Numerical Relativity: Exercises

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Listed below are the exercises that have been assigned during the course and collected according to the lecture in which they were assigned. These exercises can be solved independently or together during the exercise time. Some of these questions could be part of the oral exam.

Lecture I

1. Consider a three-dimensional hypersurface Σ_t representing the set of events having the same value of the time cooridnate t. Given the one-norm $\Omega_\mu \coloneqq A\nabla_\mu t$, prove that $\Omega_\mu\Omega^\mu = 1/g^{tt}$.

Lecture II

1. Let $\boldsymbol{n}$ be the unit timelike normal to Σ_t and $\boldsymbol{\gamma}$, $\boldsymbol{N}$ are respectively the the projector operator orthogonal to Σ_t and along $\boldsymbol{n}$, *i.e.*, $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ and $N^\mu_{\ \nu} = -n^\mu n_\nu$. Show that it is possible to split a covariant tensor $\boldsymbol{W}$ of rank 2 by applying the projection tensor separately on each component of the tensor to obtain
$$W_{\mu\nu} = An_\mu n_\nu + B_\mu n_\nu + n_\mu C_\nu + Z_{\mu\nu}\,, \tag{1}$$
where
$$A \coloneqq W_{\mu\nu}N^{\mu\nu} = W_{\mu\nu}n^\mu n^\nu\,, \qquad B_\mu \coloneqq -\gamma^\alpha_{\ \mu}W_{\alpha\beta}n^\beta\,, \tag{2}$$
$$C_\nu \coloneqq -\gamma^\alpha_{\ \nu}W_{\beta\alpha}n^\beta\,, \qquad Z_{\mu\nu} \coloneqq \gamma^\alpha_{\ \mu}\gamma^\beta_{\ \nu}W_{\alpha\beta}\,. \tag{3}$$
Further show that the decomposition (1) can be written in the so-called *irreducible* form as
$$W_{\mu\nu} = An_\mu n_\nu + B_\mu n_\nu + n_\mu C_\nu + \frac{1}{3}W_{\alpha\beta}h^{\alpha\beta}\gamma_{\mu\nu} + W_{\langle\mu\nu\rangle} + \gamma^\alpha_{\ \mu}\gamma^\beta_{\ \nu}W_{[\alpha\beta]}\,. \tag{4}$$
where $W_{\langle\mu\nu\rangle}$ is the trace-free, symmetric[1] and spatial part of the tensor $\boldsymbol{W}$, namely:
$$W_{\langle\mu\nu\rangle} \coloneqq \gamma_\mu^{\ \alpha}\gamma_\nu^{\ \kappa}W_{(\alpha\kappa)} - \frac{1}{3}W_{\alpha\kappa}h^{\alpha\kappa}\gamma_{\mu\nu}\,. \tag{6}$$

2. Prove that if $\boldsymbol{u}$ is a timelike unit four-velocity (*i.e.*, $u^\mu u_\mu = -1$), the covariant three-velocity defined as
$$v^i = -\frac{\gamma^i_{\ \mu}u^\mu}{n_\mu u^\mu}\,, \tag{7}$$
has components given by
$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)\,. \tag{8}$$

[1] I recall that it is possible to construct a symmetric or antisymmetric tensor from an arbitrary one, *i.e.*,
$$Z_{(\mu\nu)} \coloneqq \frac{1}{2}\left(Z_{\mu\nu} + Z_{\nu\mu}\right)\,, \qquad Z_{[\mu\nu]} \coloneqq \frac{1}{2}\left(Z_{\mu\nu} - Z_{\nu\mu}\right)\,, \tag{5}$$
and that an arbitrary tensor can always be decomposed into its symmetric and antisymmetric parts, *i.e.*, $Z_{\mu\nu} = Z_{(\mu\nu)} + Z_{[\mu\nu]}$.

3. Prove that the quantity $W := \alpha u^t$ is the Lorentz factor since it satisfies the identity
$$W = \left(1 - v^i v_i\right)^{-1/2} . \tag{9}$$
Compare expression (8) with the equivalent expression in special relativity.

