→ "NUMERICAL RELATIVITY" ←

L. REZZOLLA ← GERMAN - ENGLISH

more info astro.uni-frankfurt.de/rezzolla

Bibliografhy

- "Relativistic Hydrodynamics"; LR, O. Zanotti OUP, 2013
- "Introduction to NR"; Alcubierre, OUP, 2008
- "Numerical Relativity"; Baumgarte, Shapiro, CUP 2011
- "3+1 Formalism in GR"; Gourgoulhon, Springer 2012

- This course is about learning how to solve Einstein field equations and those of relativistic hydrodynamics (MHD)

- Knowledge of GR is required

- 1st part of the course <u>mostly</u> mathematical;
  2nd part of the course <u>mostly</u> numerical

- Course is part of Master in Comput. Physics

- course structured in lectures (Tue 14:00 – 16:00) and exercises (Thur. 09:00 – 11:00); Dr. Galeazzi

- final exam: oral test

- follow all lectures and most importantly ask questions!

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* How many have followed GR?

* know about NR?

* why do we ~~want~~ need to use NR?

- Einstein eqs are highly nonlinear and analytic solutions known only for highly symmetric cases (stationarity/staticity, ...)

- Hydrodynamic eqs. are also highly nonlinear and lead to shock development and discontinuities

* What can we study with NR?

- regimes of highly dynamic and curved spacetimes

- sources of gravitational waves

- high-energy astrophysical phenomena: collapse to black hole, merger of neutron stars, supernova explosions

- NR code is a "virtual laboratory" where to carry out detailed gedanken experiments.

let's recap the eqs we want to solve

Einstein Equations

Newton

(1) $$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$\nabla^2 \phi = 4\pi \rho$$

Einstein tensor

energy mom. tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$R_{\mu\nu}$: Ricci tensor (Riemann tensor)

Note : I will always use geometric units, ie

$G = c = 1$ $(M_\odot = 1)$

There is another set of eqs we need to solve in numerical relativity and these are conservation equations

$$\sum_{\mu=0}^{3} \nabla_\mu T^{\mu\nu} = 0$$

Energy and momentum conservation

(2) $$\nabla_\mu T^{\mu\nu} = 0$$

covariant derivative

Christoffel symbol

$$\nabla_\mu U^\nu = \underbrace{\partial_\mu U^\nu}_{\text{partial deriv.}} + \Gamma^\nu_{\alpha\mu} U^\alpha$$

Conservation of rest mass

$$\nabla_\mu (\rho u^\mu) = 0 \qquad (3)$$

four velocity

rest-mass density

$$\partial_t \rho + \vec{\nabla}(\rho \vec{v}) = 0$$: continuity eq.

(1), (2), (3) + an equation of state (EOS) are the set of eqs. we need to solve.

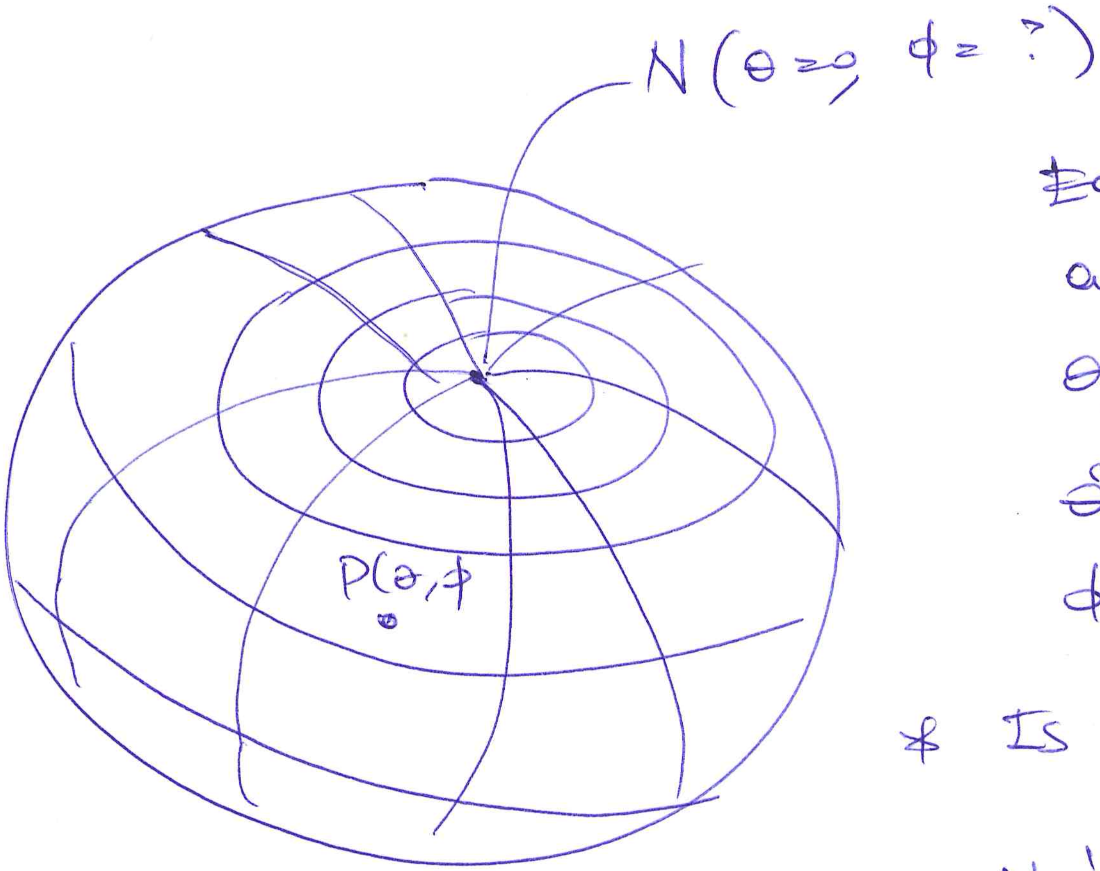
$$p = p(\rho, \epsilon, \dots)$$

- Let's learn how to solve (1). The first problem is that eqs (1) are covariant; ie eqs whose form does not depend on the coordinates chosen.
- This implies there is total freedom in choosing the coordinates and hence we first have to make a proper choice.
- Coordinates are not important in GR
  coordinates are important in NR
  because we need to express a solution in such coordinates

In what follows I will describe a couple of examples of coordinate systems that describe the same geometrical entity but with different levels of success.

I will start by describing a 2-sphere and use the metric on such surface as given by a spherical coordinate system (θ, ϕ) (latitude, ~~and~~ longitude)

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$
$$= g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$
$$= d\Omega^2$$



Each point is marked by a couple of values of θ and ϕ, where

$\theta \in [0, \pi]$

$\phi \in [0, 2\pi]$

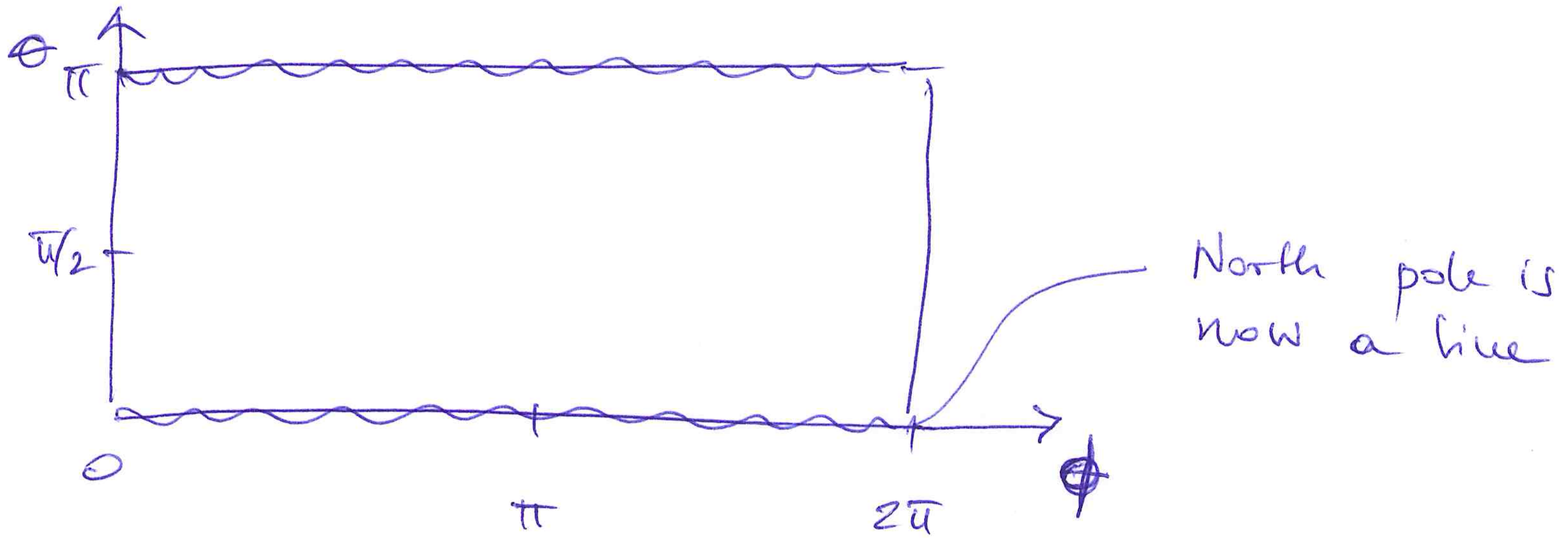
* Is everything well defined?

No! poles are degenerate

$N: (\theta=0, \phi=?)$

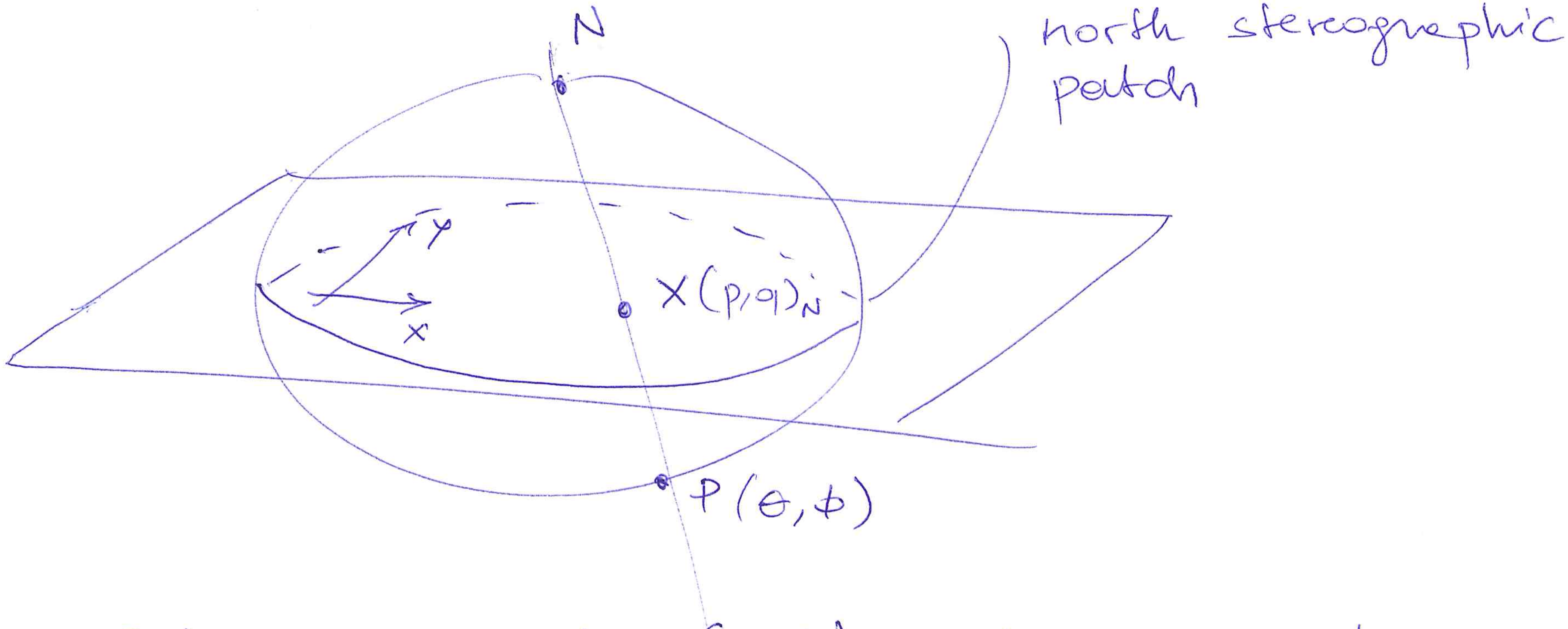
there are ∞ values of ϕ that represent the same point. This is a coordinate degeneracy

Think of stretching the coordinates in a rectangular form



There are even better coordinate systems to describe points on 2-sphere : stereographic patches

This technique maps all point on the sphere on two Cartesian patches



Point P has coordinate (θ, ϕ) and is mapped onto Cartesian coordinate system (x, y); in this way the coordinate degeneracy is removed.
Note that two patches (north and south) are needed :

$$(\theta, \phi) \longrightarrow \begin{array}{l}(p, q)_N \\ (p, q)_S\end{array}$$

The problem of incorrect coordinates can be much more serious than a matter of degeneracy. It can lead to singularity when there are none: coordinate singularity.

Example: Schwarzschild solution in Schwarzschild coordinates

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$r = 2M$ $\quad g_{rr} \to \infty$: coordinate singularity at event horizon. However nothing is wrong there and if you compute the Riemann scalar invariant

$$R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} = \frac{48 M^2}{r^6} \overset{r=2M}{=} \frac{48 M^2}{64 M^6} = \frac{3}{4} M^{-4} \neq \infty$$

$\boxed{\text{No divergence. but at } r=0}$

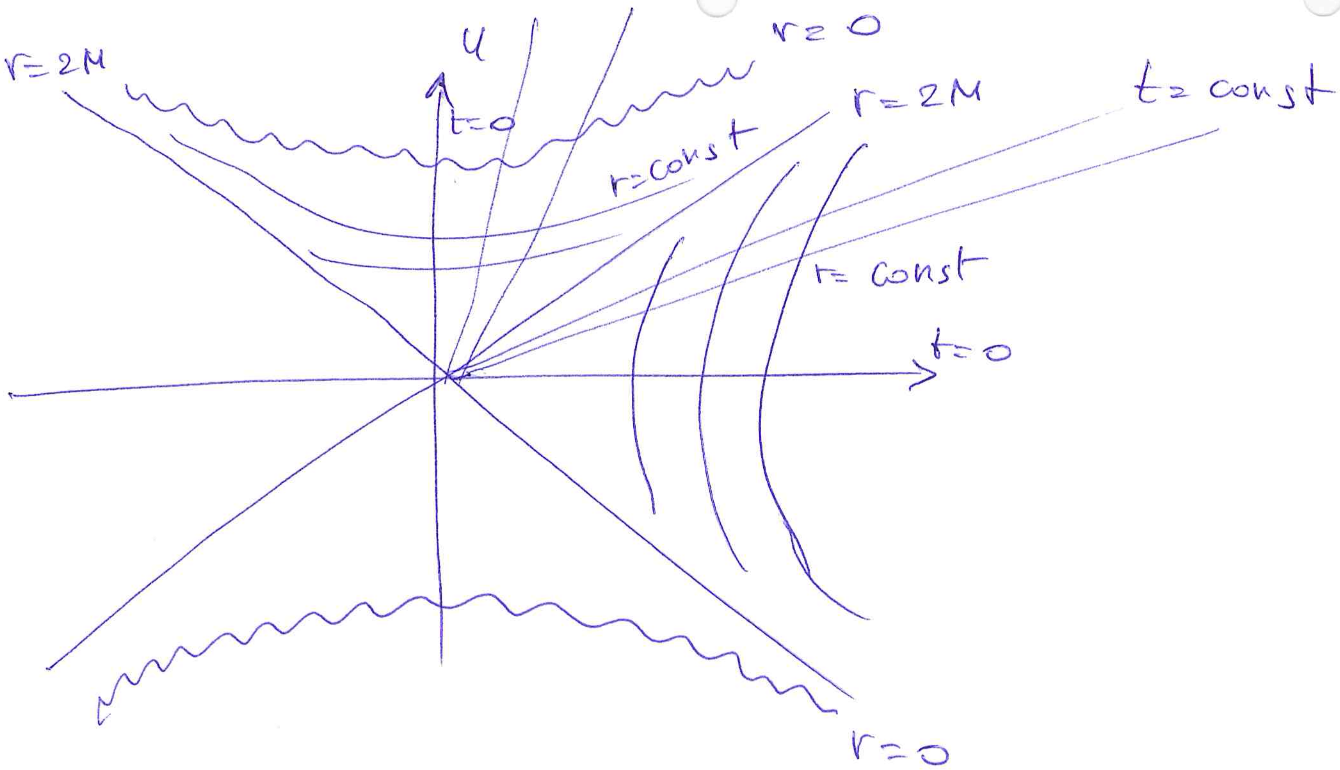
Hence there must be a better coordinate system: Kruskal-Szekeres

$$ds^2 = -\left(\frac{32 M^3}{r}\right) e^{-r/2M} \, d\tilde{v} \, d\tilde{u} + r^2 d\Omega^2$$

where

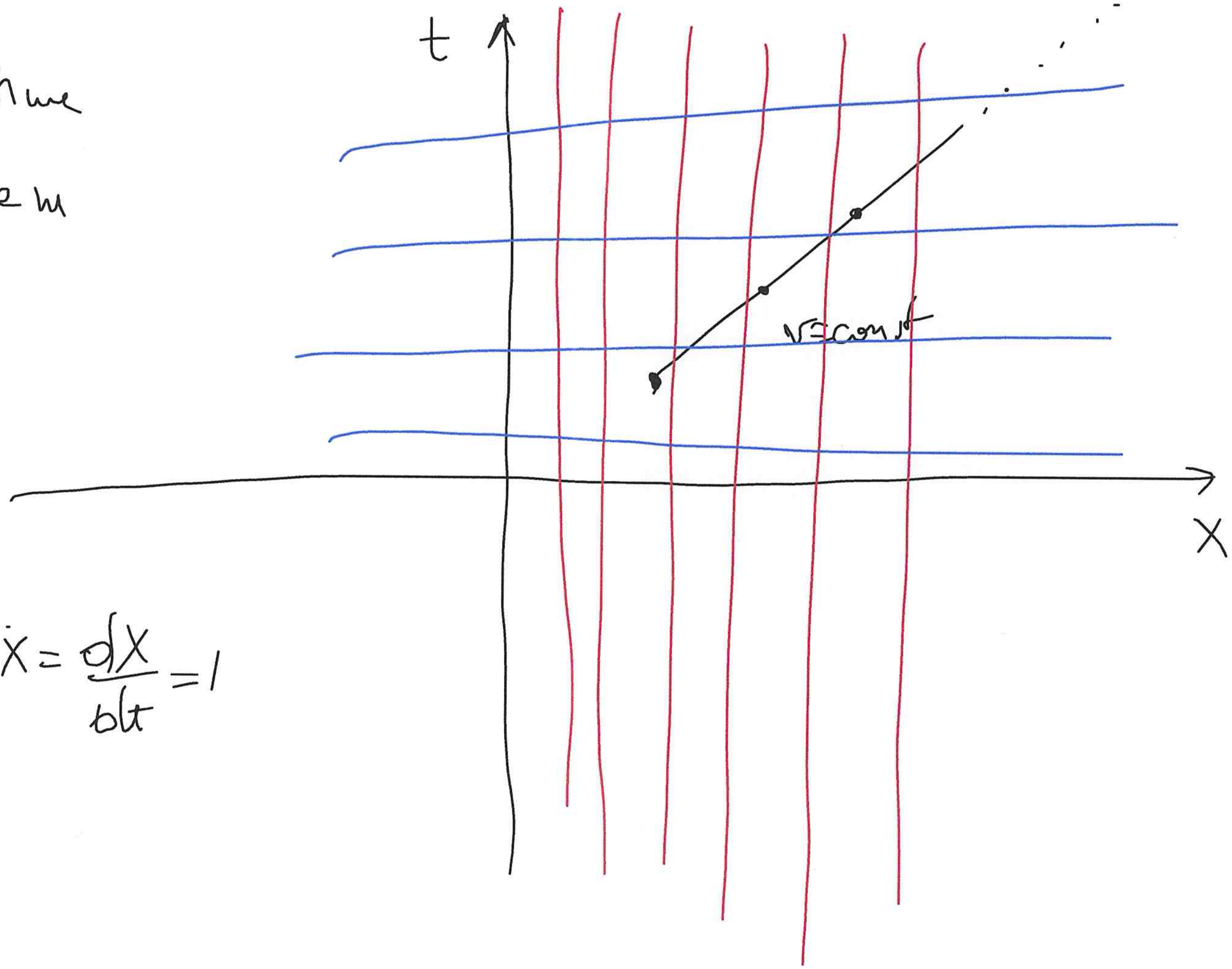
$$\tilde{u} = -\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} e^{-t/M}$$

$$\tilde{v} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} e^{t/M}$$

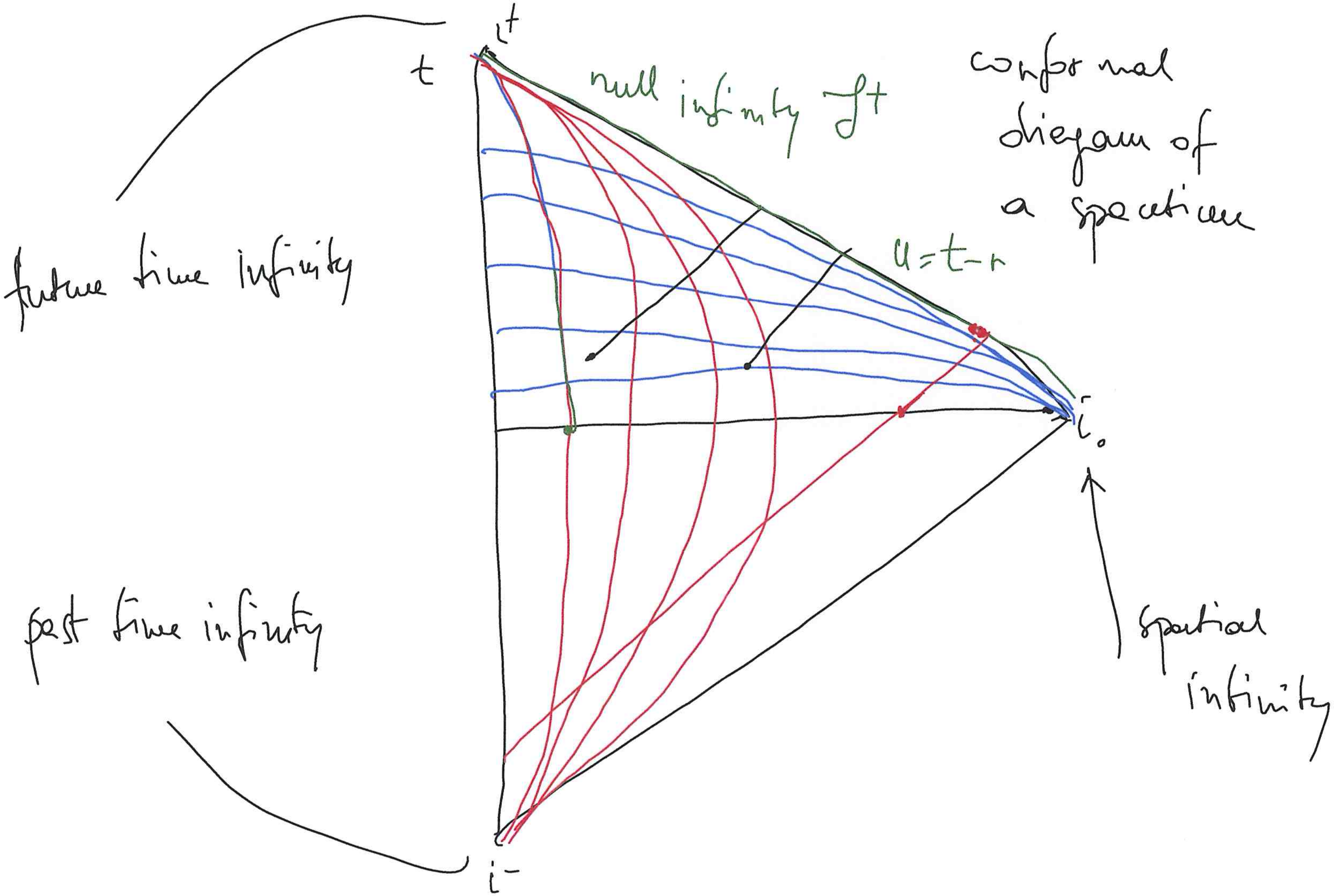


This set of coordinates can be subject to a conformal coordinate transformation and lead to a Carter Penrose diagram

Spacetime diagram

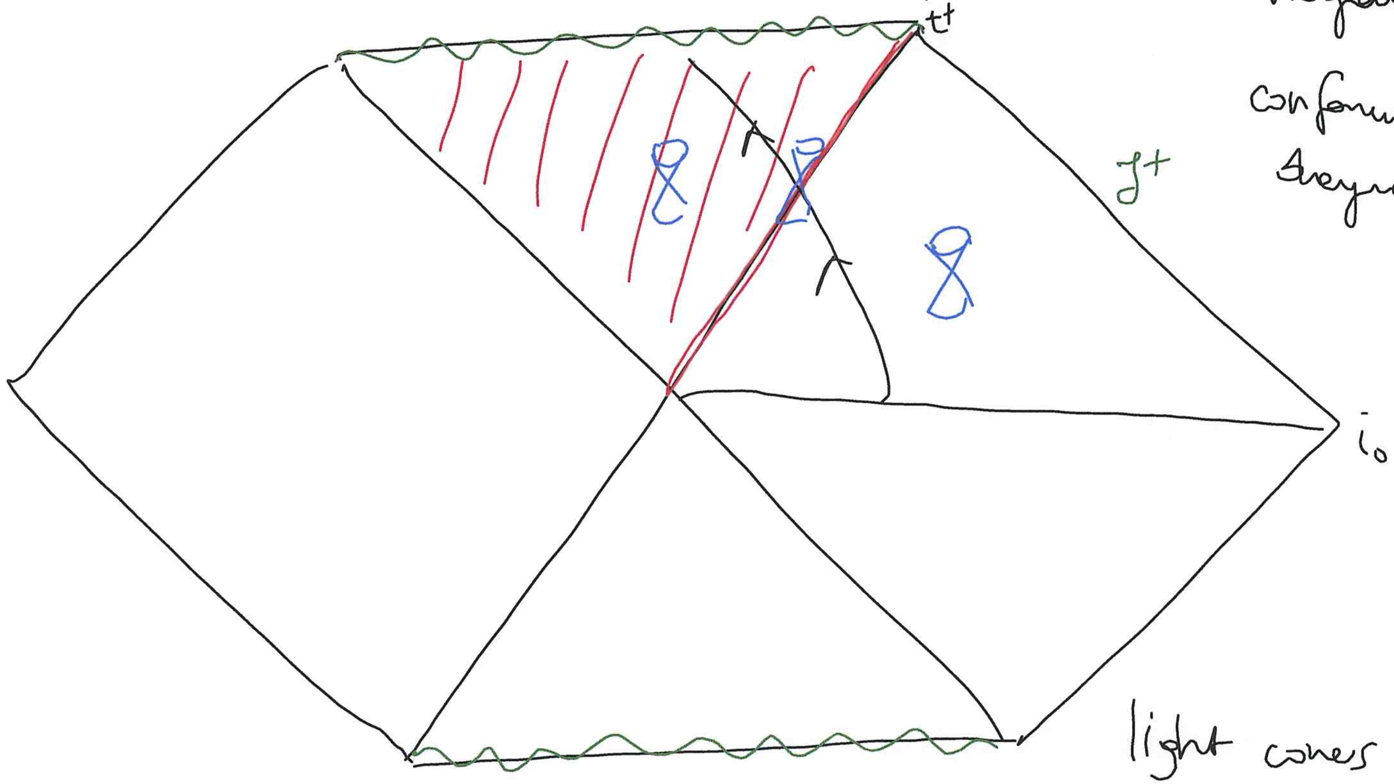


$$v = c = \dot{x} = \frac{dx}{dt} = 1$$


i^+
t
null infinity $\mathcal{J}^+$
conformal diagram of a spacetime
future time infinity
$u = t - r$
i_0
past time infinity
spatial infinity
i^-

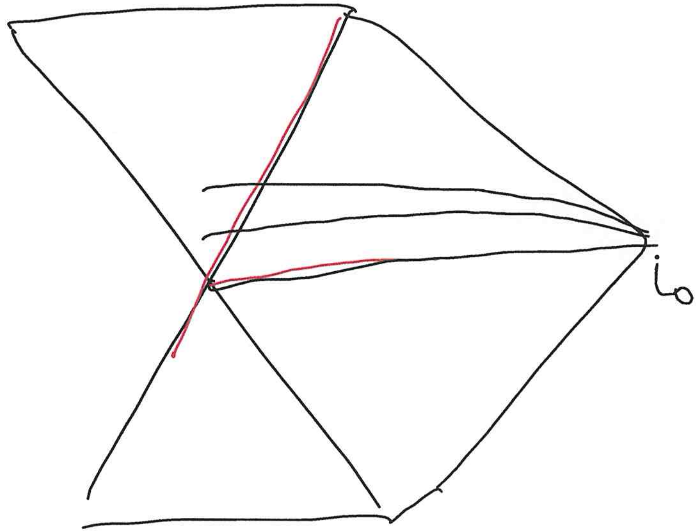
Carter Penrose diagram

conformal diagram

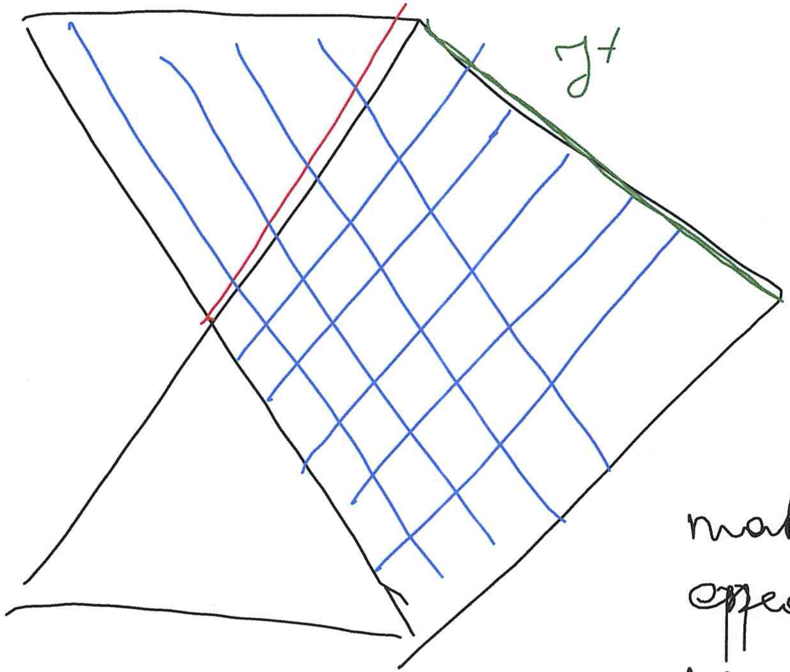


light cones are always at 45 deg

Suppose we have such a diagram what's the best set of coordinates to use?

