.

Indicating now with $n^{\hat{j}}_{\rm B} \equiv h^{\hat{j}}_{\ \hat{\alpha}}\xi^{\hat{\alpha}} = x^{\hat{j}}_{\rm B} - x^{\hat{j}}_{\rm A} = x^{\hat{j}}_{\rm B}$ the components of the separation three-vector in the positions of the two particles, the geodesic deviation equation (102) can be written as

$$\frac{D^2x^{\hat{j}}_{\rm B}}{D\tau^2} = -R^{\hat{j}}_{\ 0\hat{k}0}\,x^{\hat{k}}_{\rm B}\,. \tag{103}$$

A first simplification to these equations comes from the fact that around the particle A the affine connections vanish (*i.e.* $\Gamma^{\hat{j}}_{\hat{\alpha}\hat{\beta}} = 0$) and the covariant derivative in (103) can be replaced by an ordinary total derivative. Furthermore, because in the TT gauge the coordinate system $x^{\hat{\alpha}}$ moves together with the particle A, the proper and the coordinate time coincide at first order in the metric perturbation [*i.e.* $\tau = t$ at $\mathcal{O}(h^{\rm TT}_{\mu\nu})$]. As a result, equation (103) effectively becomes

$$\frac{d^2 x^{\hat{j}}_{\rm B}}{dt^2} = \frac{1}{2}\left(\frac{\partial^2 h^{\rm TT}_{\hat{j}\hat{k}}}{\partial t^2}\right) x^{\hat{k}}_{\rm B} \ , \tag{104}$$

and has solution

$$x^{\hat{j}}_{\rm B}(t) = x^{\hat{k}}_{\rm B}(0)\left[\delta_{\hat{j}\hat{k}} + \frac{1}{2} h^{\rm TT}_{\hat{j}\hat{k}}(t)\right] \ . \tag{105}$$

Equation (105) has a straightforward interpretation and indicates that, in the reference frame comoving with A, the particle B is seen oscillating with an amplitude proportional to $h^{\rm TT}_{\hat{j}\hat{k}}$.

Note that because these are transverse waves, they will produce a local deformation of the spacetime only in the plane orthogonal to their direction of propagation. As a result, if the two particles lay along the direction of propagation (*i.e.* if $\vec{n} \parallel \vec{k}$), then [*cf* eq. (90)]

$$h^{\rm TT}_{\hat{j}\hat{k}}\, x^{\hat{j}}_{\rm B}(0) \propto h^{\rm TT}_{\hat{j}\hat{k}}\, \kappa^{\hat{j}}_{\rm B}(0) = 0 \ , \tag{106}$$

and no oscillation will be recorded by A [*cf.* equation (93)]

Let us now consider a concrete example and in particular a planar gravitational wave propagating in the positive z-direction. In this case the only nonzero metric functions will be

$$h^{\rm TT}_{xx} = -h^{\rm TT}_{yy} = \Re\left\{A_+ \exp[-i\omega(t-z)]\right\} \ , \tag{107}$$

$$h^{\rm TT}_{xy} = h^{\rm TT}_{yx} = \Re\left\{A_\times \exp[-i\omega(t-z)]\right\} \ , \tag{108}$$

where A_+ and $A_\times$ represent the only two independent degrees of freedom.

As in classical electromagnetism, in fact, it is possible to decompose a gravitational wave in two *linearly* polarized plane waves or in two *circularly* polarized ones. In the first case, and for a gravitational wave propagating in the z-direction, the polarization *tensors* $+$ ("plus") and $\times$ ("cross") are defined as

$$\mathbf{e}_{+} \equiv \vec{e}_x \otimes \vec{e}_x - \vec{e}_y \otimes \vec{e}_y \, , \tag{109}$$

$$\mathbf{e}_{\times} \equiv \vec{e}_x \otimes \vec{e}_x + \vec{e}_y \otimes \vec{e}_y \, . \tag{110}$$