4. Recalling that the Schwarzschild solution in quasi-isotropic coordinates reads
$$ds^2 = -\left(\frac{1 - M/(2r)}{1 + M/(2r)}\right) dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right) , \tag{10}$$
compute the components of the one-form $\boldsymbol{\Omega}$, of the unit normal $\boldsymbol{n}$, of the lapse function α, of the shift vector $\boldsymbol{\beta}$, and of the three metric $\boldsymbol{\gamma}$; for all tensors compute both the covariant and the contravariant components.

Lecture III

1. The Misner-Sharp (1964) represents the simplest formulation of the Einstein equations in spherical symmetry. Derive the expressions presented at the lecture starting from the generic diagonal line element in spherical symmetry can then be written in the form
$$ds^2 = -a(r,t)^2 dt^2 + b(r,t)^2 dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{11}$$
where r and t are the radial and time coordinates is the Schwarzschild circumference coordinate, *i.e.*, $4\pi R^2 := \int_{^2\Sigma} \sqrt{^2 g}\, d\sigma$, where $d\sigma$ is the surface element over the two-sphere $^2\Sigma$ with metric determinant $^2 g$.

2. Within a 3+1 split of spacetime, prove the following expression for the extrinsic curvature $\boldsymbol{K}$
$$\mathscr{L}_{\boldsymbol{n}} \gamma_{\mu\nu} = -2K_{\mu\nu} , \tag{12}$$
where $\boldsymbol{\gamma}$ is the metric associated to Σ_t and $\boldsymbol{n}$ the corresponding unit normal.

3. Consider a cylindrical 2-surface in an embedding Eucleadian space, *e.g.*, $\mathbb{R}^3$. Compute the components of the Riemann tensor and of the extrinsic curvature. Compare with the results discussed in the lecture.

4. Consider a 2-sphere in an embedding Eucleadian space, *e.g.*, $\mathbb{R}^3$. Compute the components of the Riemann tensor and of the extrinsic curvature. Compare with the results discussed in the lecture.

Lecture IV

1. Prove that $a_\nu = D_\nu \ln \alpha$.

2. Derive the Gauss–Codazzi equations

$$\gamma^{\mu}{}_{\alpha}\,\gamma^{\nu}{}_{\beta}\,\gamma^{\rho}{}_{\delta}\,\gamma^{\sigma}{}_{\lambda}\,R_{\mu\nu\rho\sigma} = {}^{(3)}R_{\alpha\beta\delta\lambda} + K_{\alpha\delta}K_{\beta\lambda} - K_{\alpha\lambda}K_{\beta\delta}\,. \tag{13}$$

3. Derive the Codazzi-Mainardi equations

$$\gamma^{\rho}{}_{\beta}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\lambda}n^{\sigma}R_{\rho\mu\nu\sigma} = D_\alpha K_{\beta\lambda} - D_\beta K_{\alpha\lambda}\,. \tag{14}$$

4. Derive the Ricci equations

$$\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}n^{\delta}n^{\lambda}R_{\alpha\delta\beta\lambda} = \mathscr{L}_{\boldsymbol{n}}K_{\mu\nu} - \frac{1}{\alpha}D_\mu D_\nu \alpha + K^{\lambda}{}_{\nu}\,K_{\mu\lambda}\,. \tag{15}$$

5. (Optional) Given the conformal transformation $\tilde{\gamma}_{ij} = \phi^2\gamma_{ij}$, $\tilde{\gamma}^{ij} = \phi^{-2}\gamma^{ij}$, show that the second covariant derivative of the conformal factor is given by

$$\tilde{D}_i\tilde{D}_j\phi = \partial_i\partial_j\phi - \tilde{\Gamma}^k_{ij}\,\partial_k\phi\,,$$

and that the relation with the corresponding derivative in the physical metric is given by

$$D_iD_j\phi = \tilde{D}_i\tilde{D}_j\phi + \frac{2}{\phi}\partial_i\phi\,\partial_j\phi - \frac{1}{\phi}\gamma_{ij}\partial^k\phi\,\partial_k\phi\,.$$