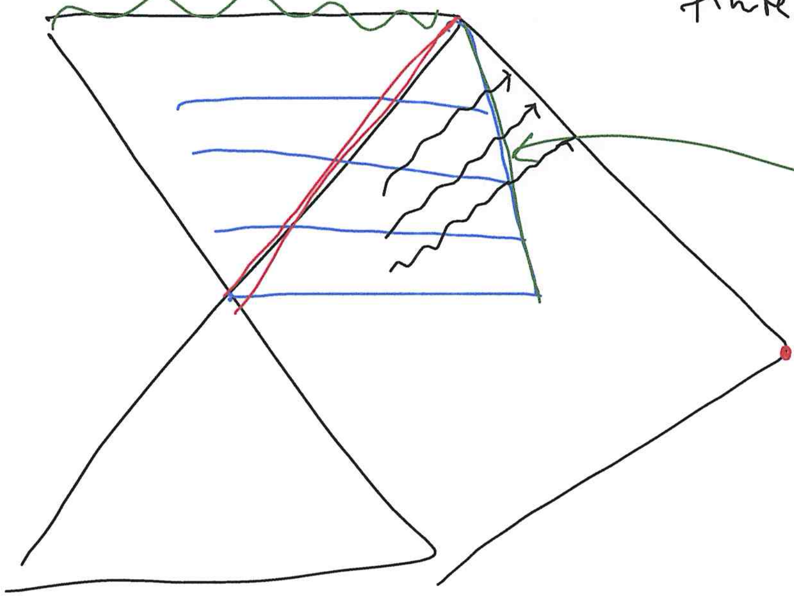


$t=$ const slicings reaching i_0



double null coordinate system

mathematically very appealing, awful to use in practice

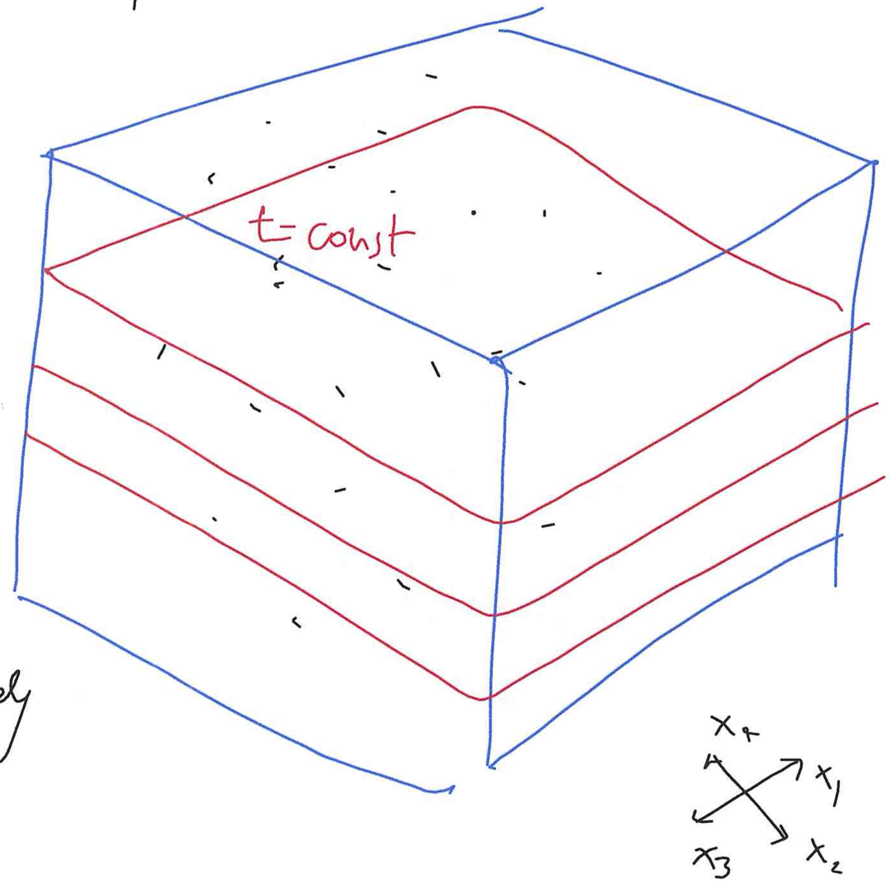


This is the typical coordinate set up we generally use: $\boxed{\text{3+1 split with timelike outer boundary}}$

I will explain what this means in practice.

3+1 split

$\mathcal{M}, g_{\mu\nu}$



It's is much more natural to think of space and time separately as this is the perception we intuitively built

- Take a manifold $\mathcal{M}$ and a metric g.
- Reduce the metric to a diagonal form
- One defines the <u>signature</u> of g the number of negative component of the metric

$$\text{sign } g = (\underbrace{-,-,- \cdots -}_{s}, \underbrace{+,+ \cdots +}_{n-s})$$

n: dimensions of $\mathcal{M}$

- If $s = 0$: the metric is said to be <u>Riemannien</u>, eg $\mathbb{R}^n$
- If $s = 1$: the metric is said to be Lorentian

$\rightarrow$ g is positive-definite ie $g(\underline{u}, \underline{u}) \geq 0$

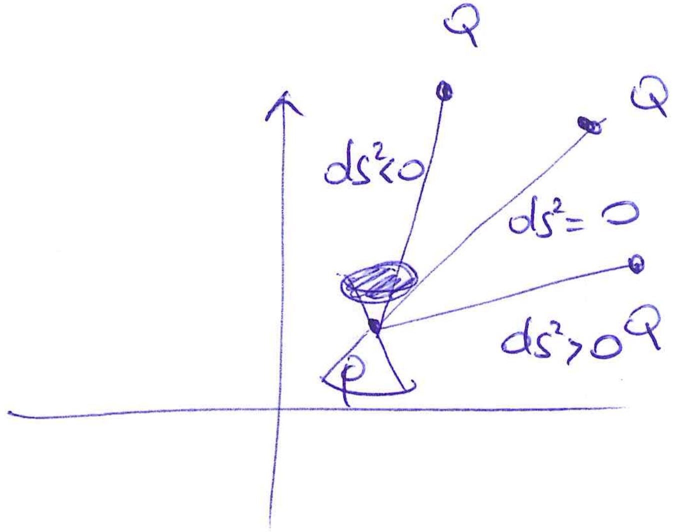
$g(u,u) = 0$ iff $\underline{u} = 0$

For a Lorentzian metric $g(\underline{u},\underline{u}) \gtrless 0$

Let $\underline{u} = d\underline{x}$: seperation vector between two events

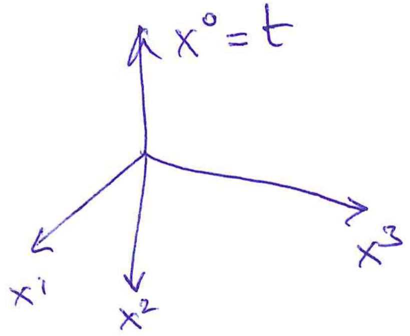
$$g(\underline{dx}, \underline{dx}) = g_{\mu\nu}\, dx^\mu dx^\nu$$

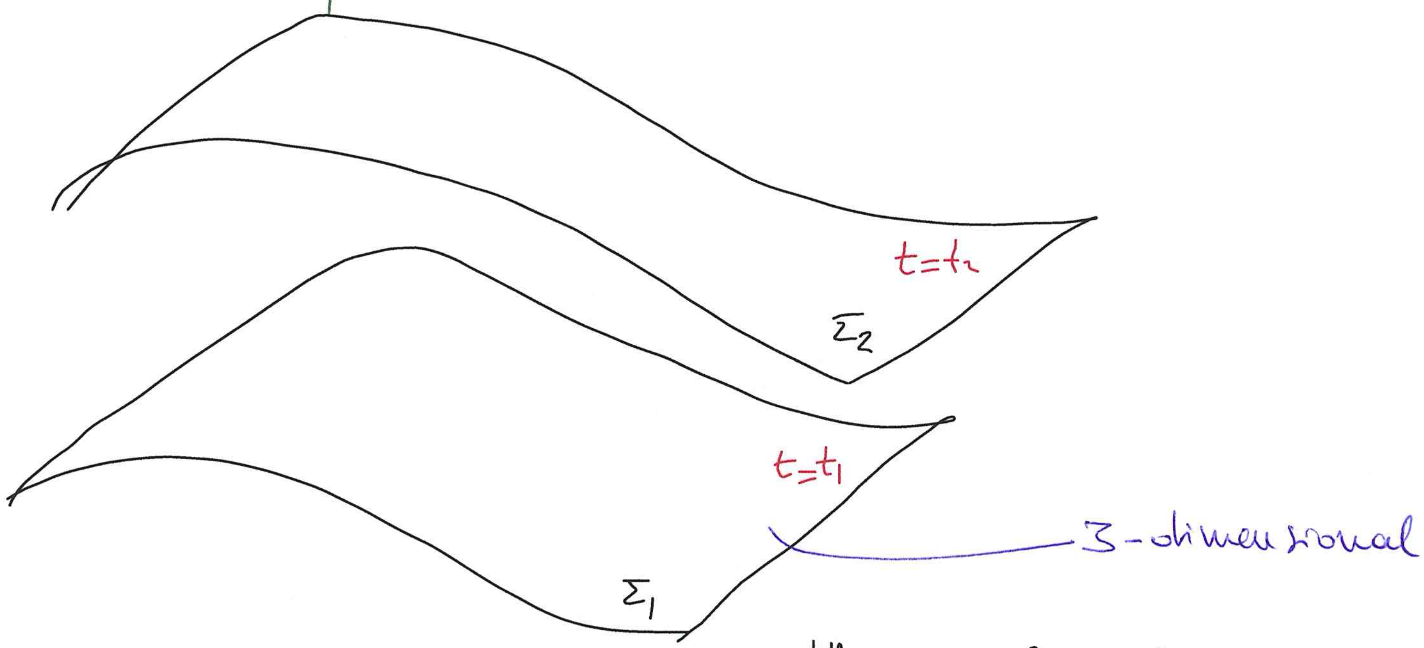
$$= \begin{cases} > 0 & : \text{spacelike} \\ = 0 & : \text{null} \\ < 0 & : \text{timelike} \end{cases}$$



The set of events at zero seperation marks the light cone

Spacelike hypersurface (3D)





The first thing to do once we select a surface is to define a normal vector

All point (events) on this hypersurface are spacelike separated

$$\Omega_\mu = \nabla_\mu t$$: one-form telling me how the time is growing

n_μ : normal vector to Σ_t

$$n_\mu = A\,\Omega_\mu$$ A: const

$$|\Omega|^2 = \Omega_\mu \Omega^\mu = g^{\mu\nu}\Omega_\mu \Omega_\nu$$
$$= g^{tt} \cancel{\Omega_t \Omega_t}$$

Clear? No?

let's do it in detail

$$\Omega_\mu \Omega^\mu = g^{\mu\nu}\,\Omega_\mu\,\Omega_\nu$$

$$= g^{\mu\nu}(\nabla_\mu t)(\nabla_\nu t)$$

$$= (g^{0\nu}\underbrace{\nabla_0 t}_{1} + g^{1\nu}\cancel{\nabla_1 t} + g^{2\nu}\cancel{\nabla_2 t} + g^{3\nu}\cancel{\nabla_3 t})(\nabla_\nu t)$$

$$= g^{0\nu}\,\nabla_\nu t$$

$$= g^{00} = g^{tt}$$

$$|\Omega|^2 = g^{tt}$$

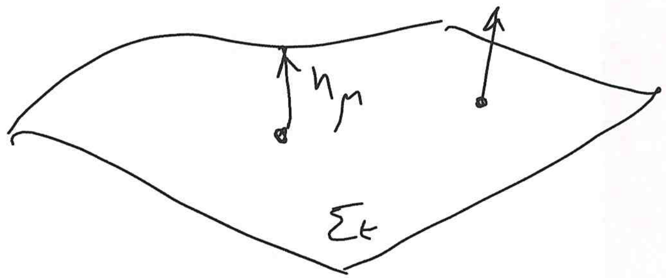
I want n_μ to be a timelike and unit vector

$$n_\mu n^\mu = A^2 \Omega_\mu \Omega^\mu = A^2 g^{tt} = -1$$

$$\Rightarrow \quad A = \pm \alpha \qquad \alpha = -1/g^{tt}$$

$A = -\alpha$ so that the normal is future directed

$$\boxed{n_\mu = -\alpha \nabla_\mu t}$$



covariant component of unit normal vector

Note that $\alpha = \alpha(x^\alpha)$: changes in space and time

Recap:

- We have considered spacetime as different. manifold $\mathcal{M}$ endowed with Lorentzian metric (signature $s=1$)

- setting t as time coordinate we have sliced spacetime in spacelike hypersurfaces $\Sigma(t)$

- Define normal unit vector $n_\mu = A\,\Omega_\mu = A\,\nabla_\mu t$

Requiring $n_\mu n^\mu = -1 \Longrightarrow$ $\boxed{n_\mu = -\alpha \nabla_\mu t}$ $\quad \alpha = -\frac{1}{g^{tt}}$

$$n^\mu = -\alpha \nabla^\mu t$$

Contravariant component

$$n^\mu = g^{\mu\nu} n_\nu = -\alpha g^{\mu\nu} \nabla_\nu t = -\alpha \nabla^\nu t$$

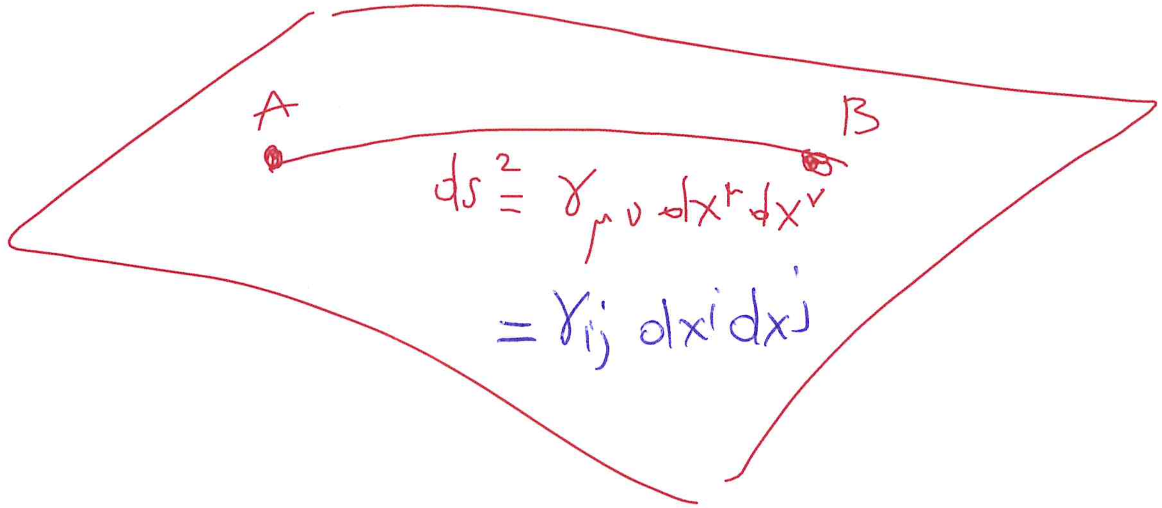
Using now the 4-metric $g_{\mu\nu}$ and the normal vector n_μ we can build the metric on the hypersurface; this will be a purely spatial metric

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

In other words, while $\mu \in [0, 3]$, the only relevant parts of $\gamma_{\mu\nu}$ are the spatial ones γ_{ij};

$$\boxed{\gamma \cdot n = 0}$$

(30)



the mixed components will be

$$\gamma^\mu{}_\nu = g^\mu{}_\nu + n^\mu n_\nu$$

$$= \delta^\mu{}_\nu + n^\mu n_\nu$$

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Since γ is purely spatial, it can be seen as a <u>spatial projector tensor</u>, ie a tensor that given a 4D object (tensor), it provides a purely spatial projection of it on Σ.

To appreciate what this means let's recall other projector tensor you know well even if you don't really call them like that.

Consider a Cartesian coordinate system (x,y) (2D) with unit vectors $\{\vec{e}_x, \vec{e}_y\}$, ie

$\vec{e}_x \cdot \vec{e}_x = 1$; $\vec{e}_x \cdot \vec{e}_y = 0$: orthonormal basis

A generic 2-vector $\vec{V}$ can then be expressed as

$$\vec{V} = V^x \vec{e}_x + V^y \vec{e}_y$$

V^x : projection of $\vec{V}$ along x direction, ie

$$\vec{V} \cdot \vec{e}_x = (V^x \vec{e}_x + V^y \vec{e}_y) \cdot \vec{e}_x$$

$$= V^x \vec{e}_x \cdot \vec{e}_x + V^y \vec{e}_x \cdot \vec{e}_y = V^x \quad \checkmark$$

Now that we know how to project on Σ we want to learn how to project out of Σ, ie we need a time projection tensor that provides the components of a tensor in the direction (locally) orthogonal to Σ

This tensor is defined as

$$N^\mu{}_\nu = -n^\mu n_\nu$$

where of course $\boxed{N \cdot \gamma = 0}$

$$N^\mu{}_\nu \, \gamma^\nu{}_\mu = -n^\mu n_\nu \left(\delta^\mu{}_\nu + n^\mu n_\nu\right)$$

$$= -\underbrace{n^\mu n_\mu}_{-1} - n^\mu n_\mu n^\nu n_\nu$$

$$= 1 - 1 = 0 \quad \checkmark$$

We have the basic tools to split not only the manifold but also any tensor in it

Eg 4-vector u^μ

$$u = \gamma \cdot u + N \cdot u = (\gamma + N) \cdot u$$

$$u^\mu = \underbrace{\gamma^\mu{}_\nu u^\nu}_{\text{purely spatial}} + \underbrace{N^\mu{}_\nu u^\nu}_{\text{purely timelike}}$$

$$V^\mu = \gamma^\mu{}_\nu u^\nu = \{0, V^1, V^2, V^3\} \quad ; \quad V^0 = 0$$

← Exercise here!

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Exercise 1

Given a vector field $\underline{u}$ with unit norm and $\underline{h}$ a projector orthogonal to $\underline{u}$, show that

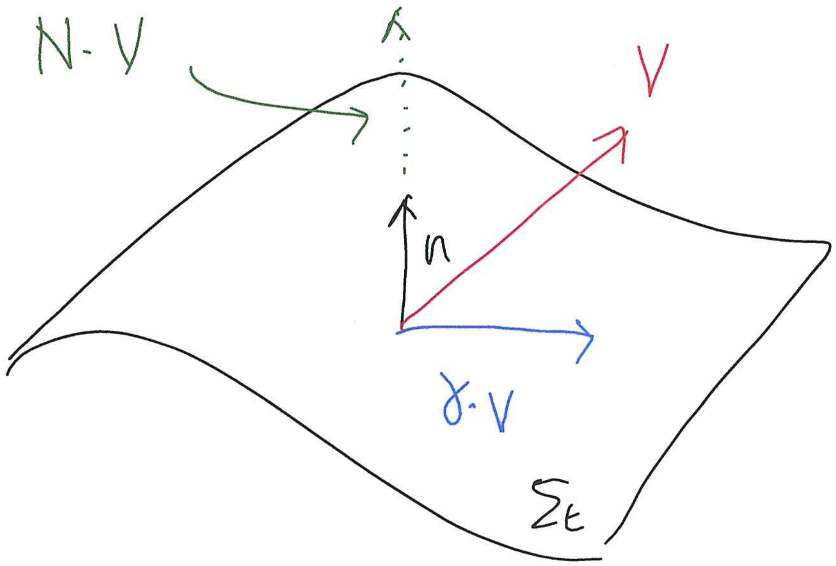
$$V^\mu = \underset{\parallel}{A U^\mu} + \underset{\perp}{B^\mu}$$

Given a covariant tensor W of rank 2 show it can be split as

$$W_{\mu\nu} = A U_\mu U_\nu + B_\mu U_\nu + U_\mu C_\nu + Z_{\mu\nu} \; ;$$

determine the parts A, B_μ, C_μ, $Z_{\mu\nu}$

35b



$$ {}^{(4)}ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu $$

$$ {}^{(3)}ds^2 = \gamma_{\mu\nu}\, dx^\mu dx^\nu $$

$$ = \gamma_{ij}\, dx^i dx^j $$

$$ N \cdot \gamma = N^{\mu\nu}\, \gamma_{\mu\nu} = 0 $$

cf. $\vec{e}_x \cdot \vec{e}_y = 0$

There is another timelike four-vector which is important and that tells me about how spatial coordinates on one hypersurface move to the next one after a given interval dt

NB recall that $\underline{n}$ is not the direction along which time evolves, ie it's not the direction of time derivatives.

This can be seen by the fact that

$$n^\mu \Omega_\mu = \frac{1}{A} n^\mu n_\mu = \frac{1}{\alpha} \neq 1$$

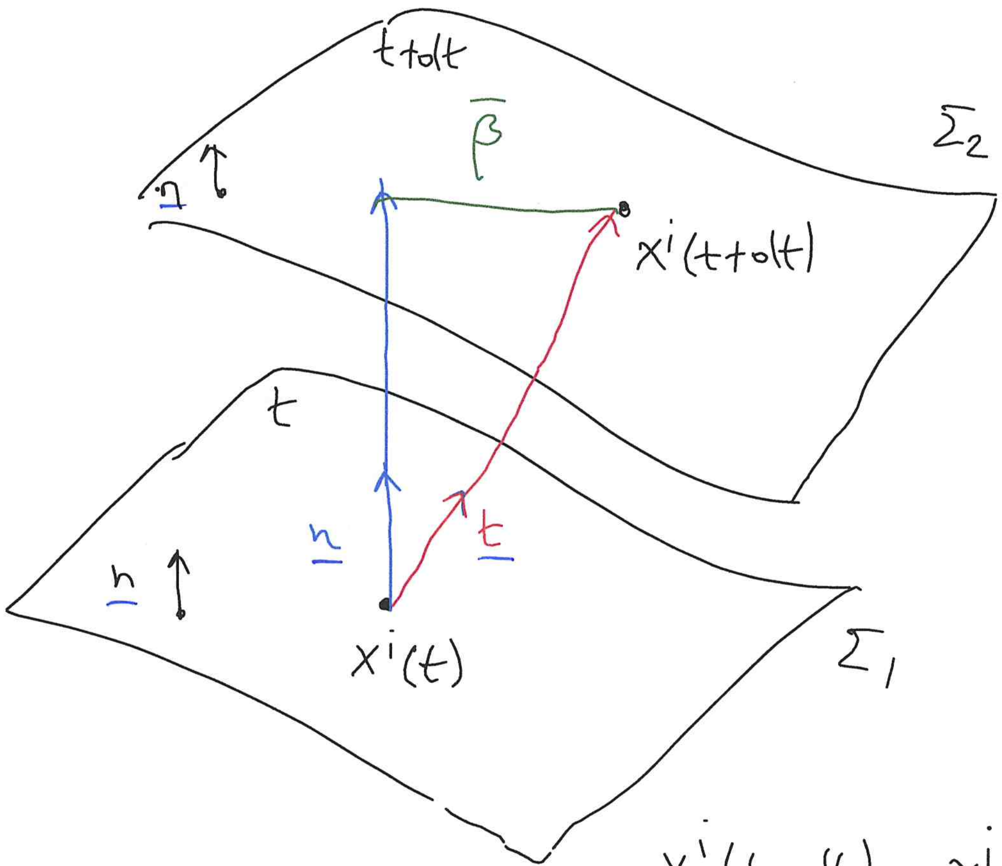
$\underline{n}$ and $\underline{\Omega}$ are not parallel because α is a function of spatial coordinates

I need a new four-vector $\underline{t}$ which is the dual to the one-form $\underline{\Omega}$, ie

$$\underline{t} \cdot \underline{\Omega} = 1$$

and that ~~is~~ acts as the time (coordinate) basis vector $\underline{e}_t$

$$\underline{t} \equiv e_t = \underbrace{\alpha \underline{n}}_{\text{part proportional to } n} + \underbrace{\beta}_{\text{purely spatial part, i.e. } n \cdot \beta = 0}$$



$$n_\mu = -\alpha \nabla_\mu t$$

$$x^i(t+dt) = x^i(t) - \beta^i(t, x^i)\,dt$$

β: shift vector ; α: lapse function

$$\boxed{t_\mu = \alpha n_\mu + \beta_\mu} \longleftrightarrow \vec{e}_t$$

$$\boxed{t^\mu \Omega_\mu = 1} \quad \text{while} \quad \boxed{n^\mu \Omega_\mu = \frac{1}{\alpha} \neq 1}$$

n and t are the same only if $\alpha = 1$, $\beta = 0$ (flat spacetime) in general they are different

proof

$$t^\mu \Omega_\mu = (\alpha n^\mu + \beta^\mu)\left(-\frac{n_\mu}{\alpha}\right)$$

$$= -n^\mu n_\mu - \frac{1}{\alpha}\beta^\mu n_\mu$$

$$= 1 \checkmark$$

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Using these functions we can write the components of $\underline{n}$ as

$$n_\mu = (-\alpha, 0, 0, 0) \quad ; \quad n^\mu = \frac{1}{\alpha}(1, -\beta^i)$$

and we can write the generic line element in terms of α, β, γ

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} dx^0 dx^0 + g_{01} dx^1 dx^0 + g_{11} dx^1 dx^1 + \dots$$

$$= -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad ; \quad \gamma_{00} = g_{00} + n_0 n_0 = g_{00} - \alpha^2$$
$$= 0$$

$\mu = i$
$\nu = j$

$$\gamma_{ij} = g_{ij} + n_i n_j$$
$$= g_{ij}$$

(41)

$$g_{\mu\nu} = \begin{pmatrix} -(\alpha^2 - \beta^i\beta_i) & \beta_i \\ \beta_i & \boxed{\gamma_{ij}} \end{pmatrix}$$

4X4 matrix

6 components

$g_{\mu\nu} = g_{\nu\mu}$: 16 ⟶ 10 indip components

$dx^i = 0$; $\beta^i = 0$ $\Longrightarrow$ $ds^2 = -\alpha^2 dt^2 = -d\tau^2$

Hence, the lapse function tells us that there are spatial differences on the proper time intervals

RECALL

$(\text{proper time})^2 = d\tau^2$

$= -ds^2$

$\Rightarrow$

$\boxed{d\tau^2 = -ds^2}$

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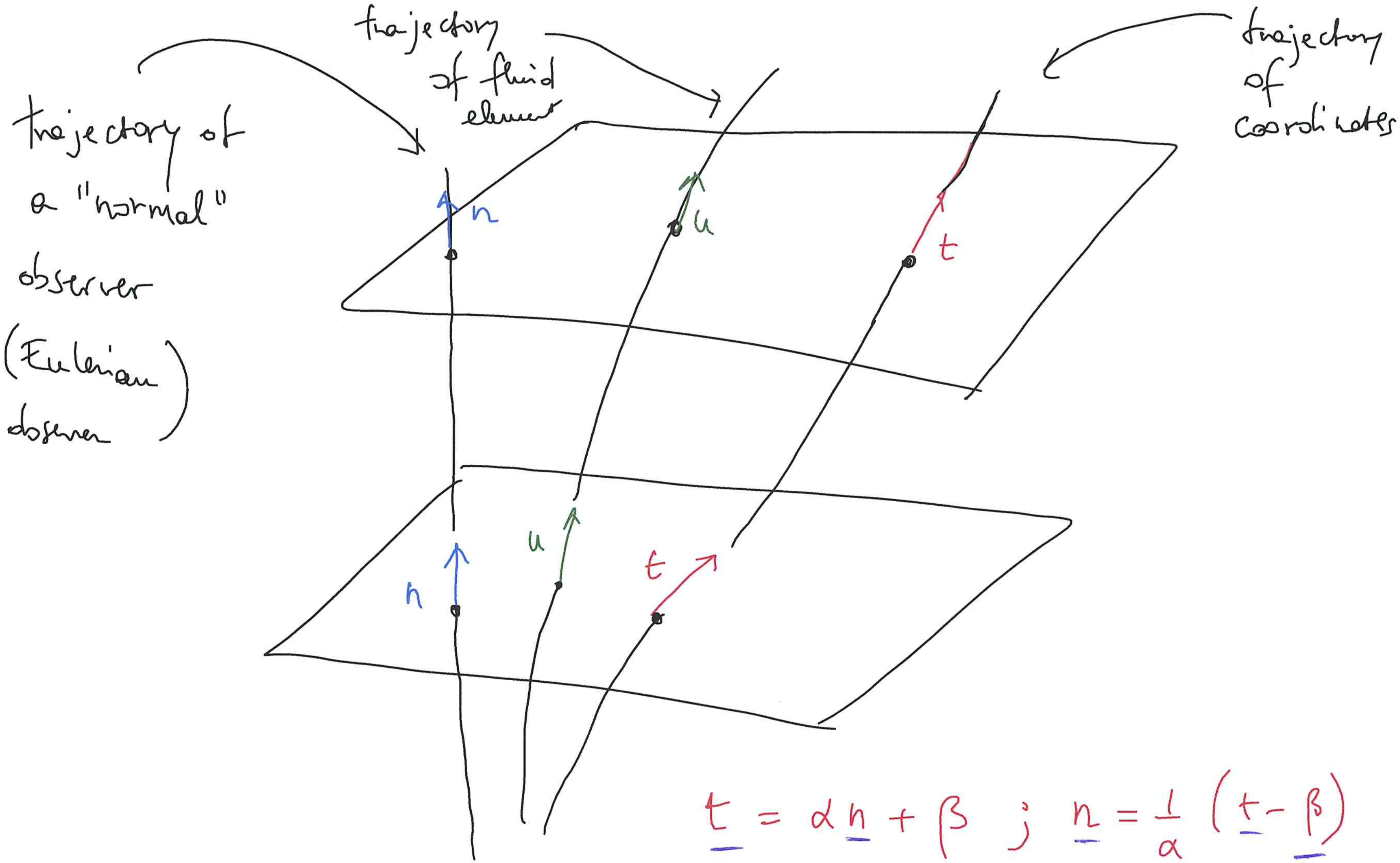
$\alpha^2 dt^2 = d\tau^2$ $\qquad$ $\underline{d\tau} = \pm \alpha \, dt$ $\qquad$ hence α is telling us about how time is changing from one slice to the next

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ & \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha \end{pmatrix}$$

(43)



$$\underline{t} = \alpha \underline{n} + \underline{\beta} \ ; \quad \underline{n} = \frac{1}{\alpha}\left(\underline{t} - \underline{\beta}\right)$$

We have mentioned that when solving the full set of the Einstein + hydrodynamic equations we need to build an energy-momentum tensor for the matter content of the spacetime. For a perfect fluid (ie without viscosity and heat transport) this is given by

$$T_{\mu\nu} = (e+p)\, u_\mu u_\nu + p\, g_{\mu\nu} = \rho h\, u_\mu u_\nu + p\, g_{\mu\nu}$$

$\underline{u}$: fluid four-velocity

p : pressure

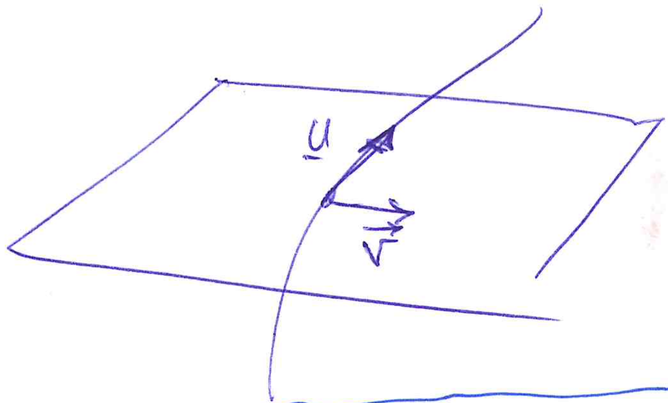
e : energy density $\quad e = \rho(1+\epsilon) \quad$ ϵ : specific internal energy

h : specific enthalpy $\quad h = \dfrac{e+p}{\rho}$

We need to characterize the spatial part of the (fluid) four-velocity $\underline{u}$ measured by Eulerian (normal) observer

$$\begin{pmatrix}\text{spatial part} \\ \text{of } \underline{u}\end{pmatrix} = \frac{(\text{projection of } \underline{u} \text{ on } \Sigma)}{(\text{projection of } \underline{u} \text{ along } n)} = \frac{(\text{space})}{(\text{time})}$$

$$v^i = \frac{\gamma^i{}_\mu u^\mu}{-u_\mu n^\mu}$$



$W = -n_\mu u^\mu = +\alpha u^t$: Lorentz factor

$W = (1 - v^i v_i)^{-1/2}$ as in special relativity

$W \to \infty$ for $v \to 1$

Exercise:
Prove
$W = (1 - v^i v_i)^{-1/2}$

Prove $W = (1 - v^i v_i)^{-1/2}$

Proof

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i\right) \; ; \; v_i = \gamma_{ij} v^j = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^0} + \beta^j\right)$$

$$v^i v_i = \frac{1}{\alpha^2}\left[\left(\frac{u^i}{u^0} + \beta^i\right)\gamma_{ij}\left(\frac{u^j}{u^0} + \beta^j\right)\right] = \frac{1}{\alpha^2}\left[\gamma_{ij}\frac{u^i u^j}{(u^0)^2} + 2\beta^i\frac{u_i}{u^0} + \beta^i\beta_i\right] =$$

$$-1 = u^\mu u_\mu = \cancel{u^0 u_0 + u^i u_i = u^0 (g_{0\mu} u^\mu) + u^i u_i = u^0 (g_{00} u^0 + g_{0i} u^i) + u^i u_i}$$

$$= g_{00}(u^0)^2 + 2 g_{0i} u^0 u^i + u^i u_i$$

$$= -(\alpha^2 - \beta^i\beta_i)(u^0)^2 + \underbrace{2\beta_i u^i u^0 + u^i u_i}_{= -1 + (\alpha u^0)^2} \Rightarrow$$

$$= \frac{1}{(\alpha u^0)^2}\left[u^i u_i + 2\beta^i u_i u^0 + (u^0)^2 \ldots\right.$$

$$= \frac{1}{(\alpha u^0)^2}\left(-1 + (\alpha u^0)^2\right) = \frac{-1 + W^2}{W^2} \Rightarrow W^2(1 - v^i v_i) = 1 \Rightarrow$$

$$W = (1 - v^i v_i)^{-1/2} \quad ✓$$

In components form

$$v^t = 0 \qquad v^i = \frac{\gamma^i{}_\mu u^\mu}{\alpha u^t} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

$$v_t = \beta_i v^i \qquad v_i = \frac{\gamma_{i\mu} u^\mu}{\alpha u^t} = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \beta^j\right)$$

or, using the Lorentz factor

$$\boxed{v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)} \; ; \; v_i = \frac{u_i}{W} = \frac{u_i}{\alpha u^t}$$

I recall that in special relativity:

$$\boxed{v^i = \frac{u^i}{u^t}} = \frac{dx^i/d\tau}{dt/d\tau} = \frac{dx^i}{dt}$$

Comparing with special relativity one notices that the three velocity gains a dependence from α (curvature) and β (shift of coordinates); the two coincide for $\alpha=1$; $\beta=0$

We can put things together and write

$$u^\mu = W\left(\underbrace{n^\mu}_{\text{purely time like}} + \underbrace{v^\mu}_{\text{purely spatial}}\right)$$

Exercises 2

1) The Schwarzschild metric in isotropic spherical coordinates reads

$$ds^2 = -\left(\frac{1 - M/(2r)}{1 + M/(2r)}\right)dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right)$$

Let Σ be a $t = $ const hypersurface. Find the the comp. components of the one-form $\underline{\Omega}$, the form of the lapse, of the normal $\underline{n}$ and of the three-metric

FORMULATION OF THE EINSTEIN EQUATIONS

We will start with spherically symmetric spacetimes as these have been traditionally the spacetimes first investigated numerically, eg to study the collapse of a fluid sphere to a black hole

We start from the line element in generic form but spherical symmetry

$$ds^2 = -a(r,t)^2 dt^2 + b(r,t)^2 dr^2 + R^2 d\Omega^2$$

However, not even r is a good coordinate and a better one is the mass μ enclosed in a shell of radius R:

$$d\mu = 4\pi\rho R^2 b\, dr$$

Setting now $d\mu = dr$ (mapping of the two coords) $\Rightarrow$ $\quad b = \dfrac{1}{4\pi R^2 \rho}$

Hence $\mu = 0$: center of the fluid sphere

$\mu = \mu_{max}$: edge of " " sphere

and we can follow each shell by simply looking at different values of μ.