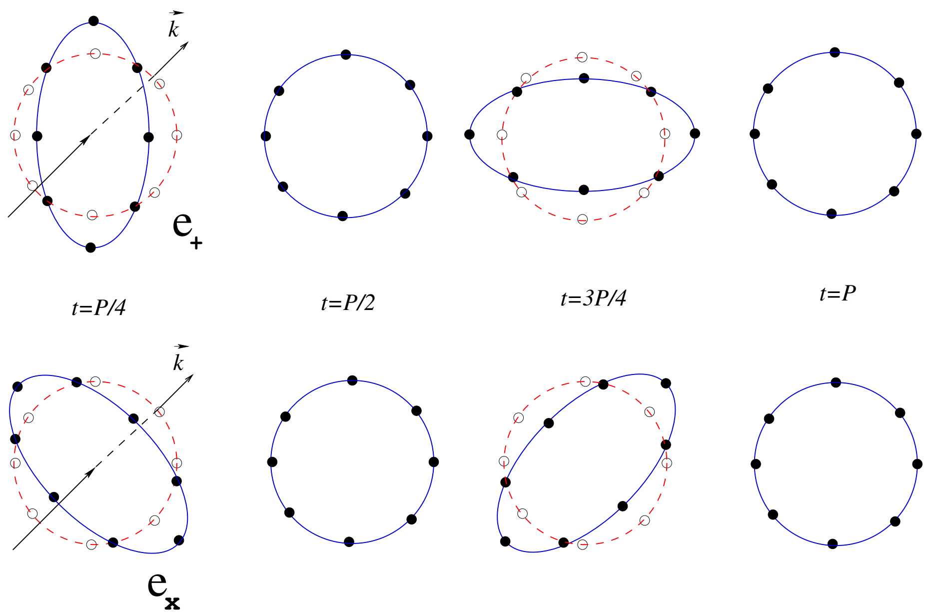

Figure 7: Schematic deformations produced on a ring of freely-falling particles by gravitational waves that are linear polarized in the "+" ("plus") and "×" ("cross") modes. The continuous lines and the dark filled dots show the positions of the particles at different times, while the dashed lines and the open dots show the unperturbed positions.

The deformations that are associated with these two modes of linear polarization are shown in Fig. 7 where the positions of a ring of freely-falling particles are schematically represented at different fractions of an oscillation period. Note that the two linear polarization modes are simply rotated of $\pi/4$.