In case you are curious, such an expression appears when splitting the three-dimensional Ricci tensor into a part containing conformal terms and another one containing space derivatives of the conformal factor, *i.e.*, ${}^{(3)}R_{ij} = {}^{(3)}\tilde{R}_{ij} + {}^{(3)}\tilde{R}^{\phi}_{ij}$, where

$${}^{(3)}\tilde{R}_{ij} := -\frac{1}{2}\tilde{\gamma}^{lm}\partial_l\partial_m\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}\left(2\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kj\,l}\right)\,, \tag{16}$$

$${}^{(3)}\tilde{R}^{\phi}_{ij} := \frac{1}{\phi^2}\left[\phi\left(\tilde{D}_i\tilde{D}_j\phi + \tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\phi\right) - 2\tilde{\gamma}_{ij}\tilde{D}^k\phi\,\tilde{D}_k\phi\right]\,. \tag{17}$$

Lecture V

1. Derive the form of the Hamiltonian and momentum constraints in a conformal and traceless formulation of the Einstein equations.

2. Let V be the volume enclosed by the three-dimensional surface Σ on a spacelike hypersurface, so that the (proper) volume element is given by

$$V = \int_{\Sigma} \sqrt{\gamma}\, d^3x \,, \tag{18}$$

where, as usual, $\gamma = \det(\gamma_{ij})$. The volume V can be thought as the volume delimited by a closed two-dimensional surface $\mathscr{S}$, where $\mathscr{S}$ is of course part of a two-dimensional surface, say, Σ_0. Show that the variation of V in time when $\mathscr{S}$ remains fixed is given by

$$\partial_t V = -\int_{\Sigma} \alpha K \sqrt{\gamma}\, d^3x \,, \tag{19}$$

where, as usual, α is the lapse and K is the trace of the extrinsic curvature. This expression shows that a slicing with $K = 0$ is "maximal" in the sense that the volume V is an extremal with respect to variations of the domain enclosed by $\mathscr{S}$.

3. The "harmonic slicing" is the slicing requiring that the "harmonic condition"

$$\Box\, x^{\alpha} = \nabla_{\mu}\nabla^{\mu}\, x^{\alpha} = 0 \,. \tag{20}$$

holds only for the time coordinate $x^0 = t$, *i.e.*, that

$$\Box\, t = 0 \,. \tag{21}$$

Show that this condition corresponds to the following prescription for the lapse

$$\left(\partial_t - \beta^i \partial_i \beta\right) \alpha = -\alpha^2 K \,. \tag{22}$$

4. The "minimal distortion condition" imposes that

$$D^j \Sigma_{ij} = 0 \,, \tag{23}$$

where[2]

$$\Sigma_{ij} = \Theta_{ij} - \frac{1}{3}\gamma_{ij}\Theta = \Theta_{ij} - \frac{1}{3}\gamma_{ij}\Theta_{kl}\gamma^{kl} = \frac{1}{2}\gamma^{1/3}\mathscr{L}_{\boldsymbol{t}}\tilde{\gamma}_{ij} \,, \tag{24}$$

is the metric distortion tensor and

$$\Theta_{ij} = \frac{1}{2}\mathscr{L}_{\boldsymbol{t}}\gamma_{ij} = \frac{1}{2}\partial_t \gamma_{ij} \,, \tag{25}$$

[2] Note that Eq. (7.132) of my book contains two (!) typos. The one reported here is the correct expression.

is the metric strain tensor. The "Gamma-freezing" shift condition, on the other hand, imposes that

$$\partial_t \tilde{\Gamma}^i = 0\,. \tag{26}$$

Show that

$$\partial_t \tilde{\Gamma}^i = 2\phi^{-2}\left[D_j \Sigma^{ij} - \tilde{\Gamma}^i{}_{jk}\Sigma^{jk} - 6\Sigma^{ij}\partial_j \phi\right]\,. \tag{27}$$

and hence that the Gamma-freezing shift condition and the minimal-distortion condition are equivalent up to terms involving the conformal factor and its derivatives.