This is therefore a Lagrangian formulation!

There are two different time and spatial operators that it is useful to introduce:

$$D_t = \frac{1}{a}\,\partial_t \qquad\qquad D_r = \frac{1}{b}\,\partial_r$$

By dividing by the metric function these operators express changes in "proper time" and "proper distance" making the equation look like the Newtonian ones

We can apply these operators to R and obtain

$$D_t R = U \qquad ; \qquad D_r R = \Gamma$$

radial component of four-velocity in coordinate frame in which R is radial coordinate

(*): hydrodynamic equations

$$\left\{\begin{array}{l} G_{\mu\nu} = 8\pi T_{\mu\nu} \\ \nabla_\mu T^{\mu\nu} = 0 \\ \nabla_\mu (\rho u^\mu) = 0 \end{array}\right. \Longrightarrow$$

$$D_t U = -\frac{\Gamma}{(e+p)} D_r p + \frac{M}{R^2} + 4\pi R p \qquad *$$

$$D_t \rho = -\left(\frac{\rho}{\Gamma R^2}\right) D_r (R^2 U) \qquad *$$

$$D_t e = \left(\frac{e+p}{\rho}\right) D_t \rho \qquad *$$

$$D_r a = -\left(\frac{a}{e+p}\right) D_r p$$

$$D_r M = 4\pi \Gamma e R^2 = 4\pi \Gamma \rho (1+\epsilon) R^2$$

$$D_t R = U \qquad *$$

MISNER and SHARP (1964)

Note the presence of evolution equations (*) and of constraint equations (not involving time derivatives); this is not an accident and we will discuss this later on.

Boundary conditions

$M = 0$; $R = 0$; $u = 0$ at $\mu = 0$

$p = 0$; $\rho = 0$; $a = 1$ at $\mu = \mu_{max}$

coordinate time is proper time of comoving observer at edge of the fluid sphere

Exercise : derive the Misner - Sharp equations

Recap:

- 3+1 split. Given Σ_t, define normal such that

$$\gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$$

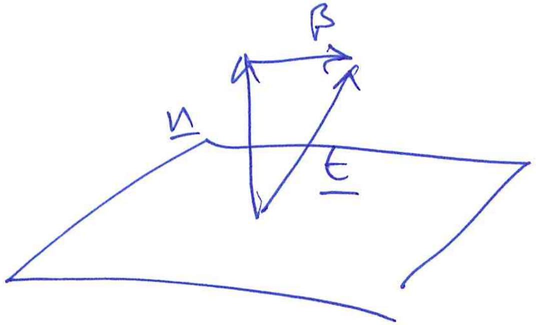
$$N^\mu{}_\nu = - n^\mu n_\nu$$

are projectors onto and out of Σ_t

- line element: $ds^2 = -(\alpha^2 - \beta^i \beta_i)\, dt^2 + 2\beta_i\, dx^i dt + \gamma_{ij}\, dx^i dx^j$

α: lapse function $\quad d\tau^2 = \alpha^2 dt^2 \quad$ for $\beta^i = 0 = dx^i$

β^i: shift vector



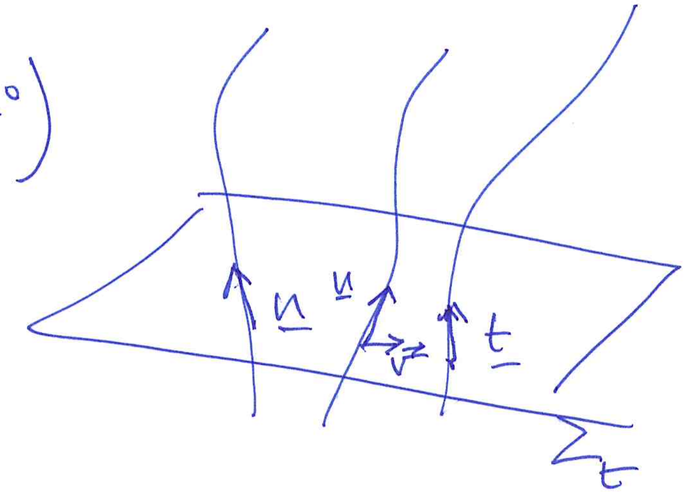
$\underline{t} = \underline{n}\alpha + \underline{\beta}$

$\beta \cdot n = 0$

- given 4-velocity $\underline{u}$ (eg tg. vector to particle or fluid-element)

$$= -\gamma^i{}_\mu u^\mu / n^\mu u_\mu$$

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i\right) = \frac{1}{W}\left(u^i + \beta^i u^0\right)$$



- simplest formulation of Einstein eqs is in spherical symmetry: Misner-Sharp (Lagrangian)

Today we will derive the most famous 3+1 formulation of the Einstein eqs, the <u>ADM formulation</u> (from Arnowitt, Deser, Misner 1962).

Notes

- the ADM formulation was not derived for numerical solutions but for Hamiltonian formulation of the Einstein eqs
- the ADM formulation is seldom used in practice and I'll explain why
- much of the formulation I present comes from the formalism introduced by York (1979)

An important first step in the derivation of the ADM eqs is the definition of the <u>spatial covariant derivative</u>

$$D_\nu := \gamma^\mu{}_\nu \nabla_\mu = (\delta^\mu{}_\nu + n^\mu n_\nu) \nabla_\mu$$

Just like the 4D cov. derivative is compatible with the 4-metri
ie

$$\nabla_\mu g^{\mu\nu} = 0$$

so is the 3D cov. derivative

$$D_\mu \gamma^{\mu\nu} = 0 .$$

In practice, the spatial covariant derivative is just the result of the projection of the 4D one, e.g.

$$\boxed{D_\alpha T^\gamma{}_\beta = \gamma^\mu{}_\alpha \gamma_\beta{}^\rho \gamma_\nu{}^\gamma \nabla_\mu T^\nu{}_\rho}$$

what is important is that the spatial covariant derivative is based on the 3D Christoffel symbols.

I recall that the chistoffel symbols or connections are first derivatives of the metric

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma} \right)$$

so that the corresponding 3D (spatial) objects are

$$^{(3)}\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} \gamma^{\alpha\delta} \left(\partial_\beta \gamma_{\gamma\delta} + \partial_\gamma \gamma_{\delta\beta} - \partial_\delta \gamma_{\beta\gamma} \right)$$

As for the 4D chistoffels, also the 3D ones follow the same properties: symmetric on lower indices and not proper tensors (do not transform as tensors)

If we want to express a 3+1 decomposition of the Einstein eqs. we have to follow the same route in 4D spacetimes:

$$g_{\mu\nu} \longrightarrow \Gamma^{\alpha}_{\mu\nu} \longrightarrow R^{\alpha}{}_{\beta\mu\nu} \longrightarrow R_{\mu\nu} \longrightarrow G_{\mu\nu}$$

(metric) (Riemann tensor) (Ricci tensor) (Einstein tensor)

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$$

What is the Riemann tensor? (curvature)

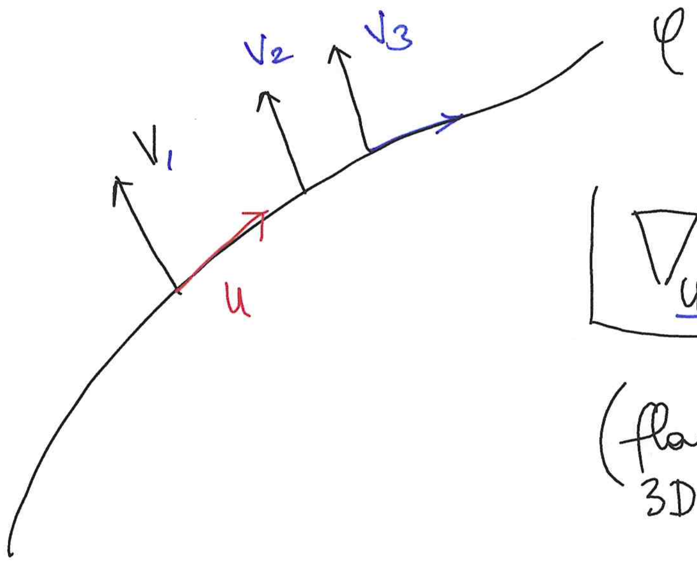
Riem. is notation for Riemann tensor

Does everybody remember how to derive it?

I am assuming that you all know in detail the properties of the Riemann (curvature) tensor. However, to make sure that nothing is lost and that you can all follow, I will make a short recap of the curvature tensor, whose definition relies on a mathematical tool of great importance: the <u>parallel transport</u>.

Consider therefore a vector field $\underline{u}$, which defines a series of curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \ldots$ having $\underline{u}$ as the local tangent vector.

A vector $\underline{V}$ is then said to be parallel transported along $\underline{u}$



$$\boxed{\nabla_{\underline{u}} \underline{V} = 0} = u^\mu \nabla_\mu V^\nu = 0$$

$$\left(\underset{3D}{\text{flat}} \quad u^i \partial_i V^j = 0\right)$$

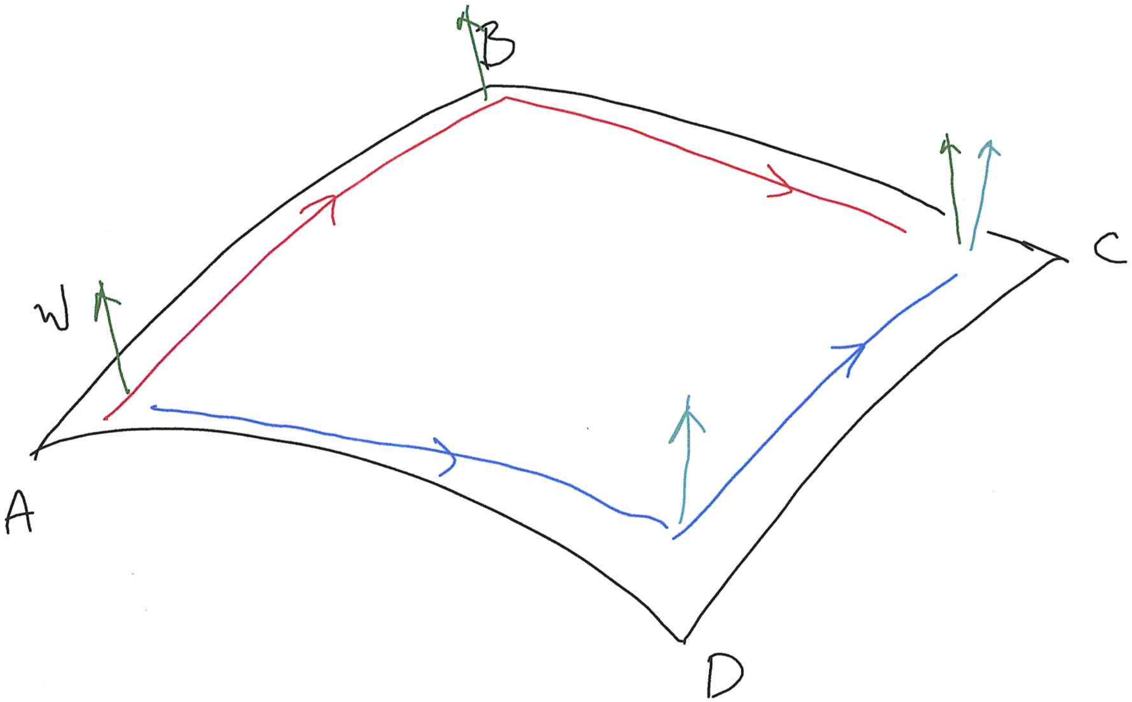
If $\vec{u}$ along say x-direction, then // transport implies $\boxed{\partial_x V^i = 0}$

Let's expand:

$$u^\mu \nabla_\mu V^\nu = 0 \iff u^\mu(\partial_\mu V^\nu + \Gamma^\nu_{\alpha\mu} V^\alpha) = 0 \implies$$

(62) $$u^\mu \partial_\mu V^\nu = - \Gamma^\nu_{\alpha\mu} V^\alpha u^\mu$$: RHS measures the changes in V resulting from the changes in

Having defined the parallel transport, there is an operational manner to measure the respective curvature by comparing two vectors that are parallel transported to the same final position when starting from the same initial position but after following two different paths.



$$2\nabla_{[\alpha}\nabla_{\beta]}W_\delta = R^\gamma{}_{\delta\beta\alpha}W_\gamma$$

- Are you familiar with the bracket notation?

antisymmetric; $T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right)$; symmetric $T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right)$

$$\nabla_{[\alpha} \nabla_{\beta]} = \frac{1}{2}\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right)$$

$\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$: ie partial derivatives commute

$\nabla_\alpha \nabla_\beta \neq \nabla_\beta \nabla_\alpha$: " covar. " do not commute

What we need is the 3D spatial equivalent, ie

$$2 D_{[\alpha} D_{\beta]} W_\delta = {}^{(3)}R^\gamma{}_{\delta\beta\alpha} W_\gamma$$

65

After a bit of algebra one obtains the following expression for the Riemann tensor

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\lambda\alpha} \Gamma^{\lambda}_{\nu\beta} - \Gamma^{\mu}_{\lambda\beta} \Gamma^{\lambda}_{\nu\alpha}$$

$$= f\left(\partial^2 g, (\partial g)^2\right) \Rightarrow \boxed{[R^{\mu}{}_{\nu\alpha\beta}] = L^{-2}}$$

The corresponding 3D (spatial) object is readily obtained

$${}^{(3)}R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha\, {}^{(3)}\Gamma^{\mu}_{\nu\beta} - \partial_\beta\, {}^{(3)}\Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\lambda\alpha} \Gamma^{\lambda}_{\nu\beta} - \Gamma^{\mu}_{\lambda\beta} \Gamma^{\lambda}_{\nu\alpha}$$

${}^{(3)}R^{\mu}{}_{\nu\alpha\beta}$ is purely spatial: ${}^{(3)}R^{\mu}{}_{\nu\alpha\beta}\, n_\mu = 0$

${}^{(3)}R_{\alpha\beta} = {}^{(3)}R^{\mu}{}_{\alpha\mu\beta}$: 3D (spatial) Ricci tensor

${}^{(3)}R = {}^{(3)}R^{\alpha}{}_{\alpha}$: 3D (spatial) Ricci scalar

$$ {}^{(3)}G_{\mu\nu} = {}^{(3)}R_{\mu\nu} - \frac{1}{2}\,{}^{(3)}R\,\gamma_{\mu\nu} $$

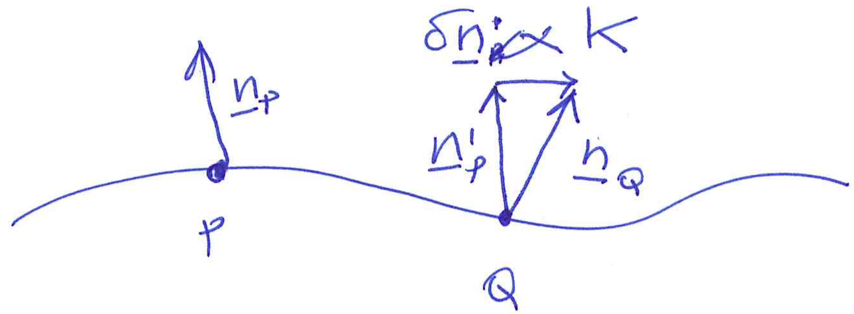
Recap: we have taken 4D curvature and obtained 3D equivalent telling us the curvature on the slice (hypersurface): intrinsic curvature

In doing so we have lost information and this is contained in another tensor that is called extrinsic curvature

The extrinsic curvature provides information on the "shape" of the hypersurface, ie on how it is "bent" with respect to the 4D embedding spacetime

(67)

It is quite intuitive that a way of measuring the extrinsic curvature is one in which we compare the difference in normals on the hypersurface Σ_t

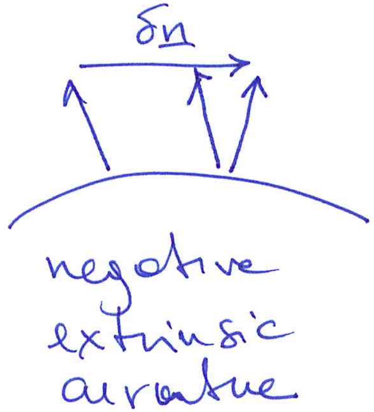


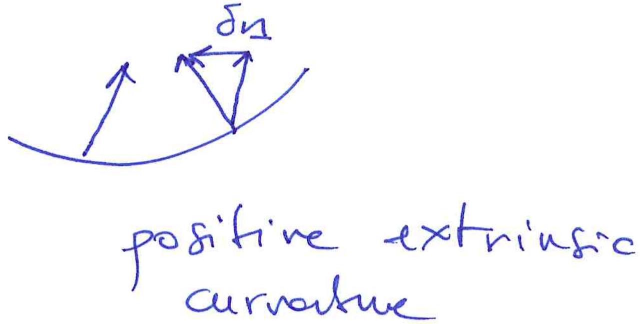
Let $\underline{n}_P$ be the normal 4 vector in P and $\underline{n}_Q$ the equivalent in Q.
Let $\underline{n}'_P$ the 4 vector $\underline{n}_P$ parallel transported at Q. We can compare $\underline{n}'_P$ and $\underline{n}_Q$
Of course we are interested in the projection of $\delta \underline{n}$ on Σ_t

$$\boxed{K = \gamma \cdot \delta n} \iff \boxed{K_{\mu\nu} = -\gamma^{\alpha}_{\mu} \nabla_{\nu} n_{\alpha}}$$

Question: why do we choose the negative sign?

Answer: the sign distinguishes "convex" and "concave" hypersurfaces

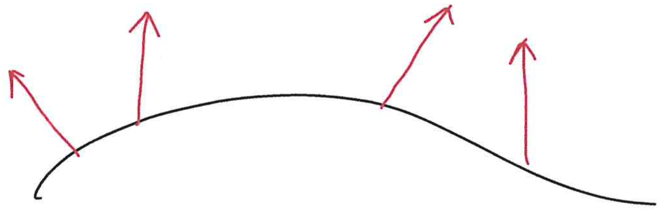




Note that this is not the only way of defining the extrinsic curvature.

other ways to measure extrinsic curvature:

- acceleration of normal observers (do normals interact or expand?)



acceleration of normal observers:

$$a_\mu = n^\nu \nabla_\mu n_\nu$$

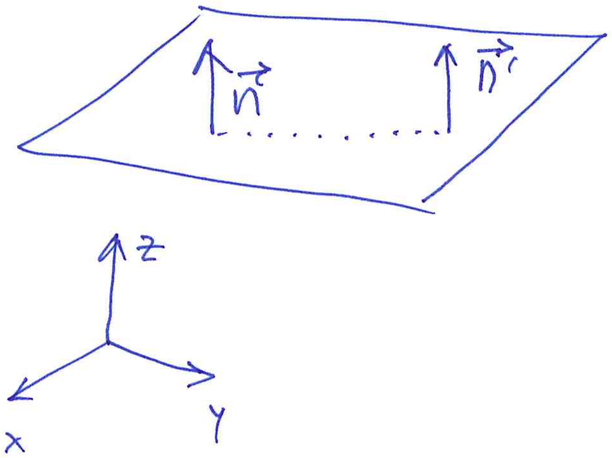
(similar to fluid $\tilde{a}_\nu = u^\mu \nabla_\mu u_\nu$)

$$\boxed{K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu}$$

The extrinsic curvature tells us about how the hypersurface is curved ("bent") with respect to the 4D manifold.

This is not easy for us to picture, but quite familiar if we restric to 2D surfaces embedded in a Euclid 3D space (no time here...)

The first example is a plane in Euclidean (flat) $\mathbb{R}^3$



Consider a Cartesian coordinate system $x^i = (x, y, z)$ so that plane is surface $z = 0$; the scalar function t defining Σ_t is simply $t = z$

The spatial metric γ_{ij} induced by $g_{\alpha\beta}$ is diagonal with components

$$\gamma_{ij} = \mathrm{diag}(1, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ds^2 = dx^2 + dy^2 \qquad \rightarrow \boxed{\mathrm{Riem} = 0}$$

obviously the Riemann tensor is zero ; The normal 3-vector is $n^i = (0, 0, 1)$; $n_i = (0, 0, 1)$

The extrinsic curvature is then

$$K_{ij} = - \gamma^k_j \nabla_i n_K \qquad i = 1, 2 ;$$

$$= - \gamma^3_j \underbrace{\nabla_i n_3}_{\partial_x z = \partial_y z = 0} \qquad \boxed{\mathrm{Riem} = 0 = K}$$

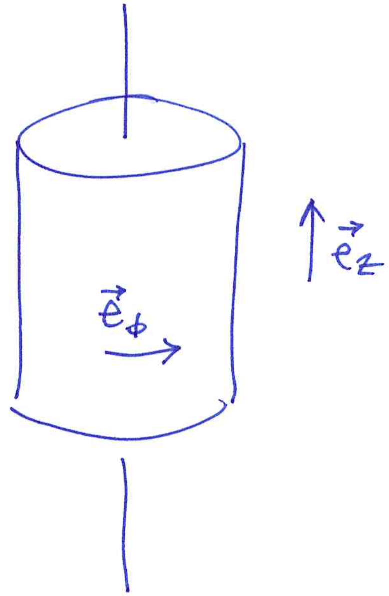
$$= 0$$

The extrinsic curvature of a plane is zero : normals can

(12) be parallel transported and do not change

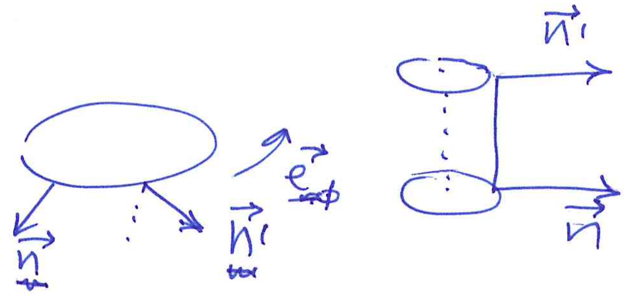
We can move on with complexity:

cylinder in $\mathbb{R}^3$
(ρ, ϕ, z)



It is not difficult to show that Riem $= 0$
(you can "cut" a cylinder and lay on a plane without wrinkles)

What about the extrinsic curvature?

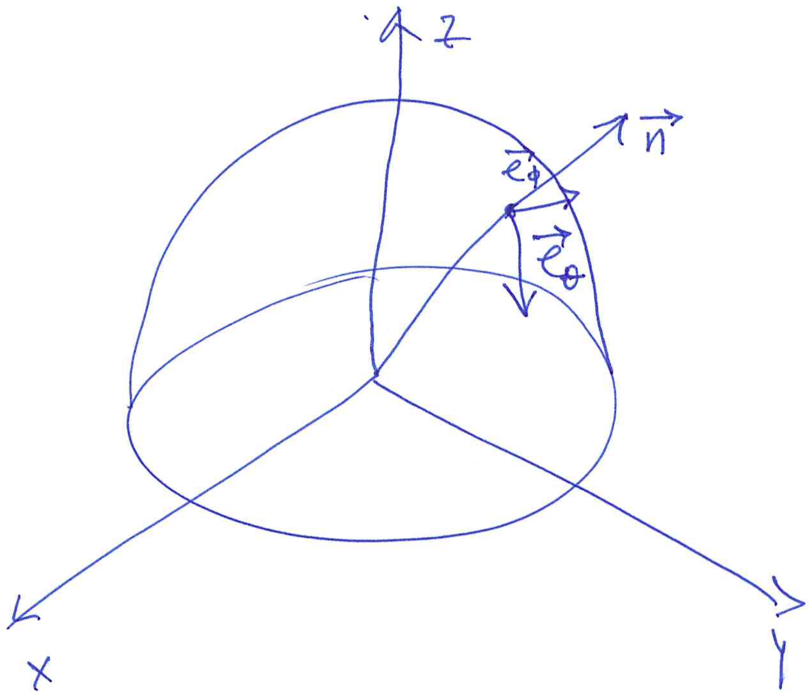


~~Exercise~~ will show that extrinsic curvature <u>vanishes</u> in the $\vec{e}_z$ direction but is <u>nonzero</u> in the $\vec{e}_\phi$ direction

What about a 2-sphere in $\mathbb{R}^3$?

You will see that Riem $\neq 0$

$K_{ij} \neq 0$ in $\vec{e}_\phi$, $\vec{e}_\theta$ directions



Exercises

(1) Compute the components of the normal and of the extrinsic curvature of a plane in cartesian coordinates in $\mathbb{R}^3$ (let the plane be the one at $z=0$)

(2) Repeat the exercise for a cylinder in $\mathbb{R}^3$

$\Sigma: \quad \rho - a = 0 \qquad$ where $\qquad \rho = \sqrt{x^2+y^2}$

Compute the extrinsic curvature also in cylindrical coordinates

(3) Repeat the exercise for a 2-sphere

$\Sigma: \quad r - a = 0 \qquad$ where $\qquad r = \sqrt{x^2+y^2+z^2}$

(4) Evaluate also the Riemann tensor the Ricci tensor and scalar

To derive the next result, ie the third way in which the extrinsic curvature can be written we need to recall the use and properties of the Lie derivative. Does anyone remember?