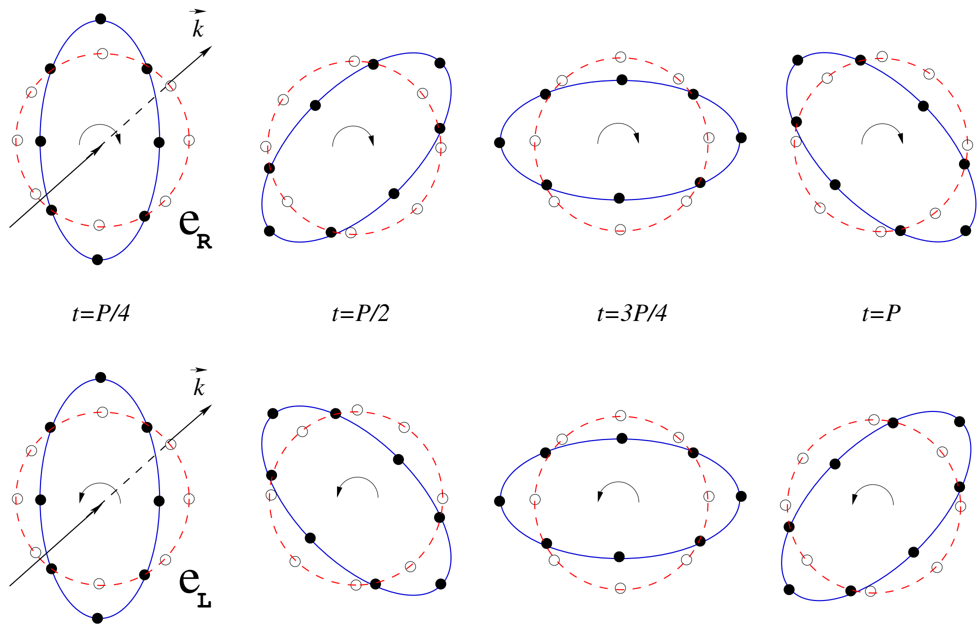

Figure 8: Schematic deformations produced on a ring of freely-falling particles by gravitational waves that are circularly polarized in the R (clockwise) and L (counter-clockwise) modes. The continuous lines and the dark filled dots show the positions of the particles at different times, while the dashed lines and the open dots show the unperturbed positions.

In a similar way, it is possible to define two *tensors* describing the two states of circular polarization and indicate with $\mathbf{e}_{\rm R}$ the circular polarization that rotates clockwise (see Fig. 8)

$$\mathbf{e}_{\rm R} \equiv \frac{\mathbf{e}_{+} + i\mathbf{e}_{\times}}{\sqrt{2}} \,, \tag{111}$$

and with $\mathbf{e}_{\rm L}$ the circular polarization that rotates counter-clockwise (see Fig. 8)

$$\mathbf{e}_{\rm L} \equiv \frac{\mathbf{e}_{+} - i\mathbf{e}_{\times}}{\sqrt{2}} \,. \tag{112}$$

The deformations that are associated to these two modes of circular polarization are shown in Fig. 8

2.3 Generation of Gravitational Waves

In what follows I briefly discuss the amounts of energy carried by gravitational waves and provide simple expressions to estimate the gravitational radiation luminosity of potential sources. Despite the estimates made here come from analogies with electromagnetism and are based on a Newtonian description of gravity, they provide a reasonable approximation to more accurate expressions from which they differ for factors of $\mathcal{O}(\text{few})$. Note also that if obtaining such a level of accuracy requires only a small effort, reaching the accuracy necessary for a realistic detection of gravitational waves is far more difficult and often imposes the use of numerical relativity calculations on modern supercomputers.

In classical electrodynamics, the energy emitted per unit time by an oscillating electric dipole d is easily estimated to be

$$L_{\text{electric dip.}} \equiv \frac{(\text{energy emitted})}{(\text{unit time})} = \frac{2}{3} q^2 \ddot{x}^2 = \frac{2}{3} (\ddot{d})^2 \,, \tag{113}$$

where

$$d \equiv qx \,, \qquad \text{and} \qquad \ddot{d} \equiv q\ddot{x} \,, \tag{114}$$

with q being the electrical charge and the number of "dots" counting the order of the total time derivative. Equally simple is to calculate the corresponding luminosity in gravitational waves produced by an oscillating mass-dipole. In the case of a system of N point-like particles of mass m_A ($A = 1, 2, \ldots, N$), in fact, the total mass-dipole and its first time derivative are

$$\vec{d} \equiv \sum_{A=1}^{N} m_A \vec{x}_A \,, \tag{115}$$

and

$$\dot{\vec{d}} \equiv \sum_{A=1}^{N} m_A \dot{\vec{x}}_A = \vec{p} \,, \tag{116}$$

respectively. However, the requirement that the system conserves its total linear momentum

$$\ddot{\vec{d}} \equiv \dot{\vec{p}} = 0 \, , \tag{117}$$

forces to conclude that $L_{\text{mass dipole}} = 0$, i.e. that there is no mass-dipole radiation in General Relativity (This is equivalent to the impossibility of having electromagnetic radiation from an oscillating electric monopole.). Next, consider the electromagnetic energy emission produced by an oscillating electric quadrupole. In classical electrodynamics, this energy loss is given by

$$L_{\text{electric quad.}} \equiv \frac{1}{20}(\dddot{Q})^2 = \frac{1}{20}(\dddot{Q}_{jk}\dddot{Q}_{jk}) \, , \tag{118}$$

where

$$Q_{jk} \equiv \sum_{A=1}^{N} q_A \left[(x_A)_j (x_A)_k - \frac{1}{3}\delta_{jk}(x_A)_i (x_A)^i \right] \, , \tag{119}$$

is the electric quadrupole for a distribution of N charges $(q_1, q_2, \ldots, q_N)$.