5. (Optional) Work out the conditions for a hyperboloidal slicing of the Minkowski spacetime.

Lecture VI

No exercises for this lecture to allow people to catch-up with unsolved exercises. Note that only the first 3 exercises for each lecture are expected to be solved. The rest is just for fun.

Lecture VII

1. Assuming for simplicity that the flow is one-dimensional (*i.e.*, for $\mu = 0, 1$) and the spacetime flat, we rewrite the conservation equations for energy and linear momentum

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{28}$$

can be written in a Cartesian coordinate system as

$$\partial_t\left[\left(e + pv^2\right)W^2\right] + \partial_x\left[\left(e+p\right)W^2 v\right] = 0\,, \tag{29}$$

$$\partial_t\left[\left(e+p\right)W^2 v\right] + \partial_x\left[\left(ev^2 + p\right)W^2\right] = 0\,, \tag{30}$$

where $u^\mu = W(1, v)$ and $W = (1 - v^2)^{-1/2}$ is the Lorentz factor.

2. Linearize Eqs. (29)–(30) by introducing perturbations of the type

$$e = e_0 + \delta e\,, \qquad p = p_0 + \delta p\,, \qquad v = v_0 + \delta v = \delta v\,, \tag{31}$$

Show that the resulting equations satisfy a wave equation

$$\Box \delta e = 0\,. \tag{32}$$

What are the assumptions needed to derive Eq. (32)? What is the speed of propagation of these waves?

3. The continuity and momentum equations can be written as

$$\partial_t(\rho W) + \partial_x(\rho W v) = 0\,, \tag{33}$$

$$W\partial_t(Wv) + Wv\partial_x(Wv) = -\frac{1}{\rho h}\left[\partial_x p + W^2 v\partial_t p + W^2 v^2 \partial_x p\right]\,. \tag{34}$$

Show that these partial differential equations (that you can try to derive or take as given) can be written as the following ordinary differential equations

$$(v-\xi)\frac{d\rho}{d\xi} + \rho[W^2 v(v-\xi)+1]\frac{dv}{d\xi} = 0\,, \tag{35}$$

$$\rho h W^2 (v-\xi)\frac{dv}{d\xi} + (1-v\xi)\frac{dp}{d\xi} = 0\,, \tag{36}$$

after introducing the similarity variable $\xi :- x/t$ and the following differential operators

$$\partial_t = -\left(\frac{\xi}{t}\right)\frac{d}{d\xi}\,, \qquad\qquad \partial_x = \left(\frac{1}{t}\right)\frac{d}{d\xi}\,. \tag{37}$$

Lecture VIII

1. Show that across a discontinuity surface moving to the right the Rankine-Hugoniot junction conditions

$$[\![\rho u^\mu]\!]\, n_\mu = 0\,, \tag{38}$$

$$[\![T^{\mu\nu}]\!]\, n_\nu = 0\,, \tag{39}$$

where $[\![\phi]\!] = \phi_a - \phi_b$, can be written as

$$J := \rho_a W_a v_a = \rho_b W_b v_b\,, \tag{40}$$

$$\rho_a h_a W_a^2 v_a^2 + p_a = \rho_b h_b W_b^2 v_b^2 + p_b\,, \tag{41}$$

$$\rho_a h_a W_a^2 v_a = \rho_b h_b W_b^2 v_b\,, \tag{42}$$

where $v_{a,b}$ are the fluid velocities as measured in the shock rest frame.