The Lie derivative expresses the covariant derivative of a vector (tensor) field relative to another vector (tensor) field

$$\mathcal{L}_{\underline{V}} \underline{U} = \underline{V} \partial \underline{U} - \underline{U} \partial \underline{V} = -[\underline{U}, \underline{V}] = [\underline{V}, \underline{U}]$$

In component form

$$(\mathcal{L}_{\underline{V}}\underline{U})^{\mu} = V^{\nu}\partial_{\nu}U^{\mu} - U^{\nu}\partial_{\nu}V^{\mu}$$

$$(\mathcal{L}_{\underline{V}}\underline{U})_{\mu} = V^{\nu}\partial_{\nu}U_{\mu} + U^{\nu}\partial_{\mu}V_{\nu}$$

Properties

- $\mathcal{L}_{\phi\underline{V}}\underline{T} = \phi\,\mathcal{L}_{\underline{V}}\underline{T}$
- $\mathcal{L}_{\underline{V}}\phi = V^{\nu}\partial_{\nu}\phi$
- $\mathcal{L}_{\underline{V}}\left(aY^{\alpha\gamma} + bZ^{\beta\gamma}\right) = a\,\mathcal{L}_{\underline{V}}Y^{\alpha\gamma} + b\,\mathcal{L}_{\underline{V}}Z^{\beta\gamma}$

- $\mathcal{L}_{\underline{V}} (Z^{\mu\nu} Y_{\alpha\beta}) = \mathcal{L}_{\underline{V}} (Z^{\mu\nu}) Y_{\alpha\beta} + Z^{\mu\nu} \mathcal{L}_{\underline{V}} Y_{\alpha\beta}$

- $\mathcal{L}_{V} T^{\alpha}{}_{\beta} = V^{\mu} \partial_{\mu} T^{\alpha}{}_{\beta} - T_{\beta}{}^{\mu} \partial_{\mu} V^{\alpha} + T^{\alpha}{}_{\mu} \partial_{\beta} V^{\mu}$

We can apply now these rules to compute the Lie derivative of the spatial metric along the normal field vector, ie

$$\mathcal{L}_{\underline{n}} \gamma_{\mu\nu} = n^{\alpha} \nabla_{\alpha} \gamma_{\mu\nu} + \gamma_{\mu\alpha} \nabla_{\nu} n^{\alpha} + \gamma_{\nu\alpha} \nabla_{\mu} n^{\alpha}$$

$$= \dots = -2 K_{\mu\nu}$$

$$\Longrightarrow \quad K_{ij} = -\frac{1}{2} \mathcal{L}_{\underline{n}} \gamma_{ij}$$

> **Exercise**
>
> Prove
>
> $\mathcal{L}_{\underline{n}} \gamma_{\mu\nu} = -2 K_{\mu\nu}$

Recalling now that $\underline{t} = \alpha \underline{n} + \underline{\beta}$

$$\mathcal{L}_{\underline{n}} = \frac{1}{\alpha} \mathcal{L}_{\alpha \underline{n}} = \frac{1}{\alpha} \left(\mathcal{L}_{\underline{t}} - \mathcal{L}_{\underline{\beta}} \right) = \frac{1}{\alpha} \left(\partial_t - \mathcal{L}_{\underline{\beta}} \right) \Rightarrow$$

$$\partial_t \gamma_{ij} = - 2 \alpha K_{ij} + \mathcal{L}_{\underline{\beta}} \gamma_{ij}$$

$$= -2 \alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\boxed{\partial_t \gamma_{ij} = - 2 \alpha K_{ij} + 2 D_{(i} \beta_{j)}}$$

Note that this can be seen both as a definition of K_{ij} and a kinematical description of the coordinates.

$$\left(\text{extrinsic curvature} \right) = \left(\begin{array}{c} \text{time derivative of} \\ \text{coordinates measured by} \\ \text{Eulerian observers} \end{array} \right)$$

Recap:

- introduced concept of spatial covariant derivative

$$D_\nu = \gamma^\mu{}_\nu \nabla_\mu$$

- reviewed the meaning of curvature tensor

$$2 \nabla_{[\alpha} \nabla_{\beta]} W_\delta = R^\gamma{}_{\delta\beta\alpha} W_\gamma$$

from which

$$2 D_{[\alpha} D_{\beta]} W_\delta = {}^{(3)}R^\gamma{}_{\delta\beta\alpha} W_\gamma$$

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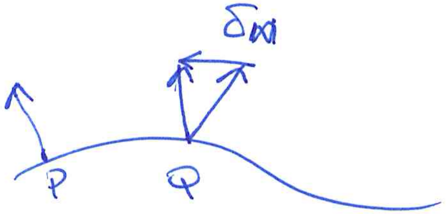
- From ${}^{(3)}R^\alpha{}_{\mu\alpha\nu} = {}^{(3)}R_{\mu\nu} \rightarrow {}^{(3)}R = R^\mu{}_\mu$

- Information missing when going from Riem to ${}^{(3)}$Riem is contained in extrinsic curvature $K_{\mu\nu}$; provides information on how Σ_t is bent with respect to embedding 4D spacetime
- Several definitions possible

(1) $K_{\mu\nu} = -\gamma^{\alpha}_{\mu} \nabla_\alpha n_\nu$



difference in normals

$\boxed{\text{check notes} \quad K_{\mu\nu} = -\gamma_\mu{}^\alpha \nabla_\nu n_\alpha}$

(2) $K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu$

where $a_\nu = n^\mu \nabla_\mu n_\nu$ acceleration normal observers

$\boxed{\text{check notes} \quad a_\nu = n^\mu \nabla_\nu n_\mu}$

(3) $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_{\underline{n}} \gamma_{\mu\nu}$

$\mathcal{L}_{\underline{n}}$: Lie derivative along $\underline{n}$

9b

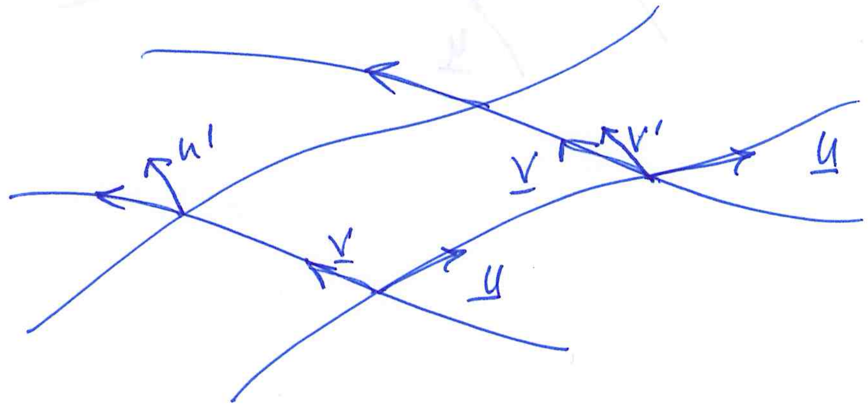
Open questions:

(1) What is the number of independent component of Riemann tensor? N^4? (N: dimension of manifold)

$$\frac{N^2(N^2-1)}{12} = \frac{\cancel{16}^{4} \cdot \cancel{15}^{5}}{\cancel{12}_{\cancel{3}}} = 20 \text{ in 4D spacetime}$$

(2) $\mathcal{L}_{\underline{u}} \underline{v} = 0 \implies$?

The vector produced when dragging $\underline{v}$ along the congruence of $\underline{u}$ is the same as the vector produced when dragging $\underline{u}$ along the congruence of $\underline{v}$



$\underline{u}' = \nabla_{\underline{v}} \underline{u} = \underline{v}' = \nabla_{\underline{u}} v$

79c

DIGRESSION ON SYMMETRIES

There is a lot more to say about symmetry generators and Lie derivatives and I expect you have already seen these discussions in the GR course. A property that is relevant to remind is that

$\underline{\xi}$ is a Killing vector field iff $\boxed{\mathcal{L}_{\underline{\xi}}\, g = 0}$ (A)

ie the metric is dragged along the congruence of $\underline{\xi}$

Using (A) it is possible to derive immediately the Killing equations

$$\nabla_{(\mu}\, \xi_{\nu)} = 0 = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

79d

Expression (Δ) can also be interpreted as indicating there are directions along which g does not vary. The corresponding coordinates are said to be cyclic.

Example

A stationary and axisymmetric spacetime is such that

$$\partial_t g_{\mu\nu} = 0 = \partial_\phi g_{\mu\nu}$$

As a result, there exist two Killing vectors

$\underline{\eta}$, $\underline{\xi}$ such that

$$\eta^\mu = \delta^\mu_t = (1,0,0,0) \; ; \qquad \xi^\mu = \delta^\mu_\phi$$

Using now the Lie derivative and the existence of a killing vector, it is possible to prove the following relation for a perfect fluid

$$\mathcal{L}_{\underline{u}}\left(h\,\underline{u}\cdot\underline{\xi}\right) = \mathcal{L}_{\underline{u}}\left(h\,u_\mu\,\xi^\mu\right) = 0 = \nabla_{\underline{u}}\left(h\,\underline{u}\cdot\underline{\xi}\right) \quad (\nabla)$$

where h: specific enthalpy $\quad h = (e+p)/e$

$\underline{u}$: fluid 4-velocity

$\underline{\xi}$: killing vector field

Expression (∇) states that the quantity $h\,\underline{u}\cdot\underline{\xi}$ is conserved along the fluid lines.

Ex. $\xi^\mu = \delta^\mu{}_\phi \qquad h\,\underline{u}\cdot\underline{\xi} = h\,u_\phi$: specific angular mom. : const

$\xi^\mu = \delta^\mu{}_t \qquad h\,\underline{u}\cdot\underline{\xi} = h\,u_t$: energy at infinity : Bernoulli's constant.

In what follows we will use a number of identities derived well before Einstein's theory of general relativity and that are generic equations of differential geometry resulting from the different possible combinations of projections that can be applied to the Riemann tensor.

• We start with the <u>Gauss-Codazzi</u> equations

$$\boxed{\gamma \cdot \gamma \cdot \gamma \cdot \gamma \cdot \mathrm{Riem}}$$

$$\gamma^{\mu}{}_{\alpha} \gamma^{\nu}{}_{\beta} \gamma^{\rho}{}_{\gamma} \gamma^{\sigma}{}_{\lambda} R_{\mu\nu\rho\sigma} =$$

$$= {}^{(3)}R_{\alpha\beta\delta\lambda} + K_{\alpha\delta} K_{\beta\lambda} - K_{\alpha\lambda} K_{\beta\delta}$$

- Next, we consider the <u>Codazzi-Meinardi</u> equations

$$\boxed{\gamma \cdot \gamma \cdot \gamma \cdot n \cdot \text{Riem}} \iff$$

$$\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\rho_\delta n^\sigma R_{\mu\nu\rho\sigma} = D_\alpha K_{\beta\lambda} - D_\beta K_{\alpha\lambda}$$

$\square$

- <u>Ricci Equations</u>

$$\boxed{\gamma \cdot \gamma \cdot n \cdot n \cdot \text{Riem}}$$

$$\gamma^\alpha_\mu \gamma^\beta_\nu n^\delta n^\lambda R_{\alpha\delta\beta\lambda} = {}^{(3)}R_{\mu\nu} + K K_{\mu\nu} - K^\lambda{}_\nu K_{\mu\lambda}$$

$$K = K^\mu{}_\mu$$

Exercises

- Derive the Gauss-Codazzi equations $\underline{\gamma \cdot \gamma \cdot \gamma \cdot \gamma \cdot \mathrm{Riem}}$

- Derive the Codazzi-Mainardi equations $\underline{\gamma \cdot \gamma \cdot \gamma \cdot n \cdot \mathrm{Riem}}$

- Derive the Ricci equations $\underline{\gamma \cdot \gamma \cdot n \cdot n \cdot \mathrm{Riem}}$

Putting things together one obtains

$$\boxed{\begin{aligned}\partial_t K_{ij} = &- D_i D_j \alpha + \beta^k \partial_k K_{ij} + K_{kj} \partial_i \beta^k \\ &+ K_{ki} \partial_j \beta^k \\ &+ \alpha \left({}^{(3)}R_{ij} + K K_{ij} - 2 K_{ik} K^k{}_j \right) \\ &+ \boxed{4\pi\alpha \left[\gamma_{ij} (S - E) - 2 S_{ij} \right]}\end{aligned}} \qquad (2)$$

$\boxed{}$: related to $T_{\mu\nu}$ and zero if $T_{\mu\nu} = 0$

Exercise :

Derive expression (2)

(83)

The matter quantities are given respectively by

$S_{\mu\nu} := \gamma^{\alpha}{}_{\mu} \gamma^{\beta}{}_{\nu} T_{\alpha\beta}$: spatial part of energy momentum tensor

$S_{\mu} := -\gamma^{\alpha}{}_{\mu} n^{\beta} T_{\alpha\beta}$: momentum density

$S := S^{\mu}{}_{\mu}$: trace of S

$E := n^{\alpha} n^{\beta} T_{\alpha\beta}$: energy density measured by Eulerian observer

In the red boxes we have the evolution part of the ADM eqs

$$\boxed{\begin{aligned} \partial_t \gamma_{ij} &= -\alpha 2 K_{ij} + \cdots \quad (I) \\ \partial_t K_{ij} &= -D_i D_j \alpha + \cdots \quad (II) \end{aligned}}$$

$\dot{x} = v$: kinematics

$\dot{v} = a$: dynamics

These the eqs. that provide the evolution of the 3-metric and extrinsic curvature

How many eqs?

Let's go back to Einstein eqs. $G_{\mu\nu} = 8\pi T_{\mu\nu}$

$G_{\mu\nu}$: 4×4 matrix $\Rightarrow$ 16 components ; symmetric tensor $\Rightarrow$

10 components $\Rightarrow$ 10 2nd-order PDEs $\Rightarrow$ 20 1st-order PDEs

simplest 2nd-order PDE : wave equation

$\Box \phi = 0$

$\partial_t^2 \phi - v^2 \partial_x^2 \phi = 0$

Define $\partial_t \phi = X \Rightarrow$

$\partial_x \phi = \psi$

$\partial_t X = v^2 \partial_x \psi$

$\partial_t \phi = X$

gone from ① 2nd-order eq. to ② first-order eqs.

(85)

We have seen that we have derived evolution eqs.

$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \dots$ 6 eqs

$\partial_t K_{ij} = -D_i D_j \alpha + \dots$ 6 eqs

12 eqs.

There are still 8 equations that we have not yet accounted for.

There are some contractions we have not considered

$\boxed{\gamma \cdot \gamma \cdot \mathrm{Riem}}$ $\quad \overbrace{\gamma^{\alpha\mu}\,\gamma^{\beta\nu}}\, R_{\alpha\beta\mu\nu} = 2 G_{\mu\nu} n^\mu n^\nu \iff$

$$(\mathrm{III}) \quad \boxed{{}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E} \quad 1 \text{ eq.}$$

$\boxed{\gamma \cdot n \cdot G}$ $\quad \gamma^{\alpha\mu} n^\nu G_{\mu\nu} = \gamma^{\alpha\mu} n^\nu R_{\mu\nu} = D^\alpha K - D_\mu K^{\alpha\mu}$

$$\iff (\mathrm{IV}) \quad \boxed{D_j \left(K^{ij} - \gamma^{ij} K\right) = 8\pi S^i} \quad 3 \text{ eqs}$$

In contrast to (I) and (II), (III) and (IV) do not have time derivatives: they are constraint equations

$$\partial_t \vec{E} = \vec{\nabla} \times \vec{B} - 4\pi \vec{j} \qquad \partial_t E_i = \epsilon_{ijk} \partial^j B^k - 4\pi J_i$$

$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E} \qquad \partial_t B_i = -\epsilon_{ijk} \partial^j E_k$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \partial_i B^i = 0$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e \qquad \partial_i E^i = 4\pi \rho_e$$

constraint eqs.

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The analogies between the ADM and the Maxwell equations will help us understand some of the problems associated with the ADM eqs and suggest possible solutions.

In practice, the ADM equations have not been used in 3D applications and their use has been abandoned when it has become clear that they are <u>weakly hyperbolic</u>.

To appreciate the implications of this statement we need a small digression.

A large class of equations in mathematical physics (eg. EFEs, eqs. of hydrodynamics and MHD) can be written in a compact form as

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = \underline{S} \qquad (*)$$

$$\partial_t U_J + (A^i)_{JK} \nabla_i U_K = S_J$$

where $\underline{U} = \{U_1, U_2, \dots U_J\}$: state vector

$\underline{S} = \{S_1, S_2, \dots S_J\}$: source term

A : matrix of coefficients and

The properties of the system (*) depend on the properties of A, and S.

(i) $\left\{ \begin{array}{l} a_{JK} \text{ : elements of } A \\ a_{JK} = \text{const.} \ ; \quad S_J = \text{const} \end{array} \right\}$ (*) is a LINEAR system of equations with constant coefficients

(ii) $\left\{ a_{JK} = a_{JK}(x,t) \ ; \quad S_J = S_J(x,t) \right\}$ (*) is a LINEAR system with variable coefficients

(iii) $A = A(\underline{u})$ (*) is a non LINEAR system (often referred to as quasi-linear)

More importantly, the system (*) is said to be (strongly) HYPERBOLIC if A is diagonalizable with a set of real eigenvalues $\lambda_1, \dots \lambda_N$ and a set of N linearly independent right eigenvectors, ie if

$$\Lambda := R^{-1} A R = \mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_N)$$

R: matrix of right eigenvectors $R^{(i)}$

$$A R^{(i)} = \lambda_i R^{(i)}$$

$\lambda_i \in \mathbb{R}$: real eigenvalues

(*) is said to be <u>STRICTLY HYPERBOLIC</u> if $\lambda_{(i)}$ are real and distinct

(*) is said to be <u>SYMMETRIC HYPERBOLIC</u> if A is symmetric, ie $A = A^T$

(*) is said to be <u>WEAKLY HYPERBOLIC</u> if A is not diagonalizable

Examples of hyperbolic equations are

- advection equation $\partial_t u + v\, \partial_x u = 0$
- wave equation $\partial_t^2 u - v^2 \partial_x^2 u = 0$
- hydrodynamic equations (inviscid)
- Einstein equations (only in harmonic coordinates $\Box x^\alpha = 0$)

The importance of hyperbolicity is strictly related with that of WELL POSEDENESS of the Cauchy initial-value problem.

$\underline{u}(x,0)$: initial data

$\underline{u}(x,t)$: solution of set (*) at time t

(*) is well posed if

$$\| u(x,t) \| \leq k e^{at} \| u(x,0) \|$$

$k, a \in \mathbb{R}$: ~~arbitrary~~ constants.

In other words the solution is always bounded by some exponential of the initial data ("it does not blow up...")

An important theorem of hyperbolic systems states

(*) a hyperbolic set of equations $\Longrightarrow$ (*) is well posed

Opposite implication not true.

It follows that a weakly hyperbolic system is not guaranteed to be well-posed and indeed the numerical solution leads to the growth of unstable modes ("codes crash"...)

In the case of the ADM equations the weak hyperbolicity comes from the mixed derivatives in the Ricci tensor for the evolution of the extrinsic curvature, ie

$$\partial_t K_{ij} = - D_i D_j \alpha + \alpha \left(R_{ij} + \dots \right)$$

$\gamma_l{}^i \gamma_m{}^j \partial_i \partial_j$ mixed second derivatives, eg

$\partial_x \partial_y$, $\partial_x \partial_z$, $\partial_y \partial_z$

while one wishes to have diagonal second derivatives, ie

$\gamma^{ij} \partial_i \partial_j$: ∂^2_x, ∂^2_y, ∂^2_z

There a number of ways around this problem and the easiest way to understand how this works is to look at Maxwell eqs.

Let's introduce vector potential $\vec{A} = \vec{\nabla} \times \vec{B}$

$$A_\mu = (-\phi, A_i) \qquad \bar{E} = -\bar{\nabla}\phi$$

Then Maxwell eqs.

$$\partial_t A_i = -E_i - D_i \phi$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

[of course $D_i \leftrightarrow \partial_i$ but I'm keeping D to highlight analogy with GR]

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The term $D_i D^j A_j$ breaks hyperbolicity and we want to get rid of it. Different ways to do this

1) Lorenz gauge $\partial_t \phi = -D^j A_j$

This is good because if I take a time derivative of $\partial_t A_i$

$$-\partial_t^2 A_i + D^j D_j A_i - \underbrace{D_i D^j A_j} = D_i \partial_t \phi - 4\pi J_i \qquad \partial_t \phi \to = \underbrace{D_i(-D^j A_j)}$$

$\Longrightarrow$

$$-\partial_t^2 A_i + D^j D_j A_i = -4\pi J_i \iff \Box A_i = 4\pi J_i$$

$$\boxed{\begin{array}{l}\text{Recall: } \Box \text{ : Dalambertian operator (wave)} \\ \Box \phi = (\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)\phi = 0 \text{ for wave eq.}\end{array}}$$

This is ~~an~~ hyperbolic equation and hence well-posed!

2) Introduce a new variable

$$\Gamma := D^i A_i \qquad : \text{scalar function}$$

then I obtain

$$\Box A_i = -D_i \Gamma - D_i \partial_t \phi + 4\pi J_i$$

Written in this way the eqs are again hyperbolic because the "principal part" ($\Box A_i$) is the same; the "nasty" term is now just a source term on the RHS.

To fix the ADM equations and remove weak hyperbolicity we do something very similar: ie we introduce new quantities.
Before doing this, we need also some other ingredients.

1) Introduce a conformally related metric,

ie

$$g_{\mu\nu} \longleftrightarrow \tilde{g}_{\mu\nu} = \phi^{n} g_{\mu\nu}$$

physical metric — conformally related metric

ϕ: conformal factor so that

$$ds^2 = \phi^{+n} g_{\mu\nu}\, dx^\mu dx^\nu = \tilde{g}_{\mu\nu}\, dx^\mu dx^\nu$$

A conformal metric allows to set some additional conditions on the properties of the determinant of the corresponding 3-metric. In particular

2) $$\tilde{\gamma}_{ij} = \phi^2 \gamma_{ij} \qquad \tilde{\gamma}^{ij} = \phi^{-2} \gamma^{ij}$$

so that we can impose the volume element ⊙ to be

$$\tilde{\gamma} := \det(\tilde{\gamma}_{ij}) = 1$$

and the conformal factor is given by

$$\phi = (\det(\gamma_{ij}))^{-1/6} = \gamma^{-1/6} \quad (\cdot\cdot)$$

⊙ Recall that $V_\Omega = \int_\Omega \sqrt{\det(\gamma_{ij})}\, d^3x$	$(\cdot\cdot)$ $\tilde{\gamma} = \phi^6 \gamma = 1 \Rightarrow \phi^6 = \gamma^{-1}$

3) Introduce the conformally related Christoffel symbols (or Christoffel symbols of conformal three metric)

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2\left(\delta^i_j \partial_k \ln\phi + \delta^i_k \partial_j \ln\phi - \gamma_{jk}\gamma^{il}\partial_l \ln\phi\right)$$

4) Introduce the conformally related extrinsic curvature

$$\tilde{A}_{ij} = \phi^2 A_{ij} = \phi^2\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) ; \quad \tilde{A}^{ij} = \phi^{-2}A^{ij}$$

so that

$$\tilde{A}_{ij}\gamma^{ij} = \tilde{A}^i{}_i = \phi^2\left(K^i{}_i - \frac{1}{3}\gamma_{ij}\gamma^{ij}K\right) = 0$$

In other words the conformal extrinsic curvature is <u>traceless</u>

5) introduce additional variables to separate mixed derivatives : "Gammas"

$$\tilde{\Gamma}^i := \tilde{\gamma}^{jk}\,\tilde{\Gamma}^i_{jk} = \tilde{\gamma}^{ij}\,\tilde{\gamma}^{kl}\,\partial_l\,\tilde{\gamma}_{jk}$$

The resulting set of equation is then:

f. DM

$$\left[\begin{array}{l} \partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + 2\tilde{\gamma}_{k(i}\,\partial_{j)}\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\,\partial_k\beta^k + \beta^k\,\partial_k\,\tilde{\gamma}_{ij} \\ \partial_t \tilde{A}_{ij} = \phi^2\left[-D_iD_j\alpha + \alpha\left({}^{(3)}R_{ij} - 8\pi S_{ij}\right)\right]^{TF} + \beta^k\,\partial_k\,\tilde{A}_{ij} + \dots \end{array}\right]$$

$$\partial_t \phi = \frac{1}{3}\phi\,\alpha K - \frac{1}{3}\partial_i\beta^i + \beta^k\,\partial_k\,\phi$$

$$\partial_t K = -D_i D^i \alpha + \alpha\left[\tilde{A}_{ij}\,\tilde{A}^{ij} + \dots\right.$$

$$\partial_t \tilde{\Gamma}^i = \tilde{\gamma}^{jk}\,\partial_j\partial_k\,\beta^i + \frac{1}{3}\tilde{\gamma}^{ik}\,\partial_k\,\partial_j\,\beta^j + \dots$$

where "TF" indicates that the trace-free part of the bracket is used, ie $\left({}^{(3)}R_{ij}\right)^{TF} \longrightarrow {}^{(3)}R_{ij} - \frac{1}{3}\gamma_{ij}\,{}^{(3)}R^{k}{}_{k}$

$\square$

These equations are those normally used and are referred to as the BSSNOK formulation

When comparing with the ADM equations we have clearly gained 5 more evolution eqs: $\partial_t \phi$, $\partial_t K$, $\partial_t \tilde{\Gamma}^i$ but the system is now hyperbolic and indeed well behaved in numerical simulations.

The additional computational costs: 12 variables $\longrightarrow$ 15 variables

$(12+5-2)$ $\tilde{\gamma}_{ij}$ $\tilde{A}_{ij}$ are traceless or with known trace

The constraint equations are then

$$\boxed{{}^{(3)}R = \tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2 + 16\pi E}$$

Proof

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$$

LHS :

$${}^{(3)}R + K^2 - \left(\tilde{A}_{ij} + \tfrac{1}{3}\gamma_{ij}K\right)\left(\tilde{A}^{ij} + \tfrac{1}{3}\gamma^{ij}K\right) =$$

$$= {}^{(3)}R + K^2 - \left(\tilde{A}_{ij}\tilde{A}^{ij} + \tfrac{2}{3}\cancel{\tilde{A}^i{}_i K} + \tfrac{1}{9}\cdot 3K^2\right)$$

$$\tilde{A}^i{}_i = \tilde{A}^{ij}\gamma_{ij} = \left(K^{ij} - \tfrac{1}{3}\gamma^{ij}K\right)\gamma_{ij} = K - K = 0$$

$$= {}^{(3)}R + K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - \tfrac{1}{3}K^2$$

$$= {}^{(3)}R - \tilde{A}_{ij}\tilde{A}^{ij} + \tfrac{2}{3}K^2 = 16\pi E \implies {}^{(3)}R = \tilde{A}_{ij}\tilde{A}^{ij} - \tfrac{2}{3}K^2 + 16\pi E$$

$$D_j\left(K^{ij} - \gamma^{ij} K\right) = 8\pi S^i \qquad \Longleftrightarrow$$

$$D_j\left(\hat{A}^{ij} + \frac{1}{3}\gamma^{ij} K - \gamma^{ij} K\right) = 8\pi S^i$$

$$D_j\left(\hat{A}^{ij} - \frac{2}{3}\gamma^{ij} K\right) = 8\pi S^i$$

$$\tilde{\gamma}^{ij} = \phi^{-2}\gamma^{ij}$$

$$\gamma^{ij} = \phi^{2}\tilde{\gamma}^{ij}$$

$$\boxed{D_j\left(\hat{A}^{ij} - \frac{2}{3}\phi^2\tilde{\gamma}^{ij} K\right) = 8\pi S^i}$$

$\square$

$$\mathcal{H} : \quad 0 = \tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2 - {}^{(3)}R + 16\pi E$$

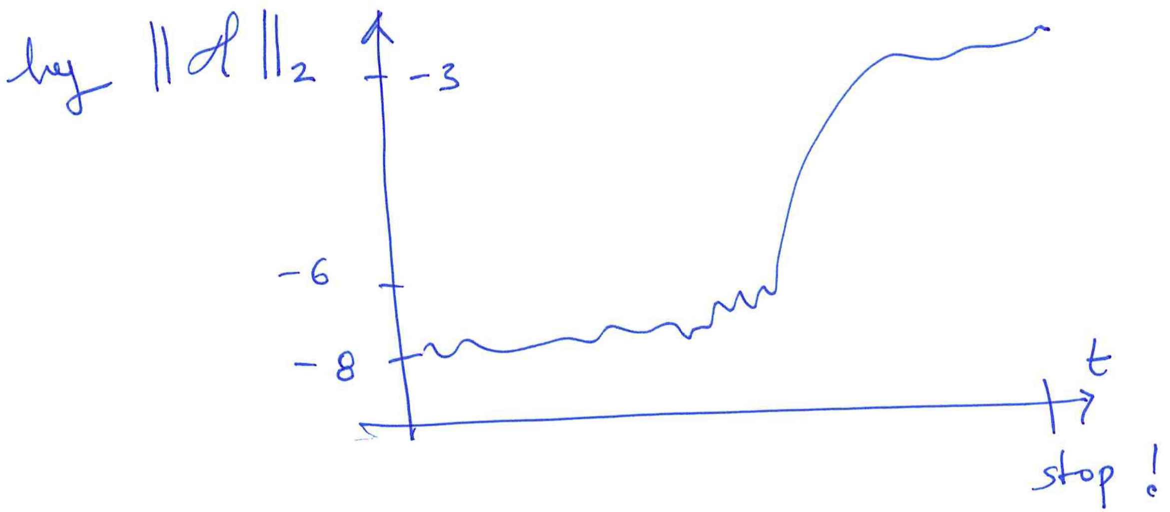
$$\mathcal{M}^i : \quad 0 = D_j\left(\hat{A}^{ij} - \frac{2}{3}\phi^2\tilde{\gamma}^{ij} K\right) - 8\pi S^i$$

How are the constraints handled?

These are 3D nonlinear elliptic equations to be solved on each slice and very expensive to solve.

Solving one of them is more expensive than computing the full set of evolution equations.

In practice the constraints are monitored as a measure of the quality of the solution



A property that the BSSNOK does not have is that of damping the constraint violations, ie the property of reducing the violations if a violation is for some reason introduced.

Let's go back to EM

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \partial_i B^i = 0$$

Let $\psi := \partial_i B^i$ and assume it is not zero initially

$$\partial_t \psi = \partial_t \partial_i B^i = \partial_i \partial_t B^i = -\partial_i \epsilon^{ijk} \partial_j E_k = -\epsilon^{ijk} \partial_i \partial_j E_k = 0$$

$$\Longleftrightarrow \underline{\psi = \text{const}}$$

So if $\psi = 0$ initially, $\psi = 0$ at all times but if $\psi \neq 0$ " , $\psi \neq 0$ at all times. This is not a desirable feature which can be compensated if we write

$$\begin{cases} \partial_t B^i = -\epsilon^{ijk}\partial_j E_k + \gamma^{ij}\partial_j \Psi \\ \partial_t \Psi = -a_1 \partial_i B^i - a_2 \Psi \end{cases} \qquad a_1 > a_2 \in \mathbb{R}^+$$

Clearly we return to the initial system if

$$\Psi(x,0) = 0 = \partial_t \Psi(x,0)$$

However, if $\partial_i B^i \neq 0$ then the action of the scalar field will be that of driving the solution exponentially fast towards $\Psi = 0 = \partial_i B^i$.

This spirit is behind new formulations of the EFEs that have been developed over the last couple of years. These are the CCZ4 and the Z4c formulation. Details can be found in the book.

$\gamma \cdot \gamma \cdot \gamma \cdot \gamma \cdot$ Riem.

$\gamma \cdot \gamma \cdot \gamma \cdot n \cdot$ Riem.

...

$\checkmark$

Recap

- we have seen that a series of projections of the Riemann tensor lead to evolution eq. for the extrinsic curvature

$$\partial_t K_{ij} = - D_i D_j \alpha + \alpha \left({}^{(3)}R_{ij} + K K_{ij} - \dots \right)$$

- in addition there are 4 constraint equations ($\vec{\nabla} \cdot \vec{B} = 0$ in Maxwell)
- presence of mixed derivatives on RHS of $\partial_t K_{ij}$ eq. makes the ADM system weakly hyperbolic

- no theorem guarantees that the system is well posed

$$\|u(t,x)\| \leq k e^{at} \|u(0,x)\| \qquad k, a \in \mathbb{R}$$

solution at given time

initial data

(1039)

- analogy with EM fields and Maxwell eqs reveals that a way to recover hyperbolicity^(•) is to introduce new variables that preserve the principal part of the eqs.