In close analogy with expression (118), the energy loss per unit time due to an oscillating mass quadrupole is calculated to be

$$L_{\text{mass quadrupole}} \equiv \frac{1}{5}\langle \,\dddot{\boldsymbol{I}}^2\, \rangle = \frac{1}{5}\langle \dddot{I\!\!\!-}_{jk}\dddot{I\!\!\!-}_{jk}\rangle \, , \tag{120}$$

where $I\!\!\!-_{jk}$ is the trace-less mass quadrupole (or "reduced" mass quadrupole), defined as

$$\begin{aligned} I\!\!\!-_{jk} &\equiv \sum_{A=1}^{N} m_A \left[(x_A)_j (x_A)_k - \frac{1}{3}\delta_{jk}(x_A)_i (x_A)^i \right] \\ &= \int \rho \left(x_j x_k - \frac{1}{3}\delta_{jk} x_i x^i \right) dV \, , \end{aligned} \tag{121}$$

and the brackets $\langle\,\rangle$ indicate a time average [Clearly, the second expression in (121) refers to a continuous distribution of particles with rest-mass density ρ.].

A crude estimate of the third derivative of the mass quadrupole of the system is given by

$$\dddot{I\!\!\!-}_{jk} \sim \frac{(\text{mass of the system in motion}) \times (\text{size of the system})^2}{(\text{time scale})^3} \sim \frac{MR^2}{\tau^3} \sim \frac{M\langle v^2\rangle}{\tau} \, , \tag{122}$$

where $\langle v\rangle$ is the mean internal velocity. Stated differently,

$$\dddot{I\!\!\!-}_{jk} \sim L_{\text{internal}} \, , \tag{123}$$

where $L_{\rm internal}$ is the power of the system flowing from one part of the system to the other.

As a result, the gravitational-wave luminosity in the quadrupole approximation can be calculated to be

$$L_{\rm mass\ quadrupole} \sim \left(\frac{M\langle v^2\rangle}{\tau}\right)^2 \sim \left(\frac{M}{R}\right)^5 \sim (L_{\rm internal})^2 \ , \tag{124}$$

where the second equality has been derived using the Virial theorem for which the kinetic energy is of the same order of the potential one and thus $M\langle v^2\rangle \sim M^2/R$ and the timescale is set by the frequency of the fundamental mode of oscillation $\tau \sim 1/f \sim \sqrt{R^3/M}$. Note that the apparently weird association of a luminosity to a luminosity squared is the consequence of the use of geometrized units in which the power is dimensionless.

Although extremely simplified, expressions (122) and (124) contain the two most important pieces of information about the generation of gravitational waves. The first one is that the conversion of any type of energy into gravitational waves is, in general, not efficient. To see this it is necessary to bear in mind that expression (120) is in geometrized units and that the conversion to conventional units occurs by multiplying (120) by the factor $L_0 \equiv c^5/G = 2.63 \times 10^{59}\ {\rm erg\ s^{-1}}$. On the other hand, expression (124) shows that $L \ll 1$ and thus the gravitational-wave luminosity in conventional units is suppressed by a coefficient of $O(10^{-59})$.

The second information is about the time variation of the mass quadrupole, which can become considerable only for very *large masses* moving at *relativistic speeds*. Clearly, these conditions cannot be reached by sources in terrestrial laboratories, but can be easily met by astrophysical compact objects, which therefore become the most promising sources of gravitational radiation.

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