2. Show that Eqs. (40)–(41) can be re-written as

$$\left[\!\left[J^2\right]\!\right] = 0\,, \qquad J^2 = -\frac{[\![p]\!]}{[\![h/\rho]\!]}\,, \qquad [\![hW]\!] = 0\,. \tag{43}$$

3. Show that Eqs (43) can be combined in the famous Taub adiabat

$$\left[\!\left[h^2\right]\!\right] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right)[\![p]\!]\,, \tag{44}$$

Derive the Newtonian limit of Eq. (44) and show it leads to the Hugoniot adiabat

$$\left[\!\left[\epsilon + \frac{p}{\rho}\right]\!\right] = \frac{1}{2}\left(\frac{1}{\rho_a} + \frac{1}{\rho_b}\right)[\![p]\!]\ . \tag{45}$$

Lecture IX

1. Using the differential relation

$$W^2 dv \pm \frac{c_s}{\rho} d\rho = 0\,,$$

and exploiting the isentropic character of simple waves, derive the following expressions for the Riemann invariants

$$\mathcal{J}_\pm := \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) \pm \int \frac{c_s}{\rho} d\rho = \text{const.} \tag{46}$$

and show that it is equivalent to

$$\int \frac{c_s}{e+p} de = \pm\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) + \text{const.} \tag{47}$$

2. Using the junction conditions

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)}\,, \tag{48}$$

$$v_b^2 = \frac{(p_a - p_b)(e_a + p_b)}{(e_a - e_b)(e_b + p_a)}\,, \tag{49}$$

and under the assumption of a highly relativistic shock, a cold fluid ahead of the shock and an ultrarelativistic one behind the shock, *i.e.*,

$$W_a \gg 1, \quad p_a \approx 0, \quad e_a \approx \rho_a, \quad p_b = \frac{e_b}{3}, \tag{50}$$

show that the energy density in the shocked fluid scales like the square of the Lorentz factor of the shock front (with respect to the unshocked fluid).

$$e_b = 2W_a^2 e_a\,. \tag{51}$$

This is a result often used in astrophysical relativistic shocks.

3. Using the junction condition

$$J := \rho_a W_a W_{\scriptscriptstyle S}(V_{\scriptscriptstyle S} - v_a) = \rho_b W_b W_{\scriptscriptstyle S}(V_{\scriptscriptstyle S} - v_b)\,, \tag{52}$$

compute the mass flux J such that the shock velocity V_s is twice the velocity v_a in the unshocked region. Compare it with the corresponding Newtonian mass flux. Which of the two is larger for the same value of v_a?

Lecture X

1. Show that the following definitions of energy-momentum tensor are equivalent

$$T^{\mu\nu} = (e + p)u^\mu u^\nu + pg^{\mu\nu} = \rho h u^\mu u^\nu + pg^{\mu\nu} \,,$$
$$T^{\mu\nu} = En^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu} \,.$$

where E, S^μ and $S^{\mu\nu}$ are the Eulerian energy density, the momentum density and the purely spatial energy-momentum tensor, respectively. Show also that the following definitions are possible for these quantities

$$S^{\mu\nu} = \rho h W^2 v^\mu v^\nu + p\gamma^{\mu\nu} \,,$$
$$S^\mu = \rho h W^2 v^\mu \,,$$
$$E = \rho h W^2 - p \,.$$

2. Show that the following identity holds when considering the left-hand side of the conservative formulation of the momentum-density equation

$$\partial_\mu(\sqrt{-g}T^\mu{}_j) = \partial_t(\sqrt{\gamma}S_j) + \partial_i[\sqrt{\gamma}(\alpha S^i{}_j - \beta^i S_j)] \,.$$

Similarly, show that the right-hand side satisfies the following identity

$$\frac{1}{2}\sqrt{-g}T^{\mu\nu}\partial_j g_{\mu\nu} = \sqrt{-g}\left(\frac{1}{2}S^{ik}\partial_j\gamma_{ik} + \frac{1}{\alpha}S_i\partial_j\beta^i - E\,\partial_j \ln\alpha\right) \,,$$