- this is done with the BSSNOK formulation of the EFEs

- the formulation is conformal and traceless.

$$\tilde{\gamma}_{ij} = \phi^2 \gamma_{ij} \qquad \tilde{\gamma}^{ij} = \phi^{-2} \gamma^{ij}$$

$$\underbrace{\tilde{\gamma} = \det(\tilde{\gamma}_{ij}) = 1}_{\text{unit volume element}} \implies \phi = (\det(\gamma_{ij}))^{-1/6} = \gamma^{-1/6}$$

(•) $\partial_t \underline{u} + A \cdot \nabla \underline{u} = \underline{S}$ is hyperbolic if A is diagonalizable with real eigenvalues and indip. set of right eigenvectors $\Lambda = R^{-1} A R = \mathrm{diag}(\lambda_1 \dots \lambda_N)$

$A R^{(i)} = \lambda_i R^{(i)}$

$$\tilde{A}_{ij} = \phi^2 \left(K_{ij} - \frac{1}{3}\gamma_{ij} K\right) \quad ; \quad \tilde{A}^i{}_i = 0$$

conformal traceless ext. curvature

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk}$$

Gammas

similar to ADM
$$\left[\begin{array}{l} \partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \dots \\ \partial_t \tilde{A}_{ij} = \phi^2 \left[-D_i D_j \alpha + \dots \right. \end{array}\right.$$

$$\partial_t \phi = \frac{1}{3} \phi \alpha K + \dots$$

$$\partial_t K = -D_i D^i \alpha + \dots$$

$$\partial_t \tilde{\Gamma}^i = \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i + \dots$$

This formulation is <u>hyperbolic</u>

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The constraint equations are then

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Proof

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$$

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$$\boxed{D_j\left(\tilde{A}^{ij} - \frac{2}{3}\phi^2 \tilde{\gamma}^{ij} K\right) = 8\pi S^i}$$

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$$\gamma^{ij} = \phi^{2}\tilde{\gamma}^{ij}$$

$\square$

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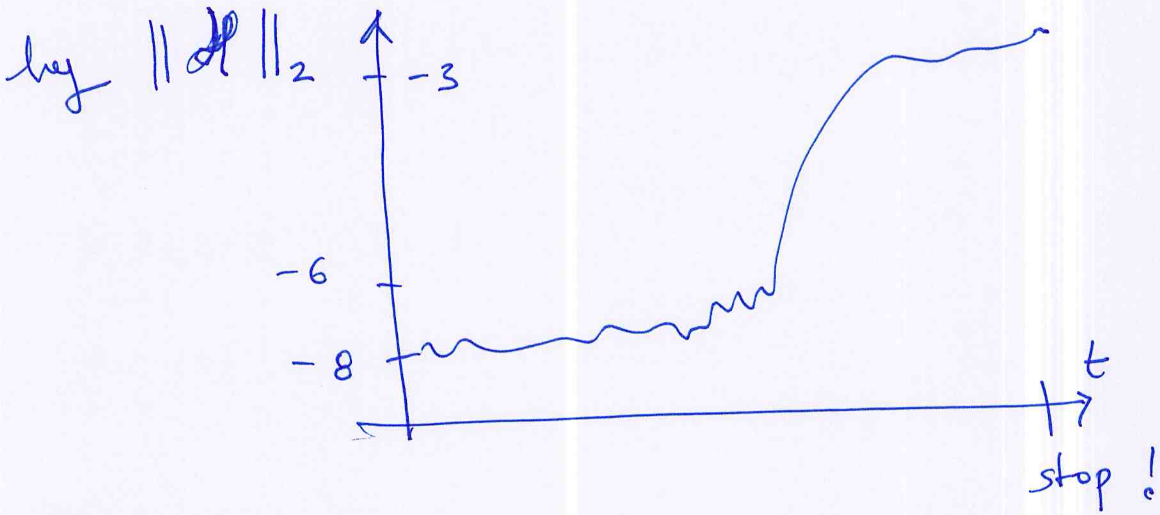

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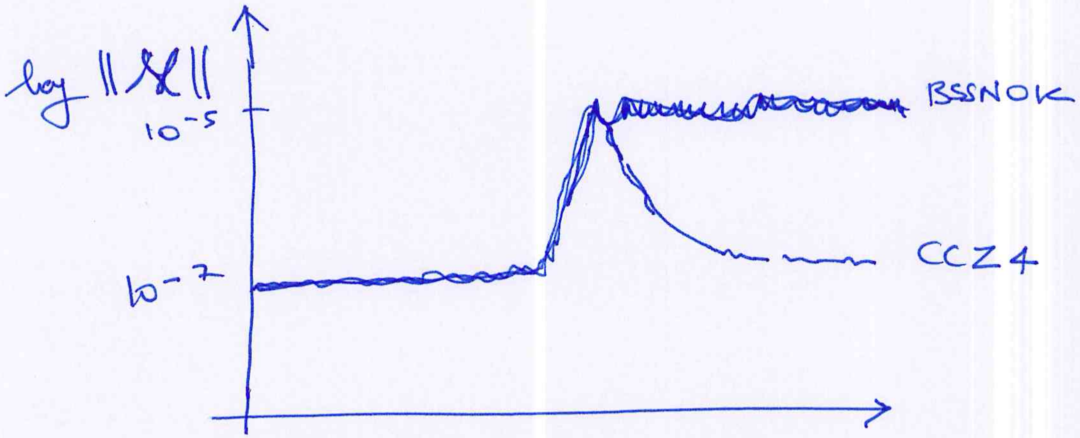


the addition of constraint damping terms in the CCZ4 formulation lead to exponential suppression of violations.

$\square$

GAUGE CONDITIONS

Let's go back to the counting of the equations:

$G_{\mu\nu} = 8\pi T_{\mu\nu}$

10 eqs, 2nd-order in time

$\Rightarrow$ 20 eqs. 1st-order in time

$\Longrightarrow$ ADM eqs

$\partial_t \gamma_{ij} = \dots$ (6)

$\partial_t K_{ij} = \dots$ (6)

${}^{(3)}R - K_{ij}K^{ij} + K^2 = \dots$ (1)

$D_j (K^{ij} - \gamma^{ij} K) = \dots$ (3)

16

4 equations still missing

The four missing equations represent the gauge freedom inherent to general relativity and correspond to the freedom in specifying the lapse function α : SLICING CONDITION and the shift vector β^i : spatial gauge condition.

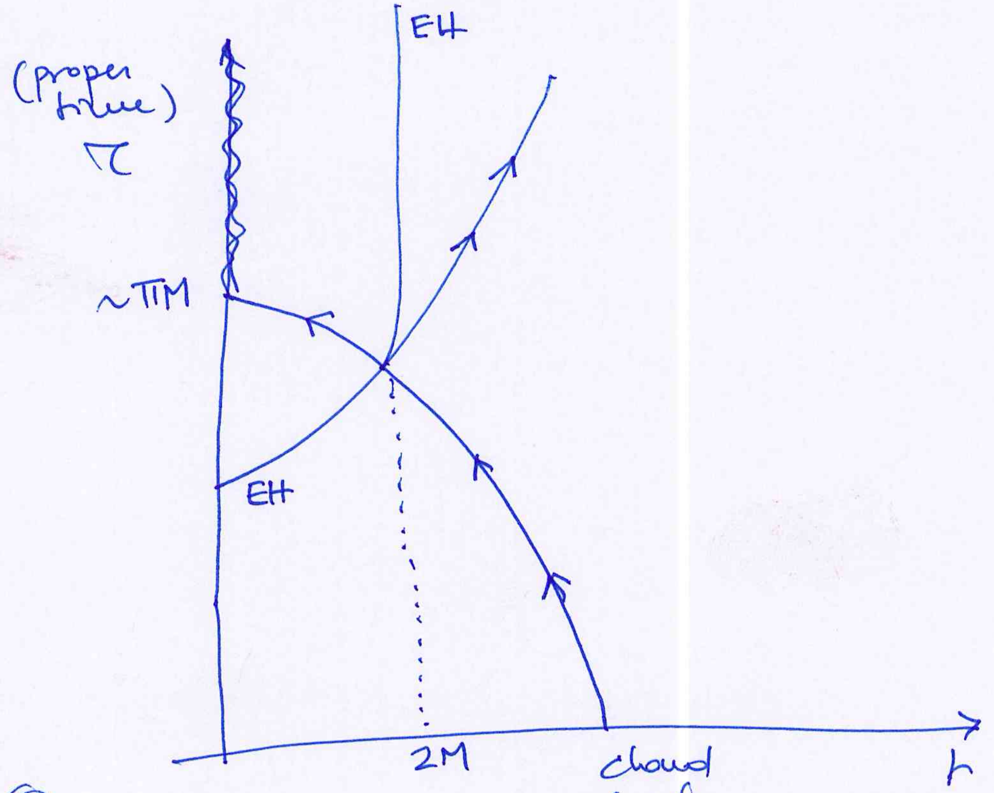
The arbitrariness of the gauges implies that the large majority of the quantities computed in a numerical relativity simulation are "gauge dependent" ie they will depend on the gauge used. This however does not prevent the existence and calculation of "gauge independent" quantities, eg. gravitational radiation, masses of spacetime, etc.

Arbitrariness of gauge choice is an advantage and a handicap at the same time. Good gauge conditions make difference between stable evolution and crash.

Requirements of good gauge choices

(i) if singularities are present or develop on a slice, these are avoided: <u>singularity avoiding property</u>

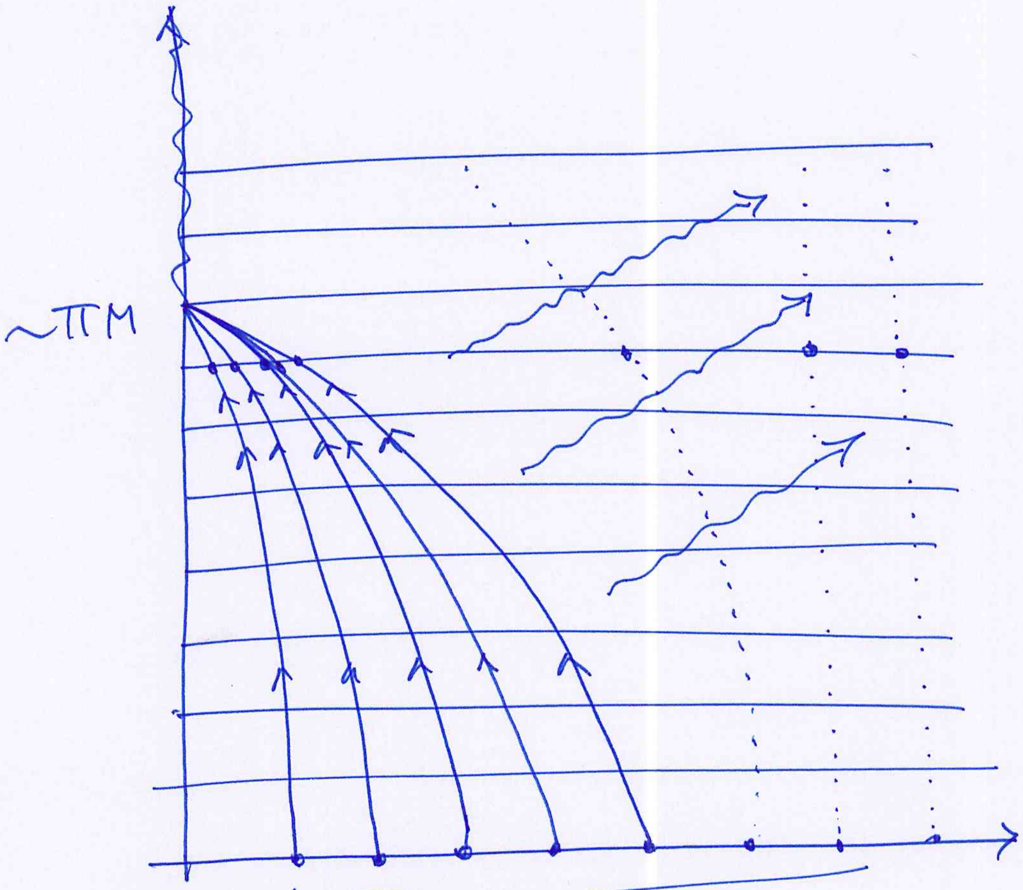
Example: Oppenheimer-Snider collapse (analytic solution of the collapse of a collisionless cloud of particles)



- In this solution a physical singularity is produced when all fluid shells reach the center. This happens at the same proper time for all shells

- An event horizon starts to grow from zero size and reaches the asymptotic value when the cloud surface crosses the r=2M surface

There are several different ways of slicing this spacetime. The simplest is called <u>geodesic slicing</u>; this is the slicing of freely falling particles : $\alpha = 1$; $\beta^i = 0$

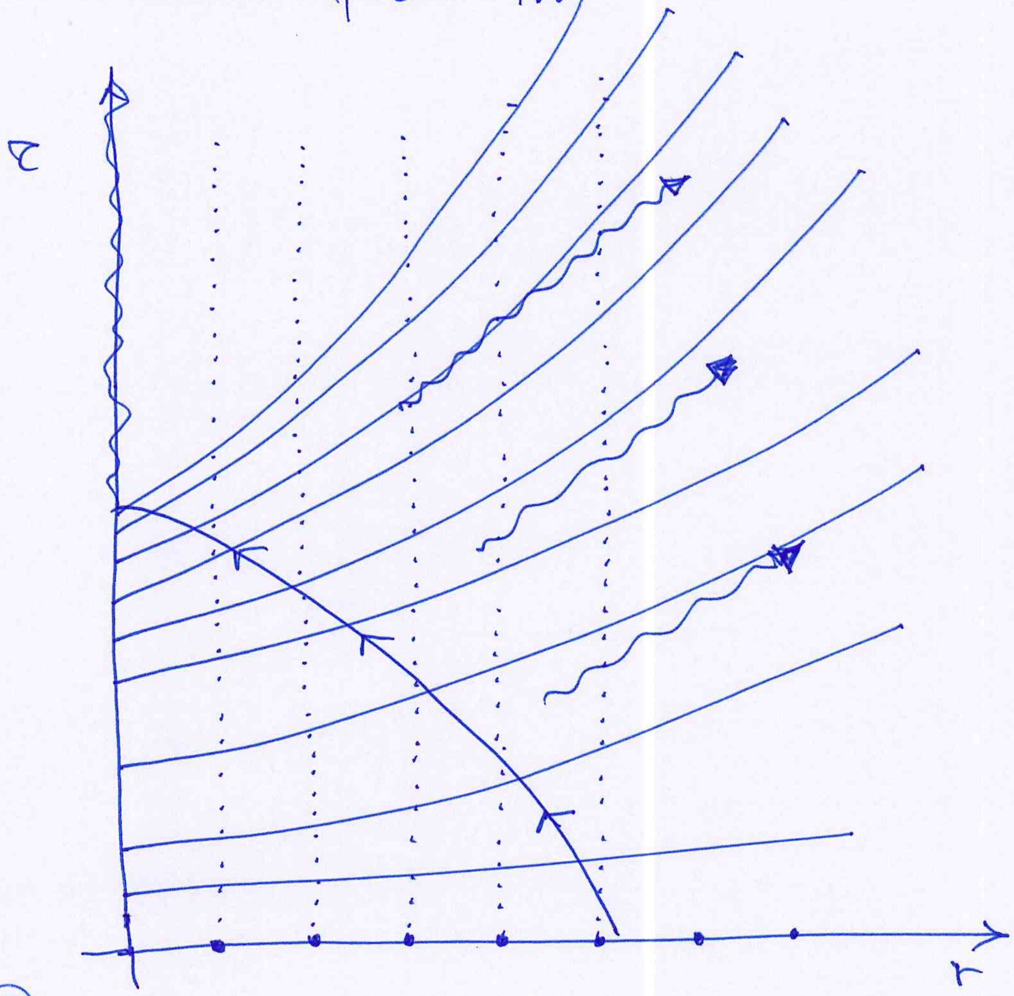


all coordinates are equally spaced

This approach has at least two undesired features.

- code crashes when the singularity develops (coordinates coincide with portion of spacetime with singular curvature); this happens <u>before</u> any radiation reaches distant observer
- nicely spaced coordinates end up concentrated and resolution is bad in outer regions.

How can we improve on this and avoid the pathologies mentioned above? The slicing should be such that the coordinate time is slowed down near the singularity.
The coordinates should be "pushed out" such that they do not fall in.



- In this way the time is "slowed down" near the singularity. It is as if you never crash against a wall towards which you are accelerating just because you reach it ever more slowly

- All the radiation that needs to reach a distant observer has the time to do so.

How do we impose this singularity-avoiding condition?

It's not difficult to show that the condition $K = 0 = \gamma^{ij} K_{ij}$ leads to coordinate volume elements that are <u>maximal</u>, ie they are at a maximum wrt small variations in time

$$V = \int \sqrt{\det(\gamma_{ij})}\, d^3x = \int \sqrt{\gamma}\, d^3x \qquad : \underline{\text{volume element}}$$

$$\partial_t V \propto \int \alpha K \sqrt{\gamma}\, d^3x \qquad \text{so that} \quad K=0 \Rightarrow \partial_t V = 0 \; \odot$$

So what we want is not only that $K=0$ but also that $\partial_t K = 0$.

$$\boxed{K = 0 = \partial_t K}$$

This condition is called maximal slicing and requires that Eulerian observers in free fall do not "focus" where extrinsic curvature increases

$\beta^i = 0$

$\odot$ **Exercise**

A bit of algebra then leads to show that the maximal slicing condition is equivalent to

$$\gamma^{ij} \partial_i \partial_j \alpha = D^2 \alpha = \alpha \left[K_{ij} K^{ij} + 4\pi (E+S) \right]$$

This is an elliptic equation that needs to be solved on every spatial slice Σ_t. As for previous elliptic equations we have discussed before, this equation is computationally too expensive to be solved on each Σ_t. In practice hyperbolic (ie time evolving) slicing conditions are used that enforce a condition similar to the maximal slicing condition. A popular choice is the so-called Bona-Massó family of slicing conditions

$$\partial_t \alpha - \beta^k \partial_k \alpha = -f(\alpha)(K - K_0)$$

where $f(\alpha) > 0$ and $K_0 := K(t=0)$ initial trace

Different values of f allow one to obtain several important and well known slicing conditions.

Example

- $f = 1 \implies \partial_t \alpha \propto -K\alpha^2$: harmonic slicing

- $f = \frac{q}{\alpha}$ $q \in \mathbb{N}$

This condition then leads to a lapse function for $q = 2$ and $\underline{\beta} = 0$

$\alpha = 1 + \log \gamma$: $1 + \log$ slicing

The harmonic slicing enforces the condition

$$\Box t = 0 = \nabla_\mu \nabla^\mu t$$

It's not difficult to show that this leads to

$$(\partial_t - \beta^k \partial_k)\alpha = -K\alpha^2$$

Exercise

Proof

Recall

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\gamma^{ij}(\partial_t \gamma_{ij} - D_i \beta_j - D_j \beta_i) = -2\alpha K_{ij} \gamma^{ij} = -2\alpha K$$

(16) $$D_j \beta^j - \alpha K = \frac{1}{2} \gamma^{ij} \partial_t \gamma_{ij}$$

Using now the theorem on determinants of a matrix A

$$\delta \ln |\det A| = \mathrm{tr}\,(A^{-1} \times \delta A) \qquad \delta: \text{ derivative}$$

we have

$$\frac{1}{2}\gamma^{ij}\partial_t \gamma_{ij} = \frac{1}{2}\partial_t \ln \gamma = \partial_t \ln \sqrt{\gamma} = \frac{1}{\sqrt{\gamma}}\partial_t \sqrt{\gamma} \Rightarrow$$

$$D_j \beta^j - \alpha K = \frac{1}{\sqrt{\gamma}}\partial_t \sqrt{\gamma} \Rightarrow$$

$$-\alpha K = -D_j \beta^j + \frac{1}{\sqrt{\gamma}}\partial_t \sqrt{\gamma}$$

$$(\partial_t - \beta^k \partial_k)\alpha = -2\alpha(K - K_0) \overset{K_0=0}{=} -2\alpha K = -2D_j\beta^j + \frac{2}{\sqrt{\gamma}}\partial_t\sqrt{\gamma}$$

$$= -2D_j\beta^j + \partial_t \ln \gamma$$

Imposing normal coords ($\beta^i = 0$)

$$\partial_t \alpha = \partial_t \ln \gamma$$

$$\boxed{\alpha = 1 + \ln \gamma} \quad \square$$

This is an <u>algebraic slicing</u> i.e. no evolution equation is required.

In practice we use evolution equation

$$(\partial_t - \beta^k \partial_k)\alpha = -2\alpha(K - K_0) \qquad \text{because } \begin{matrix}\beta \neq 0 \\ K_0 \neq 0\end{matrix}$$

Question: where is the singularity avoiding property? $\partial_t \alpha \sim -\alpha \Rightarrow \alpha = \alpha_0 e^{-t}$

- Finally if $f \to \infty$ the Bona-Massó family tends to the maximal slicing condition. □

Similar considerations apply also for the spatial gauge conditions, ie for the conditions to be imposed on the shift vector.
The second requirement of our optimal gauge conditions is that
(ii) if coordinate conditions take place (eg collapse of coordinates, stretching or rotation) these are counteracted to minimize the distortion.

Mathematically this is not difficult to impose. Consider the <u>metric strain tensor</u> which measures size and shape of volume element

$$\Theta_{ij} := \frac{1}{2} \mathcal{L}_{\underline{t}}\, \gamma_{ij} = \frac{1}{2}\left(\alpha \mathcal{L}_{\underline{n}} + \mathcal{L}_{\underline{\beta}}\right) \gamma_{ij} = -\alpha K_{ij} + \frac{1}{2} \mathcal{L}_{\underline{\beta}}\, \gamma_{ij}$$

Compute contraction $\Theta_{ij}\Theta^{ij}$ over spatial slice Σ_t and minimize its variation leads to

$$D_j \Theta_{ij} = 0 \quad \Longrightarrow \quad D^2 \beta^i + D^i D_j \beta^j + R^i{}_j \beta^j = 2 D_j \left(\alpha K^{ij}\right)$$

<u>minimal strain condition</u>. These are perfectly reasonable but three elliptic equations to be solved on Σ_t

The situation does not improve considerably when considering the <u>distortion tensor</u>, which measures only the change in shape of volume elements and not the change in size

This tensor is defined as the tracefree part of Θ_{ij}

$$\Sigma_{ij} := \Theta_{ij} - \frac{1}{3}\gamma_{ij}\Theta_{ke}\gamma^{ke}$$

Working in a similar manner, integrating positive contraction of $\Sigma_{ij}\Sigma^{ij}$ and minimizing its variations one obtains

$$D^j\Sigma_{ij} = 0 \Rightarrow D^2\beta^i + \frac{1}{3}D^iD_j\beta^j + R^i{}_j\beta^j = 2D_j(\alpha A^{ij})$$

minimal-distortion shift condition

once again, we have obtained the required condition but at the expense of a new set of three elliptic equations.

This is computationally not feasible with present solvers and computers and alternative approaches are usually employed.

In practice, most 3+1 simulations use the following hyperbolic conditions

$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \frac{3}{4} B^i$$

$$\partial_t B^i - \beta^j \partial_j \beta^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i$$

where B^i is an auxiliary variable. This gauge condition is called "Gamma-driver" shift condition and it essentially changes the shift in such a way as to "freeze" the evolution of the Gammas, ie such that $\partial_t \tilde{\Gamma}^i \simeq 0$

In turn, this amounts to a condition that tends to minimize changes in the distortion tensor(o) Exercise

(o) $\tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk} = \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \partial_l \tilde{\gamma}_{jk}$ ie $\tilde{\Gamma} \sim (\gamma\gamma D\gamma)$

$\partial_t \tilde{\Gamma} = 0 \iff \partial_t (\gamma\gamma D\gamma) = 0 \cong D(\gamma\gamma \partial_t \gamma) \sim D\Sigma = 0$

The Gamma driver works well in most cases but it still represents a tunable prescription. For example, the term η is called damping term and is used to damp large oscillation in the shift (think of a BH moving on the grid). $\eta = 2/M$, where M is the ADM mass of the spacetime, but it can be chosen also to be a function of space and time. In other words: there is a lot of flexibility in the definition of the gauges and a certain trial and error is necessary for most difficult cases.

Recap:
most commonly used gauges are

$$\begin{cases} (\partial_t - \beta^k \partial_k)\,\alpha = -2\alpha\,(K - K_0) \\ (\partial_t - \beta^k \partial_k)\,\beta^i = \frac{3}{4} B^i \\ (\partial_t - \beta^k \partial_k)\,B^i = (\partial_t - \beta^k \partial_k)\tilde{\Gamma}^i - \eta B^i \end{cases}$$

2.141

THE YORK - LICHNEROWICZ CONFORMAL TRACELESS DECOMP.

1

$$\bar{\gamma}_{ij} = \psi^{-4}\gamma_{ij} \quad \text{or} \quad \gamma_{ij} = \psi^{4}\bar{\gamma}_{ij} \qquad \bar{\gamma} \neq 1$$

$$\det(\gamma_{ij}) = \psi^{12}\det(\bar{\gamma}_{ij}) \quad \text{or} \quad \gamma = \psi^{12}\bar{\gamma}$$

$$A_{ij} = K_{ij} - \frac{1}{3}K\gamma_{ij} \quad \Leftrightarrow \quad K_{ij} = A_{ij} + \frac{1}{3}K\gamma_{ij}$$

$$R = \psi^{-4}\bar{R} - \frac{8}{\psi^5}\bar{D}_i\bar{D}^i\psi \Longrightarrow \text{~~Prove~~ Exercise 1}$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi E \qquad \text{Hamilt. Con}$$

$$\psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}_i\bar{D}^i\psi + K^2 - \left(A_{ij} + \frac{1}{3}K\gamma_{ij}\right)\left(A^{ij} + \frac{1}{3}K\gamma^{ij}\right) = 16\pi E$$

$$\psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}_i\bar{D}^i\psi + K^2 - A_{ij}A^{ij} - \frac{1}{\cancel{9}_3}K^2\cdot\cancel{3} = 16\pi E$$

$$\psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}_i\bar{D}^i\psi - A_{ij}A^{ij} + \frac{2}{3}K^2 = 16\pi E$$

$$\psi\bar{R} - 8\bar{D}_i\bar{D}^i\psi - \psi^5\left(A_{ij}A^{ij} - \frac{2}{3}K^2\right) = 16\pi E\psi^5$$

$$\boxed{8\bar{D}_i\bar{D}^i\psi - \psi\bar{R} + \psi^5\left(A_{ij}A^{ij} - \frac{2}{3}K^2\right) + 16\pi E\psi^5 = 0}$$

$$\bar{D}_j\left(A^{ij} + \frac{1}{3}K\gamma^{ij}\right) - \gamma^{ij}\bar{D}_jK = 8\pi S^i \qquad \text{Mom. Con.}$$

$$\bar{D}_jA^{ij} + \frac{1}{3}\gamma^{ij}\bar{D}_jK - \gamma^{ij}\bar{D}_jK = 8\pi S^i$$

$$\bar{D}_jA^{ij} - \frac{2}{3}\gamma^{ij}\bar{D}_jK = 8\pi S^i$$

For any symmetric traceless tensor

$$U^{ij} = U^{ij}_{TT} + U^{ij}_{L} = U^{ij}_{TT} + (\mathbb{L}W)^{ij}$$

$$\text{where} \quad \gamma_{ij}U^{ij}_{TT} = 0 \quad \& \quad D_jU^{ij}_{TT} = 0$$

$$(\mathbb{L}W)^{ij} = D^iW^j + D^jW^i - \frac{2}{3}\gamma^{ij}D_kW^k$$

tsokaros@th.physik.uni-frankfurt.de

Also $\bar{A}^{ij} = \psi^{10} A^{ij}$ then

$$\bar{A}_{ij} = \bar{\gamma}_{ia} \bar{\gamma}_{jb} \bar{A}^{ab} = \psi^{-8} \gamma_{ia} \gamma_{jb} \psi^{10} A^{ab} = \psi^{2} A_{ij}$$

Prove 7.157

$$\boxed{D_j U^{ij} = \psi^{-n} \bar{D}_j (\psi^n U^{ij}) + (10-n) U^{ij} \partial_j \ln\psi}$$

EXERCISE 2

not used.

$$D_j U^{ij} = D_j (\psi^{-n} \psi^n U^{ij}) = D_j \psi^{-n} (\psi^n U^{ij}) + \psi^{-n} D_j (\psi^n U^{ij})$$

$$= (-n) \psi^{-n-1} (D_j \psi) \psi^n U^{ij} + \psi^{-n} D_j (\psi^n U^{ij})$$

$$= (-n) \frac{1}{\psi} D_j \psi \, U^{ij} + \psi^{-n} D_j (\psi^n U^{ij})$$

$$= \psi^{-n} D_j (\psi^n U^{ij}) + (-n) U^{ij} D_j \ln\psi$$

or

$$D_i U^{ij} = \bar{D}_i U^{ij} + 10 U^{ij} \bar{D}_i \ln\psi$$

see next page.

Prove.

$$= \bar{D}_i (\psi^{-n} \psi^n U^{ij}) + 10 U^{ij} \bar{D}_i \ln\psi$$

$$= (-n) \psi^{-n-1} \bar{D}_i \psi (\psi^n U^{ij}) + \psi^{-n} \bar{D}_i (\psi^n U^{ij}) + 10 U^{ij} \bar{D}_i \ln\psi$$

$$= (-n) \frac{1}{\psi} \bar{D}_i \psi \, U^{ij} + \psi^{-n} \bar{D}_i (\psi^n U^{ij}) + 10 U^{ij} \bar{D}_i \ln\psi$$

$$\boxed{D_i U^{ij} = \psi^{-n} \bar{D}_i (\psi^n U^{ij}) + (10-n) U^{ij} \bar{D}_i \ln\psi.}$$

$$D_j A^{ij} - \frac{2}{3} \gamma^{ij} \bar{D}_j K = 8\pi J^i$$

$$\psi^{-10} \bar{D}_j (\underbrace{\psi^{10} A^{ij}}_{\bar{A}^{ij}}) - \frac{2}{3} \psi^{-4} \bar{\gamma}^{ij} \bar{D}_j K = 8\pi J^i$$

$$\boxed{\bar{D}_j \bar{A}^{ij} - \frac{2}{3} \psi^6 \bar{D}^i K = 8\pi \psi^{10} S^i}$$

$$D_a U^{ij} = \bar{D}_a U^{ij} + \Gamma^i{}_{ab} U^{bj} + \Gamma^j{}_{ab} U^{ib}$$

$$D_i U^{ij} = \bar{D}_i U^{ij} + \Gamma^i{}_{ib} U^{bj} + \Gamma^j{}_{ib} U^{ib}$$

$$\Gamma^c{}_{ab} = \frac{4}{2\psi}\left(\delta_a{}^c \bar{D}_b \psi + \delta_b{}^c \bar{D}_a \psi - \bar{\gamma}_{ab} \bar{\gamma}^{cd} \bar{D}_d \psi\right).$$

$$\Gamma^c{}_{cb} = \frac{2}{\psi}\left(3 \bar{D}_b \psi + \cancel{\bar{D}_b \psi} - \underbrace{\bar{\gamma}_{cb} \bar{\gamma}^{cd}}_{\bar{\gamma}_b{}^d} \cancel{\bar{D}_d \psi}\right) = \frac{6}{\psi} \bar{D}_b \psi.$$

$$\Gamma^i{}_{ib} = 6 \bar{D}_b \ln\psi$$

$$\Gamma^j{}_{ib} U^{ib} = \frac{2}{\psi}\left\{\delta_i^j \bar{D}_b \psi\, U^{ib} + \delta_b^j \bar{D}_i \psi\, U^{ib} - \bar{\gamma}_{ib} \bar{\gamma}^{je} \bar{D}_e \psi\, U^{ib}\right\}$$

$=0$

$$= \frac{2}{\psi}\left\{\bar{D}_b \psi\, U^{jb} + \bar{D}_i \psi\, U^{ij}\right\} = \frac{4}{\psi} U^{ij} \bar{D}_i \psi$$

$$\Gamma^j{}_{ib} U^{ib} = 4 U^{ij} \bar{D}_i \ln\psi$$

$$\Rightarrow \quad D_i U^{ij} = \bar{D}_i U^{ij} + U^{bj}\, 6 \bar{D}_b \ln\psi + 4 U^{ij} \bar{D}_i \ln\psi$$

$$= \bar{D}_i U^{ij} + 10 U^{ij} \bar{D}_i \ln\psi$$

$$A_{ij}A^{ij} = \psi^{-2}\bar{A}_{ij}\,\psi^{-10}\bar{A}^{ij} = \psi^{-12}\bar{A}_{ij}\bar{A}^{ij} \quad \Longrightarrow$$

$$(*) \quad 8\bar{D}_i\bar{D}^i\psi - \psi\bar{R} + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - \frac{2}{3}K^2\psi^5 + 16\pi E\psi^5 = 0.$$

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + (\mathbb{L}\bar{w})^{ij} \quad , \quad \bar{D}_j\bar{A}^{ij}_{TT} = 0.$$

$$\bar{D}_j\bar{A}^{ij} = \bar{D}_j(\mathbb{L}\bar{w})^{ij}$$

$$= \bar{D}_j\left(\bar{D}^i\bar{w}^j + \bar{D}^j\bar{w}^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k\bar{w}^k\right)$$

$$= \bar{D}_j\bar{D}^i\bar{w}^j + \bar{D}_j\bar{D}^j\bar{w}^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_j\bar{D}_k\bar{w}^k.$$

But

$$D_aD_bW_c - D_bD_aW_c = R^d{}_{cba}W_d \qquad (1.188)$$

$$D_aD_bW^a - D_bD_aW^a = R^{da}{}_{ba}W_d = R^d{}_bW_d.$$

$$D_aD_bW^a = D_bD_aW^a + R_{ab}W^a$$

$$\bar{D}_j\bar{A}^{ij} = \bar{D}^i\bar{D}_j\bar{w}^j + \bar{R}_j{}^i\bar{w}^j + \bar{D}_j\bar{D}^j\bar{w}^i - \frac{2}{3}\bar{D}^i\bar{D}_k\bar{w}^k$$

$$= \bar{R}^i{}_j\bar{w}^j + \frac{1}{3}\bar{D}^i\bar{D}_j\bar{w}^j + \bar{D}_j\bar{D}^j\bar{w}^i$$

thus.

$$(\star) \quad \bar{D}_j\bar{D}^j\bar{w}^i + \frac{1}{3}\bar{D}^i\bar{D}_j\bar{w}^j + \bar{R}^i{}_j\bar{w}^j - \frac{2}{3}\psi^6\bar{D}^iK = 8\pi\psi^{10}S^i$$

In $(*)$ $\quad \bar{A}_{ij} = \bar{A}_{TT\,ij} + (\mathbb{L}\bar{w})_{ij}^{*}$

free to be chosen

$$\bar A^{ij} = \bar A^{ij}_{TT} + (\mathbb{L}\bar\omega)^{ij} \longrightarrow \bar A^{ij} = \bar M^{ij} + (\mathbb{L}\bar V)^{ij}$$

i.e. $$\bar A^{ij}_{TT} + (\mathbb{L}\bar\omega)^{ij} = \bar M^{ij} + (\mathbb{L}\bar V)^{ij}$$

$$\bar A^{ij}_{TT} = \bar M^{ij} + (\mathbb{L}(\bar V - \bar\omega))^{ij} \quad \left(\equiv \bar M^{ij}_{TT}\right)$$

$$\Rightarrow D_j \bar A^{ij} = D_j \bar M^{ij} + D_j (\mathbb{L}\bar V)^{ij} \Rightarrow$$

$$\underbrace{\bar D_j \bar D^j \bar V^i + \frac{1}{3}\bar D^i \bar D_j V^j + \bar R^i{}_j V^j}_{(\bar\Delta_{\mathbb{L}}\bar V)^i} + \bar D_j \bar M^{ij} - \frac{2}{3}\psi^6 \bar D^i K = 8\pi\psi^{10} S^i$$

$$\boxed{(\bar\Delta_{\mathbb{L}}\bar V)^i + \bar D_j \bar M^{ij} - \frac{2}{3}\psi^6 \bar D^i K = 8\pi\psi^{10} S^i}$$ instead of $\bigstar$

$$\left.\begin{aligned} \gamma_{ij} &= \psi^4 \bar\gamma_{ij} \\ K^{ij} &= A^{ij} + \frac{1}{3}K\gamma^{ij} = \psi^{-10}\bar A^{ij} + \frac{1}{3}K\gamma^{ij} \\ &= \psi^{-10}\left(\bar M^{ij} + (\mathbb{L}\bar V)^{ij}\right) + \frac{1}{3}K\psi^{-4}\bar\gamma^{ij} \end{aligned}\right\} \text{in terms of } \bar\gamma_{ij}, K, \bar M_{ij}$$

For <u>Constant Mean Curvature K</u>

$$(\bar\Delta_{\mathbb{L}}\bar V)^i + \bar D_j \bar M^{ij} = 8\pi\psi^{10} S^i \Rightarrow \bar V^i = \dots$$

$$\Rightarrow \bar A^{ij} = \dots \Rightarrow \bar D_i \bar D^i \psi + \dots$$

can be solved for ψ.

For $\bar\gamma_{ij} = \delta_{ij} \Rightarrow 8\nabla^2\psi + \psi^{-7}\bar A_{ij}\bar A^{ij} - \frac{2}{3}K^2\psi^5 + 16\pi E\psi^5 = 0$

$\Rightarrow$ Time symmetry : $K_{ij} = 0 \Rightarrow$ in vacuum $\nabla^2\psi = 0$

$$\psi = 1 + \frac{M}{2r}$$

$W^i = \varepsilon^{ijk} x_j J_k$ solution of $\partial^j \partial_j W^i + \frac{1}{3} \partial^i \partial_j W^j = 0$

assuming $\partial_i J_j = 0$

$$\partial^m \partial_m \left(\varepsilon^{ijk} X_j J_k \right) + \frac{1}{3} \partial^i \partial_j \left(\varepsilon^{jmn} X_m J_n \right) =$$

$$\varepsilon^{ijk} J_k \, \partial^m \partial_m X_j + \frac{1}{3} \varepsilon^{jmn} J_n \, \partial^i \partial_j X_m =$$

$$X^i = \frac{e^i}{r^2} = \frac{x^i}{r^3} \qquad \partial_m r = \frac{x_m}{r}$$

$$\partial_m X^i = \partial_m x^i \frac{1}{r^3} + x^i \partial_m \left(\frac{1}{r^3}\right) = \delta^i_m \frac{1}{r^3} + x^i \left(-\frac{1}{r^{\cancel{6}\,4}}\right) 3 \cancel{x^2}\, \partial_m r$$

$$= \delta^i_m \frac{1}{r^3} - \frac{3}{r^4} x^i \frac{x_m}{r} = \frac{\delta^i_m}{r^3} - \frac{3}{r^5} x^i x_m$$

$$= \frac{1}{r^5} \left(r^2 \delta^i_m - 3 x^i x_m \right)$$

$$\partial^m \partial_m X^i = \delta^i_m \partial^m \left(\frac{1}{r^3}\right) - 3 \partial^m \left(\frac{1}{r^5}\right) x^i x_m - \frac{3}{r^5} \partial^m x^i x_m - \frac{3}{r^5} x^i \partial^m x_m$$

$$= \partial^i \left(\frac{1}{r^3}\right) + 3 \frac{1}{r^{10}} 5 r^{\cancel{4}\,3} \frac{x^m}{\cancel{r}} x^i x_m - \frac{3}{r^5} \delta^{mi} x_m - \frac{3}{r^5} x^i \cdot 3$$

$$= -\frac{1}{r^6} 3 r^2 \frac{x^i}{r} + \frac{15}{r^7} x^i r^2 - \frac{3}{r^5} x^i - \frac{9}{r^5} x^i$$

$$= -\frac{3x^i}{r^5} + \frac{15 x^i}{r^5} - \frac{12 x^i}{r^5} = 0.$$

$$\partial^i \partial_j X_m = \partial^i \left(\frac{\delta_{jm}}{r^3} - \frac{3}{r^5} x_j x_m \right)$$

$$= \delta_{jm} \left(-\frac{1}{r^6}\right) 3 r^2 \frac{x^i}{\cancel{r}} + \frac{3}{r^{10}} 5 r^4 \frac{x^i}{r} x_j x_m - \frac{3}{r^5} \delta^i_j x_m - \frac{3}{r^5} x_j \delta^i_m$$

$$= -\frac{3}{r^5} x^i \delta_{jm} + \frac{15}{r^7} x^i x_j x_m - \frac{3}{r^5} \delta^i_j x_m - \frac{3}{r^5} x_j \delta^i_m$$

$$\varepsilon^{jmn}\,\partial^i\partial_j x_m = -\frac{3}{r^5}\,x^i\,\cancel{\delta_{jm}\,\varepsilon^{jmn}}^{\,0}$$

$$+\frac{15}{r^7}\,x^i\,\cancel{x_j x_m\,\varepsilon^{jmn}}^{\,0}$$

$$-\frac{3}{r^5}\,\delta^i_j\,x_m\,\varepsilon^{jmn} \qquad = -\frac{3}{r^5}\,x_m\,\varepsilon^{imn}$$

$$-\frac{3}{r^5}\,\delta^i_m\,x_j\,\varepsilon^{jmn} \qquad = -\frac{3}{r^5}\,x_j\,\varepsilon^{jin}$$

$$= -\frac{3}{r^5}\,x_m\,\varepsilon^{min} = +\frac{3}{r^5}\,x_m\,\varepsilon^{imn}$$

cancel

Thus each term is zero

$$\Rightarrow \quad \partial^i\partial_j W^i + \frac{1}{3}\,\partial^i\partial_j W^j = 0 \quad \text{with } W^i = \varepsilon^{ijk}\,x_j J_k$$

$$\text{and } \partial_i J_k = 0.$$

$$\bar{A}^{ij}_{n} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\,\bar{\gamma}^{ij}\,\bar{D}_k W^k = (\mathbb{L}W)^{ij}$$

$$= \bar{D}^i\left(\bar{\varepsilon}^{jmn}\,x_m J_n\right) + \bar{D}^j\left(\bar{\varepsilon}^{imn}\,x_m J_n\right) - \frac{2}{3}\,\bar{\gamma}^{ij}\,\bar{D}_k\left(\bar{\varepsilon}^{kmn}\,x_m J_n\right)$$

$$= \bar{\varepsilon}^{jmn}\,J_n\,\bar{D}^i x_m + \bar{\varepsilon}^{imn}\,J_n\,\bar{D}^j x_m - \frac{2}{3}\,\bar{\gamma}^{ij}\,\bar{\varepsilon}^{kmn}\,J_n\,\bar{D}_k x_m$$

$$\bar{\varepsilon}^{kmn}\,\bar{D}_k x_m = \varepsilon^{kmn}\,\frac{1}{r^5}\left(r^2\delta_{km} - 3x_k x_m\right) = 0.$$

Also

$$J_n\,\bar{\varepsilon}^{jmn}\,\bar{D}^i x_m = \bar{\varepsilon}^{jmn}\,\frac{1}{r^5}\left(r^2\delta^i_m - 3x^i x_m\right)J_n$$

$$= \frac{1}{r^3}\,\varepsilon^{jin}\,J_n - \frac{3}{r^5}\,\varepsilon^{jmn}\,x^i x_m J_n$$

$$\bar{A}_L^{ij} = \frac{1}{r^3} \cancel{\varepsilon^{jin}} J_n - \frac{3}{r^5} \varepsilon^{jmn} x^i x_m J_n$$

$$+ \frac{1}{r^3} \cancel{\varepsilon^{ijn}} J_n - \frac{3}{r^5} \varepsilon^{imn} x^j x_m J_n =$$

$$= -\frac{3}{r^5} \left(x^i \varepsilon^{jmn} + x^j \varepsilon^{imn} \right) x_m J_n = -\frac{6}{r^5} x^{(i} \varepsilon^{j)mn} x_m J_n$$

- $l^i = \frac{x^i}{r}$ $\qquad$ $X^i = \frac{l^i}{r^2} = \frac{x^i}{r^3}$

$\boxed{}$

$$\bar{A}_L^{ij} = -\frac{6}{r^5} r l^{(i} \varepsilon^{j)mn} (r l_m) J_n = -\frac{6}{r^3} l^{(i} \varepsilon^{j)mn} l_m J_n$$

$$= \frac{6}{r^3} l^{(i} \varepsilon^{j)nm} J_n l_m$$

$$W^i = \varepsilon^{ijk} X_j J_k = \frac{1}{r^3} \varepsilon^{ijk} x_j J_k.$$

Let $J^i = (0, 0, J)$ $\qquad$ $\varphi^i = \frac{1}{r}(-y, x, 0)$

$$W^i = \frac{1}{r^3} \left(\varepsilon^{ijz} x_j J \right)$$

$$W^x = \frac{1}{r^3} \varepsilon^{xjz} x_j J = \frac{1}{r^3} y J$$

$$W^y = \frac{1}{r^3} \varepsilon^{yjz} x_j J = \frac{1}{r^3} (-x) J$$

$$W^z = 0$$

$$W^i = \frac{J}{r^3} \left(y, -x, 0 \right)$$

$$= -\frac{J}{r^3} \left(-y, x, 0 \right) \quad \checkmark$$

$$\bar{A}^{ij}_{h} = \frac{6}{r^3} \ell^{(i} \varepsilon^{j)nm} J_n \ell_m \qquad \ell^i = \frac{x^i}{r}$$

$$= \frac{6}{r^5} x^{(i} \varepsilon^{j)nm} J_n x_m$$

$$= \frac{6}{r^5} x^{(i} \varepsilon^{j)zm} J x_m$$

$$= \frac{6}{r^5} x^{(i} \varepsilon^{j)zy} J y + \frac{6}{r^5} x^{(i} \varepsilon^{j)zx} J x$$

$$= \frac{3}{r^5} \left(x^i \varepsilon^{jzy} + x^j \varepsilon^{izy}\right) y J + \frac{3}{r^5} \left(x^i \varepsilon^{jzx} + x^j \varepsilon^{izx}\right) J x$$

$$\bar{A}^{xx}_{h} = \frac{3}{r^5} \left(x(-1) + x(-1) \right) y J = -\frac{6}{r^5} x y J \quad \checkmark$$

$$\bar{A}^{xy} = \frac{3}{r^5} \left(0 + y(-1) \right) y J + \frac{3}{r^5} \left(x \cdot 1 + 0 \right) J x = \frac{3J}{r^5} (x^2 - y^2) \checkmark$$

$$\bar{A}^{xz} = \frac{3}{r^5} \left(0 + z(-1) \right) y J + \frac{3}{r^5} \left(0 + z \cdot 0 \right) = -\frac{3}{r^5} y z J \quad \checkmark$$

$$\bar{A}^{yy} = \frac{3}{r^5} (0 + 0) + \frac{3}{r^5} \left(y \cdot 1 + y \cdot 1 \right) J x = \frac{6 x y}{r^5} J \quad \checkmark$$

$$\bar{A}^{yz} = \frac{3}{r^5} (0 + 0) + \frac{3}{r^5} \left(y \cdot 0 + z \cdot 1 \right) J x = \frac{3 x z}{r^5} J \quad \checkmark$$

$$\bar{A}^{zz} = \frac{3}{r^5}\, 0 + 0 = 0$$

$$W_i = \frac{7}{8} V_i - \frac{1}{8}\left(\partial_i U + x^k \partial_i V_k\right)$$

$$\partial_j \partial^j W_i = \frac{7}{8} \partial_j \partial^j V_i - \frac{1}{8}\left(\partial_j \partial^j \partial_i U + \partial_j \partial^j \left(x^k \partial_i V_k\right)\right)$$

$$\partial_j \left(x^k \partial_i V_k\right) = \partial_j x^k \partial_i V_k + x^k \partial_j \partial_i V_k = \delta^k_j \partial_i V_k + x^k \partial_j \partial_i V_k$$

$$\begin{aligned}
\partial^j \partial_j \left(x^k \partial_i V_k\right) &= \delta^k_j \partial^j \partial_i V_k + \partial^j x^k \partial_j \partial_i V_k + x^k \partial^j \partial_j \partial_i V_k \\
&= \partial^k \partial_i V_k + \partial^k \partial_i V_k + x^k \partial_j \partial^j \partial_i V_k \\
&= 2 \partial_i \partial_k V^k + x^k \partial_j \partial^j \partial_i V_k
\end{aligned}$$

$$\partial_j \partial^j W_i = \frac{7}{8} \partial_j \partial^j V_i - \frac{1}{8}\left\{\partial_j \partial^j \partial_i U + 2 \partial_i \partial_k V^k + x^k \partial_j \partial^j \partial_i V_k\right\}$$

$$\begin{aligned}
\partial^i W_i &= \frac{7}{8} \partial^i V_i - \frac{1}{8}\left(\partial^i \partial_i U + \partial^i x^k \partial_i V_k + x^k \partial^i \partial_i V_k\right) \\
&= \frac{7}{8} \partial^i V_i - \frac{1}{8}\left(\partial^i \partial_i U + \partial_i V^i + x^k \partial^i \partial_i V_k\right) \\
&= \frac{6}{8} \partial^i V_i - \frac{1}{8}\left(\partial^i \partial_i U + x^k \partial^i \partial_i V_k\right)
\end{aligned}$$

$$\begin{aligned}
\partial^i \partial_j W^j &= \frac{6}{8} \partial^i \partial_j V^j - \frac{1}{8}\left(\partial^i \partial^j \partial_j U + \partial^i \left(x^k \partial^j \partial_j V_k\right)\right) \\
&= \frac{6}{8} \partial^i \partial_j V^j - \frac{1}{8}\left(\partial^i \partial^j \partial_j U + \partial^j \partial_j V_i + x^k \partial^i \partial^j \partial_j V_k\right)
\end{aligned}$$

$$\begin{aligned}
\partial^j \partial_j W^i + \frac{1}{3} \partial^i \partial_j W^j &= \underline{\frac{7}{8} \partial_j \partial^j V^i} - \frac{1}{8}\left\{\partial_j \partial^j \partial^i U + \cancel{2 \partial^i \partial_k V^k} + x^k \partial_j \partial^j \partial^i V_k\right\} \\
&\quad + \frac{2}{8} \cancel{\partial^i \partial_j V^j} - \frac{1}{3 \cdot 8}\left\{\partial^i \partial^j \partial_j U + \underline{\partial^j \partial_j V^i} + x^k \partial^i \partial^j \partial_j V_k\right\}
\end{aligned}$$

$$= \frac{5}{6} \partial_j \partial^j V^i - \frac{1}{6} \partial^i \partial^j \partial_j U - \frac{1}{6} x^k \partial^i \partial^j \partial_j V_k = 0$$

Choose U such that $\boxed{\partial^j \partial_j U = 0}$

Then $\frac{5}{6} \partial_j \partial^j V^i - \frac{1}{6} x^k \partial^i \partial^j \partial_j V_k = 0$

A solution $\boxed{\partial_j \partial^j V_i = 0}$

Assume $U = 0$ and $V_i = -\frac{2P_i}{r}$, P_i constant.

Then $W^i = \frac{7}{8} V^i - \frac{1}{8} x^k \partial^i V_k$

$$\partial^i V_k = \partial^i \left(-\frac{2P_k}{r} \right) = \frac{2P_k}{r^2} \partial^i r = \frac{2P_k}{r^2} \frac{x^i}{r}$$

$$W^i = \frac{7}{8} \left(-\frac{2P^i}{r} \right) - \frac{1}{8} x^k \frac{2P_k}{r^3} x^i = -\frac{7}{4} \frac{P^i}{r} - \frac{1}{4} \frac{x^k x^i P_k}{r^3}$$

$$W^i = -\frac{1}{4r} \left(7P^i + \frac{x^i x^k}{r^2} P_k \right) \qquad \ell^i = \frac{x^i}{r}$$

$$\boxed{W^i = -\frac{1}{4r} \left(7P^i + \ell^i \ell^k P_k \right)}$$

$$\bar{A}^{ij}_L = (\mathbb{L} W)^{ij} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3} \bar{\gamma}^{ij} \bar{D}_k W^k$$

$$\bar{D}^j W^i = + \frac{1}{4r^2} \bar{D}^j r \left(7P^i + \ell^i \ell^k P_k \right) - \frac{1}{4r} \left(0 + P_k \bar{D}^j (\ell^i \ell^k) \right)$$

$$= \frac{1}{4r^2} \frac{x^j}{r} \left(7P^i + \ell^i \ell^k P_k \right) - \frac{1}{4r} P_k \bar{D}^j (\ell^i \ell^k)$$

$$D^j \ell^i = D^j \left(\frac{x^i}{r} \right) = \frac{\delta^{ij}}{r} + x^i \left(-\frac{1}{r^2} \right) \frac{x^j}{r} = \frac{\delta^{ij}}{r} - \frac{x^i x^j}{r^3}$$

$$\bar{D}^j W^i = \frac{1}{4r^2} \left(7P^i \ell^j + \ell^i \ell^j \ell^k P_k \right) - \frac{P_k}{4r} \left(\frac{\delta^{ji}}{r} - \frac{x^j x^i}{r^3} \right) \ell^k$$

$$- \frac{P_k}{4r} \ell^i \left(\frac{\delta^{jk}}{r} - \frac{x^j x^k}{r^3} \right)$$

$$\bar{D}^j W^i = \frac{1}{4r^2}\left(7P^i l^j + l^i l^j l^k P_k\right)$$

$$- \frac{P_k}{4r}\left(\frac{1}{r} l^k \delta^{ij} - \frac{1}{r} l^i l^j l^k + \frac{1}{r} l^i \delta^{jk} - \frac{1}{r} l^i l^j l^k\right)$$

$$= \frac{1}{4r^2}\left\{7P^i l^j + l^i l^j l^k P_k - P_k l^k \delta^{ij} + 2 l^i l^j l^k P_k - l^i P^j\right\}$$

$$= \frac{1}{4r^2}\left\{7P^i l^j - l^i P^j + 3 l^i l^j l^k P_k - \delta^{ij} P_k l^k\right\}$$

$$\bar{D}_i W^i = \frac{1}{4r^2}\left\{7P_i l^i - \cancel{l_i P^i} + \cancel{3 l^k P_k} - \cancel{3 P_k l^k}\right\} = \frac{1}{4r^2} 6 P_i l^i = \frac{3 P_i l^i}{2r^2}$$

$$\bar{A}_L^{ij} = \frac{1}{4r^2}\left\{7P^i l^j - l^i P^j + 3 l^i l^j l^k P_k - \delta^{ij} P_k l^k\right.$$

$$\left. + 7 P^j l^i - l^j P^i + 3 l^j l^i l^k P_k - \delta^{ij} P_k l^k\right\}$$

$$- \frac{2}{3}\bar{\gamma}^{ij}\frac{3 P_k l^k}{2r^2}$$

$$= \frac{1}{4r^2}\left\{6 P^i l^j + 6 P^j l^i + 6 l^i l^j l^k P_k - 2\delta^{ij} P_k l^k\right\}$$

$$- \frac{1}{r^2}\bar{\gamma}^{ij} P_k l^k$$

$$= \frac{3}{2r^2}\left(P^i l^j + P^j l^i + l^i l^j l^k P_k\right) - \frac{1}{2r^2}\delta^{ij} P_k l^k - \frac{1}{r^2}\delta^{ij} P_k l^k$$

$$- \frac{3}{2r^2}\delta^{ij} P_k l^k$$

$$= \frac{3}{2r^2}\left(P^i l^j + P^j l^i + l^i l^j l^k P_k - \delta^{ij} P_k l^k\right)$$

$$\boxed{\bar{A}_L^{ij} = \frac{3}{2r^2}\left(P^i l^j + P^j l^i + \left(l^i l^j - \delta^{ij}\right) P_k l^k\right)}$$

CONFORMAL THIN-SANDWICH DECOMPOSITION

$$u_{ij} = \gamma^{\frac{1}{3}} \partial_t \left(\gamma^{-\frac{1}{3}} \gamma_{ij} \right)$$

$$= \gamma^{\frac{1}{3}} \left(-\frac{1}{3}\right) \gamma^{-\frac{1}{3}-1} \partial_t \gamma \, \gamma_{ij} + \partial_t \gamma_{ij}$$

$$= -\frac{1}{3} \frac{1}{\gamma} \partial_t \gamma \, \gamma_{ij} + \partial_t \gamma_{ij}$$ is the traceless part of the time derivative of γ_{ij}.

But

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\gamma^{ij} \partial_t \gamma_{ij} = -2\alpha K + 2 D_i \beta^i$$

But $\partial_t \gamma = \sum_{ij} \frac{\partial \gamma}{\partial \gamma_{ij}} \frac{\partial \gamma_{ij}}{\partial t} = \sum_{ij} C_{ij} \partial_t \gamma_{ij}$ but $\gamma^{ij} = \frac{C_{ij}}{\gamma}$

$$= \sum_{ij} \gamma \gamma^{ij} \partial_t \gamma_{ij} = \gamma \gamma^{ij} \partial_t \gamma_{ij}$$

$$\boxed{\gamma^{ij} \partial_t \gamma_{ij} = \frac{1}{\gamma} \partial_t \gamma = \partial_t \ln \gamma}$$ ← Exercise 3

$$\frac{1}{2} \partial_t \ln \gamma = -\alpha K + D_i \beta^i \Leftrightarrow \boxed{\partial_t \ln \gamma^{\frac{1}{2}} = -\alpha K + D_i \beta^i}$$

$$u_{ij} = -\frac{1}{3} \partial_t \ln \gamma \, \gamma_{ij} + \partial_t \gamma_{ij}$$

$$= -\frac{2}{3} \frac{1}{2} \partial_t \ln \gamma \, \gamma_{ij} + \partial_t \gamma_{ij} = -\frac{2}{3} \partial_t \ln \gamma^{\frac{1}{2}} \gamma_{ij} + \partial_t \gamma_{ij}$$

$$= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i - \frac{2}{3} \gamma_{ij} \left(-\alpha K + D_k \beta^k \right)$$

$$= -2\alpha K_{ij} + \frac{2}{3} \alpha K \gamma_{ij} + \underbrace{D_i \beta_j + D_j \beta_i - \frac{2}{3} \gamma_{ij} D_k \beta^k}$$

$$= -2\alpha \left(K_{ij} - \frac{1}{3} K \gamma_{ij} \right) + (\mathbb{L} \beta)_{ij}$$

$$\boxed{u_{ij} = -2\alpha A_{ij} + (\mathbb{L} \beta)_{ij}}$$

Define $\bar{u}_{ij} \equiv \partial_t \bar{\gamma}_{ij}$ AND $\bar{\gamma}^{ij} \bar{u}_{ij} \equiv 0$

$$\bar{\gamma}^{ij} \partial_t \bar{\gamma}_{ij} = 0 = \partial_t \ln \bar{\gamma}$$

$$\gamma_{ij} = \psi^{\lambda} \bar{\gamma}_{ij} \qquad \text{or} \qquad \gamma^{ij} = \psi^{-\lambda} \bar{\gamma}^{ij}$$

$$D_i \beta_j = \bar{D}_i \beta_j - \Gamma^m_{ij} \beta_m = \bar{D}_i \beta_j - \frac{\lambda}{2\psi} \left(\delta_i^m \partial_j \psi + \delta_j^m \partial_i \psi - \bar{\gamma}_{ij} \partial^m \psi \right) \beta_m$$

$$= \bar{D}_i \beta_j - \frac{\lambda}{2\psi} \left(\beta_i \partial_j \psi + \beta_j \partial_i \psi - \bar{\gamma}_{ij} \beta_m \partial^m \psi \right)$$

$$D_i \beta^j = \bar{D}_i \beta^j + \Gamma^j{}_{im} \beta^m = \bar{D}_i \beta^j + \frac{\lambda}{2\psi} \left(\delta^j_i \partial_m \psi + \delta^j_m \partial_i \psi - \bar{\gamma}_{im} \partial^j \psi \right) \beta^m$$

$$D_k \beta^k = \bar{D}_k \beta^k + \frac{\lambda}{2\psi} \left(3 \partial_m \psi + \cancel{\partial_m \psi} - \cancel{\partial_m \psi} \right) \beta^m = \bar{D}_k \beta^k + \frac{3\lambda}{2\psi} \beta^m \partial_m \psi$$

thus

$$D_i \beta_j + D_j \beta_i - \frac{2}{3} \gamma_{ij} D_k \beta^k =$$

$$= \bar{D}_i \beta_j - \frac{\lambda}{2\psi} \left(\beta_i \partial_j \psi + \beta_j \partial_i \psi \right) + \bar{\gamma}_{ij} \frac{\lambda}{2\psi} \beta_m \partial^m \psi$$

$$+ \bar{D}_j \beta_i - \frac{\lambda}{2\psi} \left(\beta_j \partial_i \psi + \beta_i \partial_j \psi \right) + \bar{\gamma}_{ji} \frac{\lambda}{2\psi} \beta_m \partial^m \psi$$

$$- \frac{2}{3} \psi^{\lambda} \bar{\gamma}_{ij} \left[\bar{D}_k \beta^k + \frac{3\lambda}{2\psi} \beta^m \partial_m \psi \right]$$

Note $\boxed{\beta^i = \bar{\beta}^i}$ thus

$$\beta_i = \gamma_{ij} \beta^j = \gamma_{ij} \bar{\beta}^j = \psi^{\lambda} \bar{\gamma}_{ij} \bar{\beta}^j = \psi^{\lambda} \bar{\beta}_i$$

$$\boxed{\beta_i = \psi^{\lambda} \bar{\beta}_i}$$

therefore

$$D_i \beta_j = \bar{D}_i (\psi^{\lambda} \bar{\beta}_j) - \frac{\lambda}{2\psi} \left(\beta_i \partial_j \psi + \beta_j \partial_i \psi - \bar{\gamma}_{ij} \beta_m \partial^m \psi \right)$$

$$= \lambda \psi^{\lambda-1} \bar{D}_i \psi \, \bar{\beta}_j + \psi^{\lambda} \bar{D}_i \bar{\beta}_j - \frac{\lambda}{2\psi} \left(\psi^{\lambda} \bar{\beta}_i \partial_j \psi + \psi^{\lambda} \bar{\beta}_j \partial_i \psi - \bar{\gamma}_{ij} \psi^{\lambda} \bar{\beta}_m \partial^m \psi \right)$$

$$= \lambda \psi^{\lambda-1} \bar{D}_i \psi \, \bar{\beta}_j - \frac{\lambda}{2} \psi^{\lambda-1} \left(\bar{\beta}_i \partial_j \psi + \bar{\beta}_j \partial_i \psi - \bar{\gamma}_{ij} \bar{\beta}_m \partial^m \psi \right) + \psi^{\lambda} \bar{D}_i \bar{\beta}_j$$

$$D_i b_j = \lambda \psi^{\lambda-1}\left\{ \bar{b}_j \partial_i \psi - \frac{1}{2}\bar{b}_i \partial_j \psi - \frac{1}{2}\bar{b}_j \partial_i \psi + \frac{1}{2}\bar{\gamma}_{ij}\bar{b}_m \partial^m \psi \right\} + \psi^\lambda \bar{D}_i \bar{b}_j$$

$$= \lambda \psi^{\lambda-1}\left\{ \frac{1}{2}\bar{b}_j \partial_i \psi - \frac{1}{2}\bar{b}_i \partial_j \psi + \frac{1}{2}\bar{\gamma}_{ij}\bar{b}_m \partial^m \psi \right\} + \psi^\lambda \bar{D}_i \bar{b}_j$$

$$D_i b_j = \frac{\lambda \psi^{\lambda-1}}{2}\left(\bar{b}_j \partial_i \psi - \bar{b}_i \partial_j \psi + \bar{\gamma}_{ij}\bar{b}_m \partial^m \psi \right) + \psi^\lambda \bar{D}_i \bar{b}_j$$

$$D_i b_j + D_j b_i = \lambda \psi^{\lambda-1} \bar{\gamma}_{ij}\bar{b}_m \partial^m \psi + \psi^\lambda \left(\bar{D}_i \bar{b}_j + \bar{D}_j \bar{b}_i \right)$$

$$\gamma^{ij}\left(D_i b_j + D_j b_i \right) = \gamma^{ij} \lambda \psi^{\lambda-1} \bar{\gamma}_{ij}\bar{b}_m \partial^m \psi + \psi^\lambda \gamma^{ij}\left(\bar{D}_i \bar{b}_j + \bar{D}_j \bar{b}_i \right)$$

$$2 D_i b^i = \psi^{-\lambda}\bar{\gamma}^{ij} \lambda \psi^{\lambda-1}\bar{\gamma}_{ij}\bar{b}_m \partial^m \psi + \bar{\gamma}^{ij}\left(\bar{D}_i \bar{b}_j + \bar{D}_j \bar{b}_i \right)$$

$$2 D_i b^i = \frac{\lambda}{\psi} \cdot 3\, \bar{b}_m \partial^m \psi + 2 \bar{D}_i \bar{b}^i$$

$$\boxed{D_i b^i = \frac{3\lambda}{2\psi}\bar{b}_m \partial^m \psi + \bar{D}_i \bar{b}^i}$$