3. Show that the following identity holds when considering the left-hand side of the conservative formulation of the energy-density equation

$$-\sqrt{-g}\nabla_\mu(T^{\mu\nu}n_\nu) = \partial_t(\sqrt{\gamma}E) + \partial_i[\sqrt{\gamma}(\alpha S^i - \beta^i E)] \,.$$

Similarly, show that the right-hand side satisfies the following identity

$$-\sqrt{-g}T^{\mu\nu}\nabla_\mu n_\nu = \sqrt{-g}\left(K_{ij}S^{ij} - S^i\partial_i \ln\alpha\right) \,.$$

Lectures XI-XII

1. Write explicitly the conservative formulation of the relativistic hydrodynamic equations in the Valencia formulation

$$\partial_t\big(\sqrt{\gamma}\,\boldsymbol{U}\big) + \partial_i\big(\sqrt{\gamma}\,\boldsymbol{F}^i\big) = \boldsymbol{S}\,,$$

where the vector of *conserved variables* $\boldsymbol{U}$ and the corresponding *flux vector* in the i-direction $\boldsymbol{F}^i$ are given by

$$\boldsymbol{U} = \begin{pmatrix} D \\ \\ S_j \\ \\ E \end{pmatrix} := \begin{pmatrix} \rho W \\ \\ \rho h W^2 v_j \\ \\ \rho h W^2 - p \end{pmatrix}\,, \qquad \boldsymbol{F}^i := \begin{pmatrix} \alpha v^i D - \beta^i D \\ \\ \alpha S^i{}_j - \beta^i S_j \\ \\ \alpha S^i - \beta^i E \end{pmatrix}\,,$$

while the source vector has components

$$\boldsymbol{S} := \sqrt{\gamma} \begin{pmatrix} 0 \\ \\ \frac{1}{2}\alpha S^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - E\partial_j \alpha \\ \\ \alpha S^{ij} K_{ij} - S^j \partial_j \alpha \end{pmatrix}\,.$$

2. Show that gloabal order of local accuracy for three numerical solutions at resolutions h, $k = h/2$ and $\gamma = k/2$ is given by the simple expression

$$\tilde{p} = \log_2 |R(h, h/2, h/4)|\,.$$

where $R(h, h/2, h/4)$ is the error ratio for the three resolutions.

3. Show that the second-order accurate finite-difference representation of the second spatial derivative is given by

$$\partial^2_x u\big|^n_j = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)\,.$$

What is the expression for the third derivative $\partial^3_x u\big|^n_j$? What is the accuracy order of your expression?

Lecture XIII

1. Considering the physical and conformal three metrics $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$ with $D_i \gamma_{jk} = 0 = \bar{D}_i \bar{\gamma}_{jk}$, prove that the corresponding Ricci scalars $R, \bar{R}$ associated with the original, and the conformal 3-geometry are related according to the following identity
$$R \; = \; \psi^{-4} \bar{R} - \frac{8}{\psi^5} \bar{D}_i \bar{D}^i \psi \, .$$

2. Prove that for any symmetric trace-free tensor U^{ij} we have
$$D_j U^{ij} \; = \; \psi^{-n} \bar{D}_j (\psi^n U^{ij}) + (10 - n) U^{ij} \bar{D}_j \ln \psi \, .$$

3. Given the *longitudinal operator* $\boldsymbol{L}$ defined as
$$(\boldsymbol{L}W)^{ij} \coloneqq 2D^{(i} W^{j)} - \frac{2}{3} \gamma^{ij} D_k W^k \, .$$
where W^i is a generic vector, show that the following identity holds
$$(\boldsymbol{L}\beta)^{ij} \; = \; \psi^{-4} (\bar{\boldsymbol{L}}\beta)^{ij} \, .$$
Similarly, show that for $\bar{\beta}_i = \psi^{-4} \beta_i$ we have
$$(\boldsymbol{L}\beta)_{ij} \; = \; \psi^{4} (\bar{\boldsymbol{L}}\bar{\beta})_{ij} \, .$$