Hence,

$$(\mathbb{L} b)_{ij} = D_i b_j + D_j b_i - \frac{2}{3}\gamma_{ij} D_k b^k$$

$$= \lambda \psi^{\lambda-1}\bar{\gamma}_{ij}\bar{b}_m \partial^m \psi + \underline{\psi^\lambda\left(\bar{D}_i \bar{b}_j + \bar{D}_j \bar{b}_i \right)} - \frac{2}{3}\psi^\lambda \bar{\gamma}_{ij}\left[\underline{\bar{D}_k \bar{b}^k} + \frac{3\lambda}{2\psi}\bar{b}^m \partial_m \psi \right]$$

$$= \psi^\lambda \left[\bar{D}_i \bar{b}_j + \bar{D}_j \bar{b}_i - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_k \bar{b}^k \right]$$

$$+ \lambda \psi^{\lambda-1}\cancel{\bar{\gamma}_{ij}\bar{b}_m \partial^m \psi} - \lambda \psi^{\lambda-1}\cancel{\bar{\gamma}_{ij}\bar{b}^m} \partial_m \psi$$

$$\boxed{(\mathbb{L} b)_{ij} = \psi^\lambda (\bar{\mathbb{L}} \bar{b})_{ij}}$$

Also

$$(\mathbb{L}\beta)^{ij} = D^i\beta^j + D^j\beta^i - \frac{2}{3}\gamma^{ij} D_k\beta^k$$

$$D^i\beta^j = \gamma^{im} D_m\beta^j = \psi^{-\lambda}\bar{\gamma}^{im}\left[\bar{D}_m\beta^j + \Gamma^j{}_{mk}\beta^k\right]$$

$$= \psi^{-\lambda}\bar{\gamma}^{im}\left[\bar{D}_m\beta^j + \frac{\lambda}{2\psi}\left(\delta^j_m\partial_k\psi + \delta^j_k\partial_m\psi - \bar{\gamma}_{mk}\partial^j\psi\right)\beta^k\right]$$

$$= \psi^{-\lambda}\left[\bar{D}^i\beta^j + \frac{\lambda}{2\psi}\left(\bar{\gamma}^{ij}\partial_k\psi + \delta^j_k\partial^i\psi - \delta^i_k\partial^j\psi\right)\beta^k\right]$$

$$D^i\beta^j = \psi^{-\lambda}\bar{D}^i\beta^j + \frac{\lambda}{2\psi^{\lambda+1}}\left(\bar{\gamma}^{ij}\beta^k\partial_k\psi\right) + \frac{\lambda}{2\psi^{\lambda+1}}\left(\beta^j\partial^i\psi - \beta^i\partial^j\psi\right)$$

thus

$$D^i\beta^j + D^j\beta^i = \psi^{-\lambda}\left(\bar{D}^i\beta^j + \bar{D}^j\beta^i\right) + \frac{\lambda}{\psi^{\lambda+1}}\bar{\gamma}^{ij}\beta^k\partial_k\psi$$

Also

$$\gamma_{ij}D^i\beta^j = D_i\beta^i = \gamma_{ij}\psi^{-\lambda}\bar{D}^i\beta^j + \gamma_{ij}\frac{\lambda}{2\psi^{\lambda+1}}\bar{\gamma}^{ij}\beta^k\partial_k\psi$$

$$+ \frac{\lambda}{2\psi^{\lambda+1}}\left(\cancel{\beta^j\partial_j\psi - \beta^i\partial_i\psi}\right)$$

0

$$D_i\beta^i = \bar{\gamma}_{ij}\bar{D}^i\beta^j + \frac{\lambda}{2\psi}3\beta^k\partial_k\psi$$

$$D_i\beta^i = \bar{D}_i\beta^i + \frac{3\lambda}{2\psi}\beta^k\partial_k\psi$$

Hence.

$$(\mathbb{L}\beta)^{ij} = \psi^{-\lambda}\left(\bar{D}^i\beta^j + \bar{D}^j\beta^i\right) + \frac{\lambda}{\psi^{\lambda+1}}\bar{\gamma}^{ij}\cancel{\beta^k\partial_k\psi}$$

$$- \frac{2}{3}\bar{\gamma}^{ij}\left[\bar{D}_k\beta^k + \frac{3\lambda}{2\psi}\cancel{\beta^k\partial_k\psi}\right] \qquad \& \qquad \gamma^{ij} = \psi^{-\lambda}\bar{\gamma}^{ij}$$

Exercise (4)

$$(\mathbb{L}\beta)^{ij} = \psi^{-\lambda}\left(\bar{D}^i\beta^j + \bar{D}^j\beta^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k\beta^k\right) = \psi^{-\lambda}(\bar{\mathbb{L}}\beta)^{ij}$$

i.e. $(\mathbb{L}b)^{ij} = \psi^{-\lambda} (\bar{\mathbb{L}} b)^{ij} = \psi^{-\lambda} (\bar{\mathbb{L}} \bar{b})^{ij} \qquad (b^i = \bar{b}^i)$

$(\mathbb{L}b)_{ij} = \psi^{\lambda} (\bar{\mathbb{L}} \bar{b})_{ij}$ and $\bar{b}_i = \psi^{-\lambda} b_i$

$$\bar{u}_{ij} \equiv \partial_t \bar{\gamma}_{ij} = \partial_t (\psi^{-\lambda} \gamma_{ij}) = -\lambda \psi^{-\lambda-1} \partial_t \psi \, \gamma_{ij} + \psi^{-\lambda} \partial_t \gamma_{ij}$$

$$= \psi^{-\lambda} \left(\partial_t \gamma_{ij} - \lambda \frac{1}{\psi} \partial_t \psi \, \gamma_{ij} \right)$$

$$= \psi^{-\lambda} \left(\partial_t \gamma_{ij} - \lambda \partial_t \ln \psi \, \gamma_{ij} \right)$$

But

$$\gamma_{ij} = \psi^{\lambda} \bar{\gamma}_{ij} \Leftrightarrow \gamma = \psi^{3\lambda} \bar{\gamma}$$

$$\partial_t \gamma = 3\lambda \psi^{3\lambda-1} \partial_t \psi \, \bar{\gamma} + \psi^{3\lambda} \cancel{\partial_t \bar{\gamma}} \rightarrow 0 \text{ by definition}$$

$$\partial_t \gamma = 3\lambda \psi^{3\lambda} \partial_t \ln \psi \, \bar{\gamma}$$

$$\partial_t \gamma = 3\lambda \frac{\gamma}{\cancel{\bar{\gamma}}} \partial_t \ln \psi \, \cancel{\bar{\gamma}} \Leftrightarrow \partial_t \ln \gamma = 3\lambda \, \partial_t \ln \psi$$

hence

$$\bar{u}_{ij} = \psi^{-\lambda} \underbrace{\left(\partial_t \gamma_{ij} - \frac{1}{3} \partial_t \ln \gamma \, \gamma_{ij} \right)}_{u_{ij} \text{ by def}} \Rightarrow \boxed{\bar{u}_{ij} = \psi^{-\lambda} u_{ij}}$$

FROM NOW ON USE $\lambda = 4$

It is $u^{ij} = -2\alpha A^{ij} + (\mathbb{L}b)^{ij}$

and $A^{ij} = \psi^{-10} \bar{A}^{ij}$. Also

$$u^{ij} = \gamma^{ia} \gamma^{jb} u_{ab} = \psi^{-4} \bar{\gamma}^{ia} \psi^{-4} \bar{\gamma}^{jb} \psi^{4} \bar{u}_{ab}$$

$$u^{ij} = \psi^{-4} \bar{u}^{ij}$$

$$\Rightarrow \quad \psi^{-4}\,\bar{u}^{ij} = -2\alpha\,\psi^{-10}\,\bar{A}^{ij} + \psi^{-4}(\bar{\mathbb{L}}\beta)^{ij}$$

$$\bar{u}^{ij} = -2\alpha\,\psi^{-6}\,\bar{A}^{ij} + (\bar{\mathbb{L}}\beta)^{ij}$$

$$\frac{2\alpha}{\psi^6}\,\bar{A}^{ij} = (\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij}$$

$$\boxed{\bar{A}^{ij} = \frac{\psi^6}{2\alpha}\left((\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij}\right)}$$

Define $\frac{\alpha}{\psi^6} = \bar{\alpha}$

$$\boxed{\bar{A}^{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij}\right)}$$

Momentum Constraint : $\bar{D}_j\bar{A}^{ij} - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_jK = 8\pi\psi^{10}S^i$

$$\bar{D}_j\left(\frac{1}{2\bar{\alpha}}\right)\left[(\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij}\right] + \frac{1}{2\bar{\alpha}}\left[\bar{D}_j(\bar{\mathbb{L}}\beta)^{ij} - \bar{D}_j\bar{u}^{ij}\right] - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_jK =$$
$$= 8\pi\psi^{10}S^i$$

$$\bar{D}_j\left(\frac{1}{\bar{\alpha}}\right)(\bar{\mathbb{L}}\beta)^{ij} - \underline{\bar{D}_j\left(\frac{1}{\bar{\alpha}}\right)\bar{u}^{ij}} + \frac{1}{\bar{\alpha}}\underbrace{\bar{D}_j(\bar{\mathbb{L}}\beta)^{ij}} - \underline{\frac{1}{\bar{\alpha}}\bar{D}_j\bar{u}^{ij}} - \frac{4}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_jK$$
$$= 16\pi\psi^{10}S^i$$

$$(\bar{\Delta}_L\beta)^i = \bar{D}_j\bar{D}^j\beta^i + \frac{1}{3}\bar{D}^i(\bar{D}_j\beta^j) + \bar{R}^i{}_j\beta^j$$

$$\frac{1}{\bar{\alpha}}(\bar{\Delta}_L\beta)^i + \bar{D}_j\left(\frac{1}{\bar{\alpha}}\right)(\bar{\mathbb{L}}\beta)^{ij} - \bar{D}_j\left(\frac{1}{\bar{\alpha}}\bar{u}^{ij}\right) - \frac{4}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_jK =$$
$$= 16\pi\psi^{10}S^i$$

$$\frac{1}{\bar{\alpha}}(\bar{\Delta}_L\beta)^i - \frac{1}{\bar{\alpha}^2}\bar{D}_j\bar{\alpha}\,(\bar{\mathbb{L}}\beta)^{ij} = \bar{D}_j\left(\frac{1}{\bar{\alpha}}\bar{u}^{ij}\right) + \frac{4}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_jK$$
$$+ 16\pi\psi^{10}S^i$$

$$\boxed{(\bar{\Delta}_L \beta)^i - \bar{D}_j \ln\bar{\alpha}\, (\bar{\mathbb{L}}\beta)^{ij} = \bar{\alpha}\, \bar{D}_j\left(\frac{1}{\bar{\alpha}}\bar{u}^{ij}\right) + \frac{4}{3}\bar{\alpha}\psi^6 \bar{\gamma}^{ij}\bar{D}_j K + 16\pi\bar{\alpha}\psi^{10} S^i}$$

Momentum Constraint

Exercise 5

The Hamiltonian Constraint is:

$$\bar{D}\psi - \frac{1}{8}\psi\bar{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8\psi^7}\bar{A}_{ij}\bar{A}^{ij} = -2\pi\psi^5 \rho_H$$

E

and $\bar{A}^{ij} = \frac{1}{2\bar{\alpha}}\left((\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij}\right)$

$$\bar{A}_{ij} = \bar{\gamma}_{ia}\bar{\gamma}_{jb}\bar{A}^{ab} = \psi^{-4}\gamma_{ia}\psi^{-4}\gamma_{jb}\left(\psi^{10}A^{ab}\right)$$

$$= \psi^2 A_{ij}$$

$$\bar{A}_{ij}\bar{A}^{ij} = \psi^2 A_{ij}\,\psi^{10}A^{ij} \Leftrightarrow \boxed{\bar{A}_{ij}\bar{A}^{ij} = \psi^{12}A_{ij}A^{ij}}$$

CTS decomposition : Freely specifiable variables

$\bar{\gamma}_{ij}$, $\bar{u}_{ij}$, K, $\bar{\alpha}$

$5 + 5 + 1 + 1 = 12$

Total number of variables involved : $\bar{\gamma}_{ij}, \bar{u}_{ij}, K, \bar{\alpha}, \psi, \beta^i$

$5 + 5 + 1 + 1 + 1 + 3 = 16$

4 variables (ψ & β^i) are determined by constraint eqs.

CTT deals with quantities intrinsic to the slice

CTS takes into account the evolution of the metric off the slice. The extra 4 variables that appear reflect this.

It is $\partial_t K = -D^2\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho+S)\right) + \beta^i D_i K$

It is

$$D_i D^i \alpha = D^2\alpha = \psi^{-4}\left[\bar{D}_i\bar{D}^i\alpha + \frac{2}{\psi}\bar{D}_i\psi\,\bar{D}^i\alpha\right]$$

Also

$$\bar{D}_i\bar{D}^i(\alpha\psi) = \bar{D}_i\bar{D}^i\alpha\,\psi + 2\bar{D}^i\alpha\,\bar{D}_i\psi + \alpha\,\bar{D}^i\bar{D}_i\psi$$

$$D^2\alpha = -\partial_t K + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho+S)\right) + \beta^i D_i K$$

$$\psi^{-4}\left[\bar{D}_i\bar{D}^i\alpha + \frac{2}{\psi}\bar{D}_i\psi\,\bar{D}^i\alpha\right] = -\partial_t K + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho+S)\right) + \beta^i D_i K$$

$$\bar{D}^2\alpha + \frac{2}{\psi}\bar{D}_i\psi\,\bar{D}^i\alpha = -\psi^4\partial_t K + \alpha\psi^4\left(K_{ij}K^{ij} + 4\pi(\rho+S)\right) + \beta^i\psi^4 D_i K$$

$$\underbrace{\psi\bar{D}^2\alpha + 2\bar{D}_i\psi\,\bar{D}^i\alpha}_{\bar{D}^2(\alpha\psi) - \alpha\bar{D}^2\psi} = -\psi^5\partial_t K + \alpha\psi^5\left(K_{ij}K^{ij} + 4\pi(\rho+S)\right) + \beta^i\psi^5 D_i K$$

$$\bar{D}^2(\alpha\psi) = \alpha\bar{D}^2\psi - \psi^5\partial_t K + \alpha\psi^5\left(K_{ij}K^{ij} + 4\pi(\rho+S)\right) + \beta^i\psi^5 D_i K$$

$$= \alpha\bar{D}^2\psi - \psi^5\partial_t K + \alpha\psi^5 K_{ij}K^{ij} + 4\pi\alpha\psi^5(\rho+S) + \beta^i\psi^5 D_i K$$

$$K_{ij}K^{ij} = \left(A_{ij} + \frac{1}{3}\gamma_{ij}K\right)\left(A^{ij} + \frac{1}{3}\gamma^{ij}K\right)$$

$$= A_{ij}A^{ij} + \frac{1}{9}\cdot 3K^2 = A_{ij}A^{ij} + \frac{1}{3}K^2$$

$$= \psi^{-12}\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2$$

$$\bar{D}^2(\alpha\psi) = \alpha\bar{D}^2\psi - \psi^5\partial_t K + \alpha\psi^5\left[\psi^{-12}\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2\right]$$

$$+ 4\pi\alpha\psi^5(\rho+S) + \beta^i\psi^5 D_i K$$

$$= \alpha\bar{D}^2\psi - \psi^5\partial_t K + \alpha\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\alpha\psi^5 K^2$$

$$+ 4\pi\alpha\psi^5(\rho+S) + \beta^i\psi^5 D_i K$$

and with the use of Hamiltonian Constrain.

$$\bar{D}^2(\alpha\psi) = \alpha\left\{\frac{1}{8}\psi\bar{R} + \frac{1}{12}\psi^5K^2 - \frac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - 2\pi\psi^5\rho_H\right\}$$

$$-\psi^5\partial_t K + \alpha\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\alpha\psi^5K^2 + 4\pi\alpha\psi^5(\rho_H+S) + \beta^i\psi^5 D_i$$

$$\bar{D}^2(\alpha\psi) = \frac{1}{8}\alpha\psi\bar{R} + \frac{5}{12}\alpha\psi^5K^2 + \frac{7}{8}\alpha\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} - \psi^5\partial_t K + \beta^i\psi^5\bar{D}_i K$$

$$+ 2\pi\alpha\psi^5(\rho_H + 2S)$$

$$\boxed{\begin{aligned}\bar{D}^2(\alpha\psi) = \alpha\psi\left\{\frac{\bar{R}}{8} + \frac{7}{8}\psi^{-8}\bar{A}_{ij}\bar{A}^{ij} + \frac{5}{12}\psi^4K^2 + 2\pi\psi^4(\overset{E}{\rho_H}+2S)\right\}\\ -\psi^5\partial_t K + \beta^i\psi^5\bar{D}_i K\end{aligned}}$$

Exercise 6

and $\alpha\psi = \bar{\alpha}\psi^7$

EXTENDED CTS

ASSUMPTIONS : $\bar{u}_{ij} = 0$ & $\partial_t K = 0$ & $K = 0$

$$\Downarrow \quad \bar{A}^{ij} = \frac{1}{2\bar{\alpha}}(\bar{\mathbb{L}}\beta)^{ij}$$

$$\bar{D}^2\psi = \frac{1}{8}\psi\bar{R} - \frac{1}{8}\psi^{-7}\bar{A}^{ij}\bar{A}_{ij} - 2\pi\psi^5\rho_H$$

$$(\bar{\Delta}_L\beta)^i = (\bar{\mathbb{L}}\beta)^{ij}\bar{D}_j\ln\bar{\alpha} + 16\pi\bar{\alpha}\psi^{10}S^i = 2\bar{A}^{ij}\bar{D}_j\bar{\alpha} + 16\pi\alpha\psi^4 S^i$$

$$\bar{D}^2(\alpha\psi) = \alpha\psi\left\{\frac{\bar{R}}{8} + \frac{7}{8}\psi^{-8}\bar{A}_{ij}\bar{A}^{ij} + 2\pi\psi^4(\rho_H+2S)\right\}$$

III. CONFORMAL DECOMPOSITIONS

Let $g_{\alpha\beta} = \psi^\lambda\, \tilde{g}_{\alpha\beta}$. Then $\nabla_\alpha \omega_\beta = \tilde{\nabla}_\alpha \omega_\beta - C^\gamma{}_{\alpha\beta}\omega_\gamma$, where

$$C^\gamma{}_{\alpha\beta} \;=\; \frac{1}{2} g^{\gamma\mu}(\tilde{\nabla}_\alpha g_{\mu\beta} + \tilde{\nabla}_\beta g_{\alpha\mu} - \tilde{\nabla}_\mu g_{\alpha\beta}) \;=\; \frac{\lambda}{2}(\delta^\gamma_\alpha \tilde{\nabla}_\beta \ln\psi + \delta^\gamma_\beta \tilde{\nabla}_\alpha \ln\psi - \tilde{g}_{\alpha\beta}\tilde{\nabla}^\gamma \ln\psi)$$

Useful relations are

$$C^\gamma{}_{\gamma\beta} \;=\; \frac{\lambda N}{2}\tilde{\nabla}_\beta \ln\psi, \qquad\qquad \tilde{g}^{\alpha\beta} C^\gamma{}_{\alpha\beta} \;=\; \frac{\lambda(2-N)}{2}\tilde{\nabla}^\gamma \ln\psi$$

Defining $(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\omega_\gamma \;=\; R_{\alpha\beta\gamma}{}^\delta \omega_\delta$ we have

$$R_{\alpha\beta\gamma}{}^\delta \;=\; \tilde{R}_{\alpha\beta\gamma}{}^\delta + \tilde{\nabla}_\beta C^\delta{}_{\alpha\gamma} - \tilde{\nabla}_\alpha C^\delta{}_{\beta\gamma} + C^\delta_{\beta\mu}C^\mu_{\alpha\gamma} - C^\delta_{\alpha\mu}C^\mu_{\beta\gamma}$$

and

$$\begin{aligned}
R_{\alpha\beta\gamma}{}^\delta \;&=\tilde{R}_{\alpha\beta\gamma}{}^\delta \;+\; \lambda\left\{\delta^\delta_{[\alpha}\delta^\epsilon_{\beta]}\delta^\rho_\gamma \;-\; \tilde{g}_{\gamma[\alpha}\delta^\epsilon_{\beta]}\tilde{g}^{\delta\rho}\right\}\tilde{\nabla}_\epsilon\tilde{\nabla}_\rho \ln\psi \\
&\quad+\; \frac{\lambda^2}{2}\left\{-\delta^\delta_{[\alpha}\delta^\epsilon_{\beta]}\delta^\rho_\gamma \;+\; \tilde{g}_{\gamma[\alpha}\delta^\epsilon_{\beta]}\tilde{g}^{\delta\rho} \;-\; \tilde{g}_{\gamma[\alpha}\delta^\delta_{\beta]}\tilde{g}^{\epsilon\rho}\right\}\tilde{\nabla}_\epsilon \ln\psi \tilde{\nabla}_\rho \ln\psi \\
R_{\alpha\gamma} \;&=\tilde{R}_{\alpha\gamma} \;-\; \frac{\lambda}{2}\left\{(N-2)\delta^\epsilon_\alpha\delta^\rho_\gamma \;+\; \tilde{g}_{\alpha\gamma}\tilde{g}^{\epsilon\rho}\right\}\tilde{\nabla}_\epsilon\tilde{\nabla}_\rho \ln\psi \;+\; \frac{\lambda^2(N-2)}{4}\left\{\delta^\epsilon_\alpha\delta^\rho_\gamma \;-\; \tilde{g}_{\alpha\gamma}\tilde{g}^{\epsilon\rho}\right\}\tilde{\nabla}_\epsilon \ln\psi\tilde{\nabla}_\rho \ln\psi \\
R \;&=\psi^{-\lambda}\left\{\tilde{R} \;-\; \lambda(N-1)\tilde{\nabla}_\mu\tilde{\nabla}^\mu \ln\psi \;-\; \frac{\lambda^2(N-1)(N-2)}{4}\tilde{\nabla}_\mu \ln\psi\tilde{\nabla}^\mu \ln\psi\right\}
\end{aligned}$$

or equivalently

$$\begin{aligned}
R_{\alpha\beta\gamma}{}^\delta \;&=\; \tilde{R}_{\alpha\beta\gamma}{}^\delta \;+\; \frac{\lambda}{\psi}\left\{\delta^\delta_{[\alpha}\delta^\epsilon_{\beta]}\delta^\rho_\gamma \;-\; \tilde{g}_{\gamma[\alpha}\delta^\epsilon_{\beta]}\tilde{g}^{\delta\rho}\right\}\tilde{\nabla}_\epsilon\tilde{\nabla}_\rho\psi \\
&\quad+\; \frac{\lambda}{\psi^2}\left\{-\left(\frac{\lambda}{2}+1\right)\delta^\delta_{[\alpha}\delta^\epsilon_{\beta]}\delta^\rho_\gamma \;+\; \left(\frac{\lambda}{2}+1\right)\tilde{g}_{\gamma[\alpha}\delta^\epsilon_{\beta]}\tilde{g}^{\delta\rho} \;-\; \frac{\lambda}{2}\tilde{g}_{\gamma[\alpha}\delta^\delta_{\beta]}\tilde{g}^{\epsilon\rho}\right\}\tilde{\nabla}_\epsilon\psi\tilde{\nabla}_\rho\psi \\
R_{\alpha\gamma} \;&=\; \tilde{R}_{\alpha\gamma} \;-\; \frac{\lambda}{2\psi}\left\{(N-2)\delta^\epsilon_\alpha\delta^\rho_\gamma \;+\; \tilde{g}_{\alpha\gamma}\tilde{g}^{\epsilon\rho}\right\}\tilde{\nabla}_\epsilon\tilde{\nabla}_\rho\psi \\
&\quad+\; \frac{\lambda}{2\psi^2}\left\{(N-2)\left(1+\frac{\lambda}{2}\right)\delta^\epsilon_\alpha\delta^\rho_\gamma \;+\; \left(1-\lambda\frac{N-2}{2}\right)\tilde{g}_{\alpha\gamma}\tilde{g}^{\epsilon\rho}\right\}\tilde{\nabla}_\epsilon\psi\tilde{\nabla}_\rho\psi \\
R \;&=\; \psi^{-\lambda}\left\{\tilde{R} \;-\; \frac{\lambda(N-1)}{\psi}\tilde{\nabla}_\mu\tilde{\nabla}^\mu\psi \;-\; \frac{\lambda(N-1)[\lambda(N-2)-4]}{4\psi^2}\tilde{\nabla}_\mu\psi\tilde{\nabla}^\mu\psi\right\}
\end{aligned}$$

Recap

- We have seen that it is often convenient to introduce additional terms to the equations that damp violations of the constraints

  Eg. addition of term proportional to $\vec{\nabla}\cdot\vec{B}$ in induction eq. decreases possible violations of $\vec{\nabla}\cdot\vec{B}=0$ condition

- This logic can be adopted also for Einstein eqs, eg generalized harmonic formulation or CCZ4.

  The additional terms disappear if constraints are satisfied exactly

- Gauge conditions: express arbitrariness in choice of spatial and time coordinates

- choice for α: slicing condition

  " " $\underline{\beta}$: spatial gauge condition.

Conditions that avoid singularities and distortion of coordinates can be derived (maximal slicing, minimal distortion) but lead to elliptic eqs., too expensive to solve

- solutions involve evolution equations for α, β

$$
\begin{cases}
(\partial_t - \beta^k \partial_k)\,\alpha = -\,2\alpha\,(K - K_0) & \text{1+log} \\
(\partial_t - \beta^k \partial_k)\,\beta^i = \frac{3}{4} B^i & \\
(\partial_t - \beta^k \partial_k)\,B^i = (\partial_t - \beta^k \partial_k)\,\tilde{\Gamma}^i - \eta\, B^i & \text{Gamma driver}
\end{cases}
$$

INITIAL DATA

So far we have concentrated on the evolution equations for the EFEs but of course we need to start from somewhere i.e from the solution of EFEs on the initial slice Σ_0.

Since we do not want to solve evolution equations, we obviously have to solve the constraint equations

$$ {}^{(3)}R + K^2 - K_{ij} K^{ij} = 16\pi E $$

$$ D_j (K^{ij} - \gamma^{ij} K) = 8\pi S^i $$

which need to be solved after a proper specification of the source functions for the matter E, S^i, which themselves could be the solution of some equilibrium condition (eg TOV etc)

Relativistic hydrodynamics

we have so far dealt with the EFEs $\quad G_{\mu\nu} = 8\pi T_{\mu\nu}$

but we need evolution equations also for the RHS, ie for the matter sources. These equations can be derived in a number of different manners (Bianchi identities, kinetic theory) but one that is certainly intuitive is the requirement

conservation of energy momentum and rest mass

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu(\rho u^\mu) = \nabla_\mu(J^\mu) = 0$$

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

$$\begin{cases} u_\nu \nabla_\mu T^{\mu\nu} = 0 & \text{energy conservation} \\ h^i_{\ \nu} \nabla_\mu T^{\mu\nu} = 0 & \text{momentum conservation} \end{cases}$$

Let's go back to Newtonian hydrodynamics

$$\begin{cases} \partial_t \rho + v^i \partial_i \rho + \rho \partial_i v^i = 0 \\ \partial_t v^i + v^j \partial_j v^i + \frac{1}{\rho} \partial^i p = 0 \\ \partial_t s + v^i \partial_i s = 0 \end{cases} \iff \begin{cases} \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \\ \partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho} \vec{\nabla} p = 0 \\ \partial_t s + (\vec{v} \cdot \vec{\nabla}) s = 0 \end{cases}$$

s: specif entropy $= \frac{S}{M}$

This set can be written in the general form (see previous lecture)

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = 0 \qquad (*)$$

where $\underline{u} = \{\rho, v^i, s\}^T = \begin{pmatrix} \rho \\ v^1 \\ v^2 \\ v^3 \\ s \end{pmatrix}$: <u>state vector</u>

$$
A^1 = \begin{pmatrix}
r' & e & 0 & 0 & 0 \\
\frac{1}{e}\frac{\partial p}{\partial e} & r' & 0 & 0 & \frac{1}{e}\partial_s p \\
 & & r' & & \\
 & 0 & & r' & 0 \\
 & & & & r'
\end{pmatrix}
$$

$$A^2 = \ldots$$

$$A^3 = \ldots$$

We have already discussed that the system (*) is <u>nonlinear</u> if the coefficients of A are functions of the state vector $a_{jk} = a_{jk}(U$

while the system is <u>linear</u> if the coefficients are constant.

There is no better way to appreciate the difference between linear and nonlinear hyperbolic equations than to consider some examples.

Let's start with a <u>linear</u> hyperbolic equation: advection equation in 1+1 spacetime

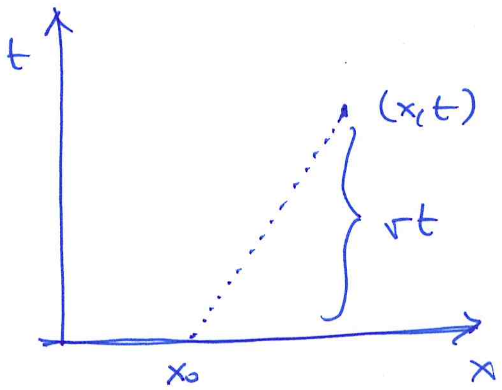
$$\boxed{\partial_t u + v\, \partial_x u = 0} \qquad v = \text{const.}$$

A has only one component $a_{11} = v = \text{const}$ (linear!)

This equation has the simple solution

$$u(x,t) = u(x_0, 0) = u_0(x_0) = u_0(x - vt)$$

$$x_0 = x - vt$$



$$v = \frac{dx}{dt}$$

In other words the solution at any new time and position $u(x,t)$ can be computed from the initial solution u_0 at the position x_0 suitably translated in space-time.

We can actually think that the initial solution is simply translated in spacetime along suitable directions.

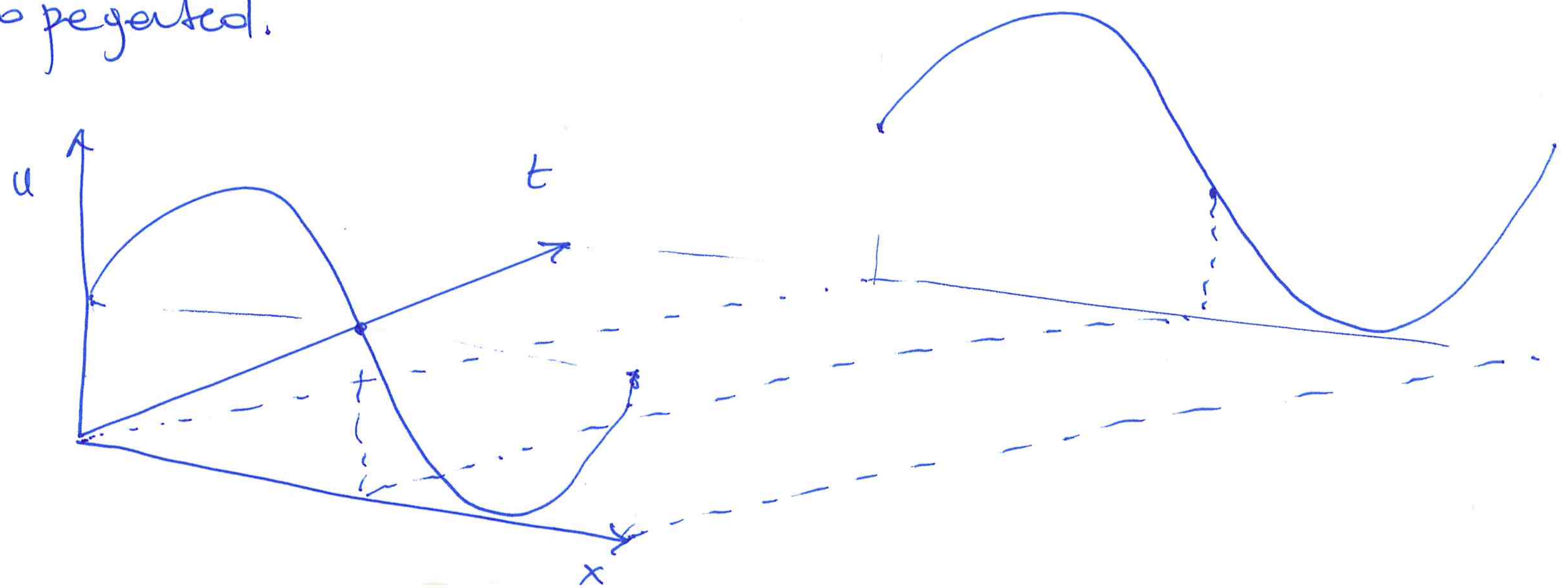
What are these directions?

$$d_t u = \frac{du}{dt} = \partial_t u + \frac{dx}{dt}\,\partial_x u = 0 \quad \text{if} \quad \frac{dx}{dt} = \lambda = v = \text{const.}$$

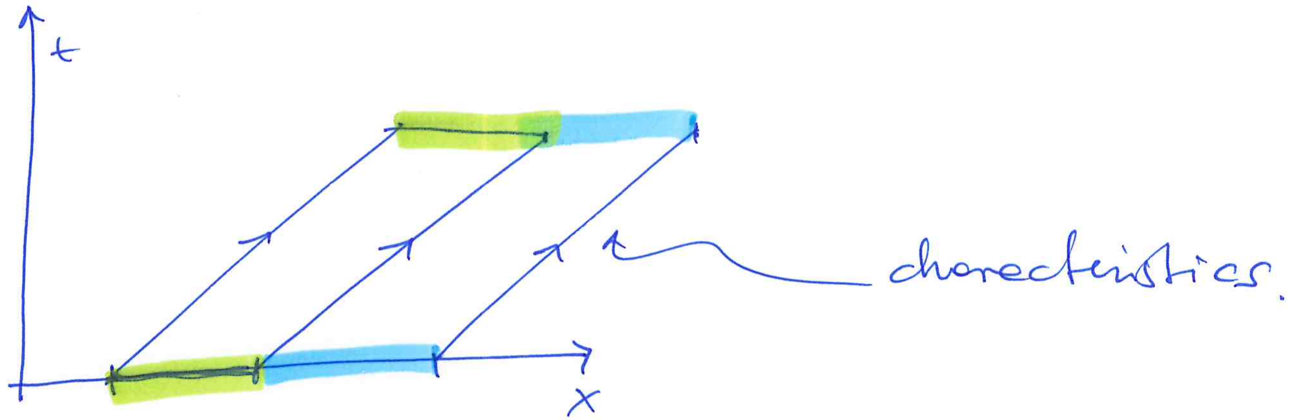
the direction (straight line) $\frac{dx}{dt} = \lambda = v$ is called characteristic direction and it corresponds with the direction in spacetime along which the solution is propagated.

Eg

$u_0 = u_0(x) =$
$\sin(4\pi x) + 1$



Because the solution is transported everywhere with the same velocity (ie along characteristic curves that are parallel) the solution is not distorted. This is all very clear and familiar.



We can generalize these results to a system of linear hyperbolic equations

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = 0$$

where A is a matrix of constant coefficients.

(130) (ie the system is linear)

Under these conditions we can define the vector of characteristic variables

$$\underline{W} := R^{-1} \underline{U}$$

where R is the matrix of right eigenvectors. Multiplying (*) by R^{-1} we obtain

$$R^{-1} \partial_t \underline{U} + R^{-1} A \cdot \nabla \underline{U} = 0$$

the right eigenvectors are constant

Now $$R^{-1} \partial_t \underline{U} = \partial_t (R^{-1} \underline{U}) = \partial_t \underline{W}$$

and $$R^{-1} A \cdot \nabla \underline{U} = R^{-1} A \cdot \nabla (R \underline{W}) = R^{-1} A R \cdot \nabla \underline{W} = \Lambda \cdot \nabla \underline{W}$$

where $\Lambda = R^{-1} A R$: diagonal ⊙ matrix with constant coefficients.

⊙ diagonal because hyperbolic system.

Putting things together

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = 0 \quad \Longleftrightarrow \quad \boxed{\partial_t \underline{W} + \Lambda \cdot \nabla \underline{W} = 0} \quad (**)$$

(**) are called <u>characteristic equations</u> and state that the characteristic vector is conserved along the directions given by the eigenvalues of A, ie

$$d\underline{W} = \partial_t \underline{W}\, dt + \partial_{\vec{x}} \underline{W}\, d\vec{x}$$

$$= \left(\partial_t W + \frac{d\vec{x}}{dt} \partial_{\vec{x}} \underline{W} \right) dt$$

$$\frac{d}{dt} \underline{W} = \partial_t W + \Lambda \frac{\partial \underline{W}}{\partial \vec{x}} = 0$$

along $\Lambda = \frac{\partial \vec{x}}{\partial t}$

Note: here $\vec{x}$ represents the spatial coordinates of an arbitrary coordinate system. and is a matrix, like Λ is a matrix.

Since Λ is a diagonal matrix with coefficients λ_i the characteristic vector $\underline{W}$ is conserved along the directions

$$\lambda_{(i)} = \frac{\partial}{\partial t} \vec{x}_{(i)}$$: characteristic curves (characteristics)

Note that (**) are N independent <u>advection</u> equations and hence the solution (or value of $\underline{W}$) at any given time can be computed from the initial solution, i.e. the solution at $t=0$.

$$\boxed{W^i(x^j,t) = W^i(x^j - \lambda_i t, 0)}$$

As a result, also the original state vector can be computed rather trivially as

$$(\square) \qquad \underline{U}(x^j, t) = \sum_{i=1}^{N} W^i (x^j, t) \underline{R}^{(i)} = \sum_{i=1}^{N} W^i (x^j - \lambda_i t, 0) \underline{R}^{(i)}$$

In other words, once $\underline{W}(x^j, 0)$ is known, $\underline{U}$ can be computed at any position in space and time.

This is a very powerful result, which is however restricted to linear problems as in this case the characteristics do not intersect and hence the expression $(\square)$ is not double valued.

What happens therefore in the case of <u>nonlinear</u> hyperbolic systems?

Once again it is simpler to understand this by starting from a simple example.

The simplest nonlinear hyperbolic equation is offered by the inviscid$^{\odot}$ Burgers equation

$$\partial_t u + u\,\partial_x u = 0$$

Clearly, in this case the matrix A in (*) has a non-constant coefficient $a_{11} = u(x,t)$: function of space and time!

As for the advection equation we can write

$$\frac{d}{dt}u = \partial_t u + \frac{dx}{dt}\partial_x u = 0$$

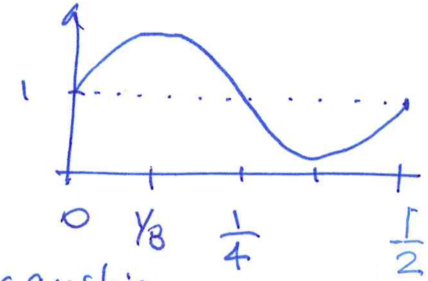
u is conserved along the direction $\lambda = \frac{dx}{dt} = u(x,t) = \text{const}$
$$= u(x_0,0) = u(x-\lambda t, 0)$$

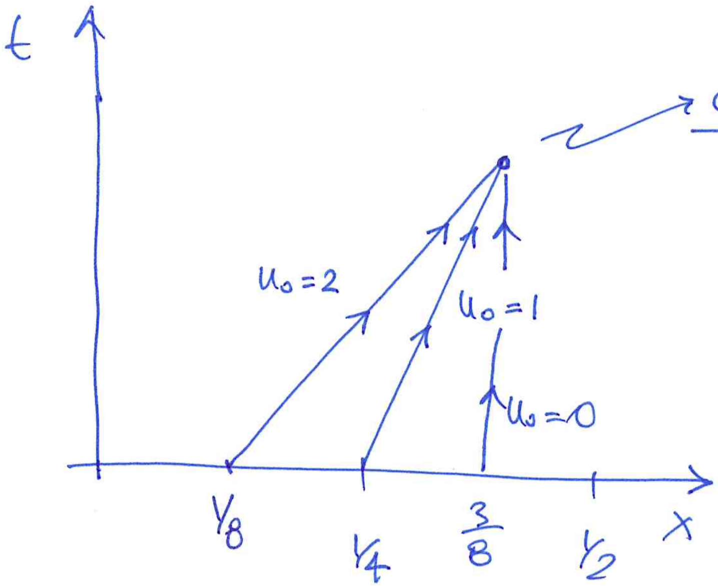
$^{\odot}$ The (viscous) Burgers equation is given by $\partial_t u + u\,\partial_x u = \eta\,\partial_x^2 u$

The characteristics are still straight lines, but they are no longer parallel !

Let's consider again the same initial data

$u_0 = \sin(4\pi x) + 1$





$$\frac{dx}{dt} = \lambda = u(x,t)$$

characteristics intersect !

When does this happen ?

[one part of the solution moves faster than the other! Traffic.]

To calculate this we can write the implicit Burgers equation whose solution is

$$u(x,t) = u_0(x - \lambda t) = u_0(x - u(x,t)t)$$

Taking a time derivative yields ⊙

$$\partial_t u_0 = [-u(x,t) - t\,\partial_t u]\,\partial_x u_0 \Rightarrow$$

$$\partial_t u(1 + t\,\partial_x u_0) = -u\,\partial_x u_0 \Rightarrow$$

$$\partial_t u = -\frac{u\,\partial_x u_0}{1 + t\,\partial_x u_0}$$; a caustic will be formed when the RHS will diverge,

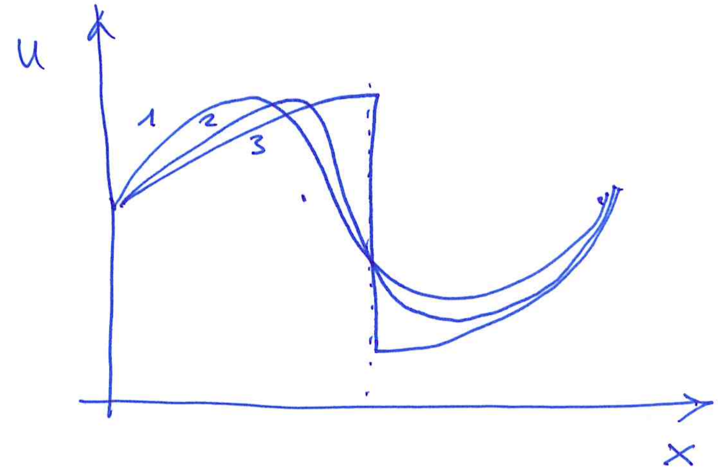
which will diverge for $t = -\dfrac{1}{\min(\partial_x u_0)} = \dfrac{1}{4\pi} \sim \dfrac{1}{0.08}$

$\min(4\pi \cos(4\pi x)) = -4\pi$

-1

$$\boxed{\begin{aligned} \odot\quad \partial_t u_0 &= \partial_x u_0\, \partial_t x = \partial_x u_0\,(\partial_t(-ut)) \\ &= \partial_x u_0\,(-u - t\,\partial_t u) \end{aligned}}$$

How does the solution change?



what is this?

This is a discontinuity or <u>shock</u>! The solution is mathematically discontinuous (double valued.)

In other words, the development of a caustic is equivalent to the development of a discontinuity or shock!

This process is called "wave steepening" and is typical of nonlinear hyperbolic problems.

Note that the wave steepening is unavoidable and occurs also from data that is initially smooth.

Recap

- relativistic hydrodynamics $\overset{\text{RHD}}{\vee}$ as conservation equations

$$\nabla_\mu (\rho u^\mu) = 0 \; ; \quad \nabla_\mu (T^{\mu\nu}) = 0$$

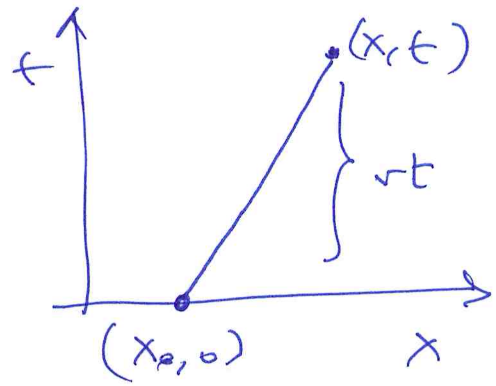
- RHD eqs. are hyperbolic.

Simplest hyperbolic eq. is the advection equation: linear

$$\partial_t u + v\, \partial_x u = 0 \qquad v: \text{const.}$$

$$u = u(x,t) = u(x_0, 0) = u(x - vt, 0) = u_0$$

The solution is translated in spacetime



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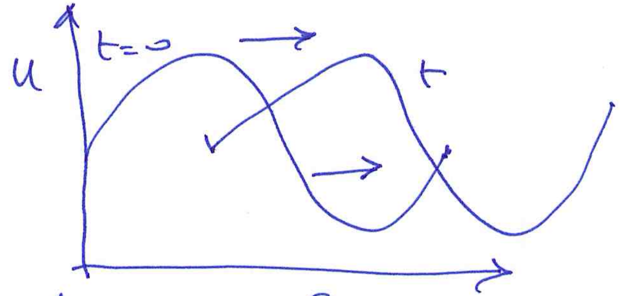
u is conserved along direction $\frac{dx}{dt} = v$: characteristic

$$d_t u = \partial_t u + \frac{dx}{dt}\partial_x u = \partial_t u + v\,\partial_x u = 0$$

characteristics are straight lines and parallel



- generic system can be written in characteristic form

$$\partial_t \underline{U} + A\cdot\nabla\underline{U} = 0 \iff \partial_t \underline{W} + \Lambda\cdot\nabla\underline{W} = 0$$

n decoupled advection eqs.

$\underline{W} = R^{-1}\underline{U}$: characteristic vector

$$W^i(x^j, t) = W^i(x^j - \lambda_i t, 0)$$

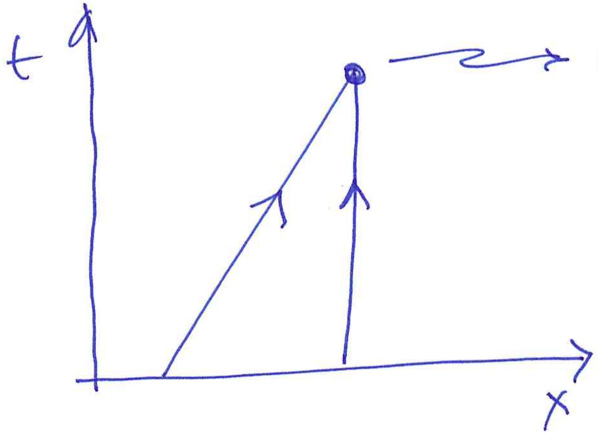
$$\underline{U}(x^j, t) = \sum_{i=1}^{N} W^i(x^j, t)\,\underline{R}^{(i)} = \sum_{i=1}^{N} W^i(x^j - \lambda_i t, 0)\,\underline{R}^{(i)}$$

- simplest nonlinear hyperbolic eq. : Burgers equation

$$\partial_t u + u\, \partial_x u = 0$$

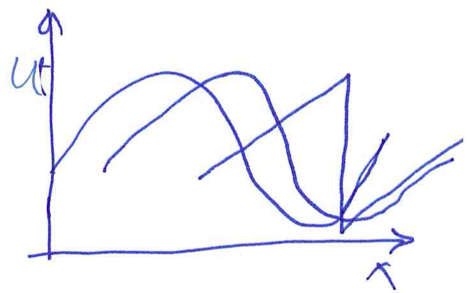
$$d_t u = \partial_t u + \frac{dx}{dt}\, \partial_x u = 0 \qquad \lambda = \frac{dx}{dt} = u(x,t) = u(x_0,0) = u(x-\lambda t, 0)$$

characteristics are still straight lines but not parallel



At caustic characteristics cross and solution is double valued

- nonlinearity implies different parts of the solution propagate at different speeds. Any wave will steepen and shock



Shocks are a well known example of nonlinear waves but not the only one. However, before diving in the topic of nonlinear waves it's useful to review what we know of linear waves in a fluid. Consider therefore a perfect fluid ⊙ and for simplicity a flat spacetime in 1+1 dimensions.

⊙ There are several different ways of defining a perfect fluid but the most effective one is that of a fluid for which viscous terms and heat fluxes are zero and the pressure tensor is diagonal.

Such a fluid is described by an energy-momentum tensor we have already encountered

$$T_{\mu\nu} = (e+p)\,u_\mu u_\nu + p\,g_{\mu\nu}$$

$$= h e\, u_\mu u_\nu + p\,g_{\mu\nu}$$

Let's consider therefore the equations of conservation of energy and momentum

$$\nabla_\mu T^{\mu\nu} = 0$$

which are given by $(\mu = 0,1)$

$$\begin{cases} \partial_t \left[(e + p v^2) W^2 \right] + \partial_x \left[(e+p) W^2 v \right] = 0 \\ \partial_t \left[(e+p) W^2 v \right] + \partial_x \left[(e v^2 + p) W^2 \right] = 0 \end{cases}$$

Exercise

where $u^\mu = W(1, v)$; $W = (1-v^2)^{-1/2}$

Let now e_0, p_0 and $v_0 = 0$ be the values of the energy density, pressure and velocity (fluid at rest) and introduce first-order perturbations of the type

$$e = e_0 + \delta e \,, \quad p = p_0 + \delta p \,, \quad v = v_0 + \delta v = \delta v \,.$$

The resulting set of perturbation equations will be

$$\begin{cases} \partial_t (\delta e) + (e_0 + p_0)\, \partial_x \delta v = 0 \\[2ex] \partial_t \delta v + \dfrac{1}{e_0 + p_0}\, \partial_x \delta p = 0 \end{cases}$$

where we have used

$$\left.\begin{aligned} \partial_t e_0 = 0 = \partial_x e_0 \\ \partial_t p_0 = 0 = \partial_x p_0 \end{aligned}\right\} \text{ stationary and uniform flow.}$$

Taking an additional time derivative and combining terms we obtain

$$(*)\qquad \partial_t^2 \delta e - \frac{\delta p}{\delta e} \partial_x^2 \delta e = 0 \iff \Box \delta e = 0$$

Similarly $\begin{cases} \Box \delta e = 0 \\ \Box \delta p = 0 \end{cases}$: wave equation with speed c_s.

$c_s^2 = \delta p / \delta e$

Eqs $(*)$ show that perturbations propagate as waves with speed c_s. These are <u>acoustic waves</u> (or sound waves) and c_s is the <u>sound speed</u>. More generally

$$c_s^2 = (\partial p / \partial e)_s \qquad s: \text{specific entropy}$$

The linearization approach has clearly removed all the nonlinearities, but as mentioned before, the hydrodynamic equations are intrinsically nonlinear and generically lead to nonlinear waves. These can be distinguished as follows:

- simple waves

  are the nonlinear equivalent of sound waves but solutions of the full nonlinear eqs. They are always associated to a single eigenvalue, for which some quantities of the flow (called Riemann invariants) are conserved (see book for details). A theorem by Friedrichs states that: "any one-dimensional smooth solution neighbouring a constant state must be a simple wave". Examples of simple waves are rarefaction/compression waves

- <u>discontinuous waves</u>

are regions of the flow in which some of the fluid properties (e.g., velocity, rest-mass density, etc) are <u>taken</u> to be discontinuous. More on this later. They can be further distinguished into

- <u>contact waves</u>: surfaces seperating two parts of the flow with different properties but without flow through the surface (contact discontinuities)
- <u>shock waves</u>: same as above but with flow across the surface; fluid on either side have different properties but no chemical/physical change (density, energy, etc) takes place across surface.

- reaction fronts : same as shock waves but with chemical/physical changes taking place across surface (eg detonations, deflagrations).

□

Having completed this classification we can now concentrate on the mathematical properties of simple waves and in particular of rarefaction waves.

They are simple waves that are characterized by the fact that the pressure and rest-mass density decrease in the region where the wave propagates. As simple waves they are adjacent to a constant state and are isentropic ($ds = 0$). The corresponding eigenvalues are the sonic ones

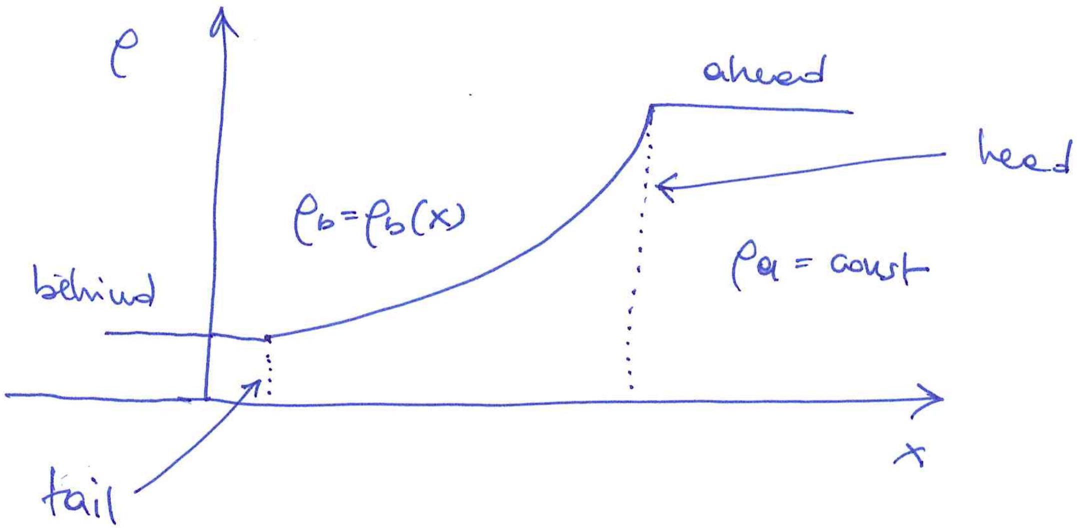
$$\lambda_{\pm} = \frac{v \pm c_s}{1 \pm v c_s}$$

(cf velocity composition law in special relativity) with Riemann invariants

$$J_\pm = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) \pm \int \frac{c_s}{e}\, de = \text{const.}$$

RWs have a head moving in the unperturbed medium and a tail, representing the portion of the flow moving with slowest velocity

Exercise



An important property of simple waves and hence of RWs is that in the case of one-dimensional flows they can be written in a self-similar form ie in a form in which all quantities depend only on a self similar variable

$$\xi := \frac{x}{t}$$

ξ is dimensionally a velocity but can be seen as a position at any given time or as a time at any given position.

To derive the relevant expressions let's consider a one-dimensional flow in flat spacetime

$$u^\mu = W(1, v, 0, 0)\,, \qquad W = (1-v^2)^{-1/2}$$

Conservation of rest-mass (continuity eq.) and conservation of momentum

$$\begin{cases} \nabla_\mu J^\mu = \nabla_\mu (\rho u^\mu) = 0 \\ h^\nu{}_\lambda \nabla_\mu T^{\mu\lambda} = 0 \end{cases} \iff$$

Exercise

$$\begin{cases} \partial_t (\rho W) + \partial_x (\rho W v) = 0 & \text{(RW1)} \\ W \partial_t (W v) + W v \, \partial_x (W v) = -\frac{1}{\rho h} \left(\partial_x p + W^2 v \partial_t p + W^2 v^2 \partial_x p \right) & \text{(RW2)} \end{cases}$$

We now express the differential operators ∂_t and ∂_x as

$$\partial_t := -\frac{\xi}{t} \frac{d}{d\xi} \qquad \partial_x := \frac{1}{t} \frac{d}{d\xi}$$

so that the adiabaticity condition $u^\mu \nabla_\mu s = 0 = W \partial_t s + W v \partial_x s = 0$

can be written as

Exercise

$$(v-\xi)\frac{ds}{d\xi}=0.$$

Similarly, eqs (RW1) and (RW2) can be written as

Exercise

$$\begin{cases} (v-\xi)\dfrac{de}{d\xi}+e\left[W^2v(v-\xi)+1\right]\dfrac{dv}{d\xi}=0 \\ ehW^2(v-\xi)\dfrac{dv}{d\xi}+(1-v\xi)\dfrac{dp}{d\xi}=0 \end{cases}$$

The energy conservation eq. $u_\nu\nabla_\mu T^{\mu\nu}=0$ is most easily deduced recalling the first law of thermodynamics for isentropic flows

$$de=h\,dp+eT\,ds=h\,dp$$ from which we obtain

(RW3) $$\frac{dp}{d\xi} = h c_s^2 \frac{d\rho}{d\xi}$$

A self-similar solution will exist if the determinant of the system (RW1) - (RW3) vanishes, ie if

$$c_s^2 = \left(\frac{v-\xi}{1-v\xi}\right)^2 \implies$$

$$c_s = \left|\frac{v-\xi}{1-v\xi}\right| = \frac{|v-\xi|}{1-v\xi} = \pm\frac{v-\xi}{1-v\xi}$$ where the $\pm$ signs have the following meaning

+: for $v > \xi$, ie $\mathcal{R}_{\leftarrow}$ left-propagating RW wrt fluid

−: for $v < \xi$, ie $\mathcal{R}_{\rightarrow}$ right-propagating RW wrt fluid

We can invert this relation to get

$$\xi = \frac{v \mp c_s}{1 \mp v c_s} \qquad \text{where now} \quad \begin{matrix} + : & R_{\rightarrow} \\ - & R_{\leftarrow} \end{matrix}$$

This expression allows us to compute the speeds of the head and tail of the RW.

Examples. Let $v_a = 0$ (fluid at rest)

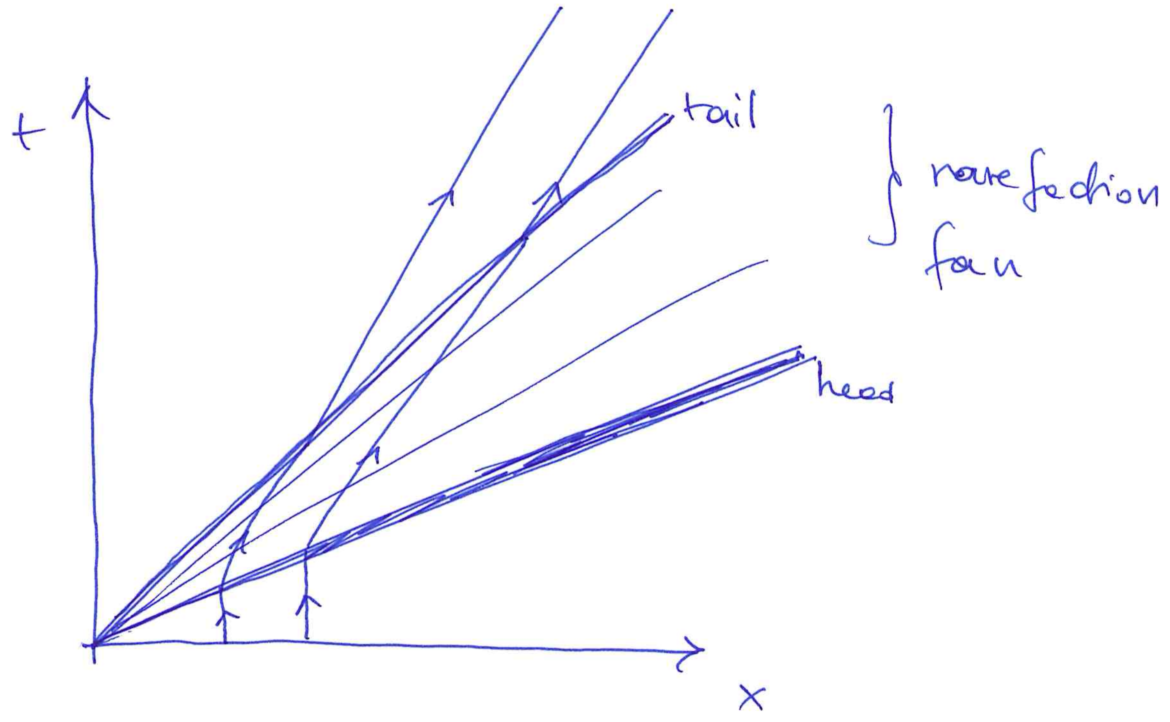
$$\xi_{head} = \mp c_s \quad \begin{matrix} R_{\leftarrow} \\ R_{\rightarrow} \end{matrix}$$

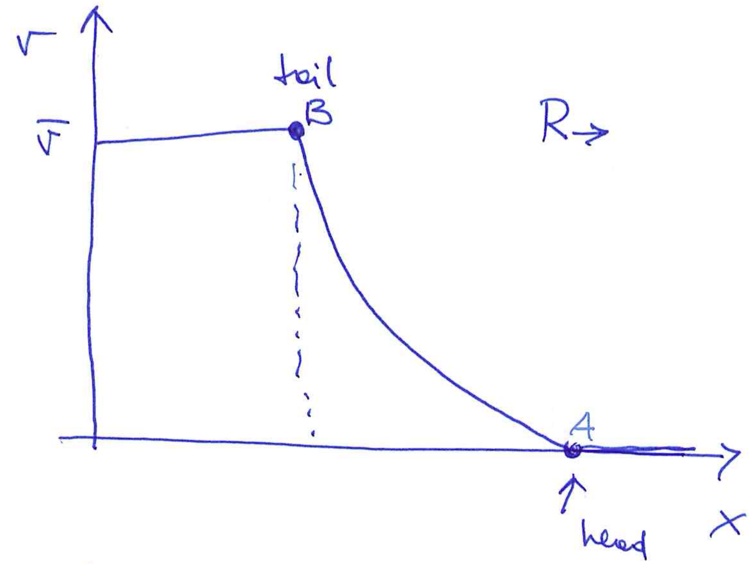
in other words: the head of the RW moves at the local speed of sound

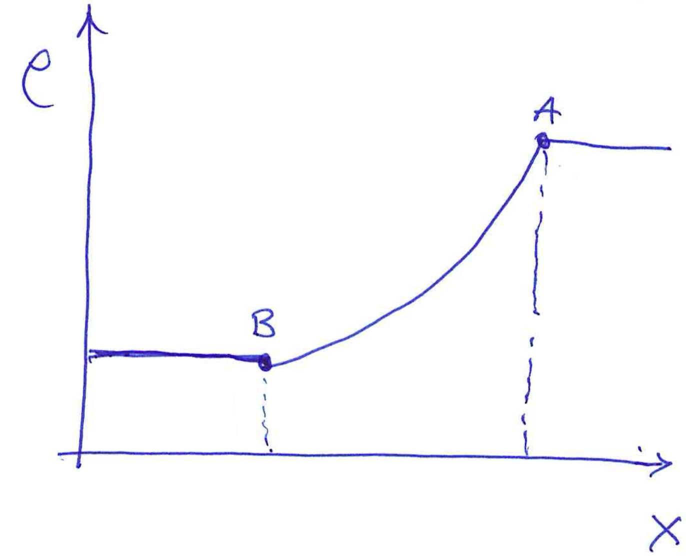
Similarly, let $v_b = \bar{v}$: const. velocity after the RW

$$\xi_{tail} = (\bar{v} \mp c_s)/(1 \mp \bar{v} c_s)$$

Some diagrams will help fix ideas



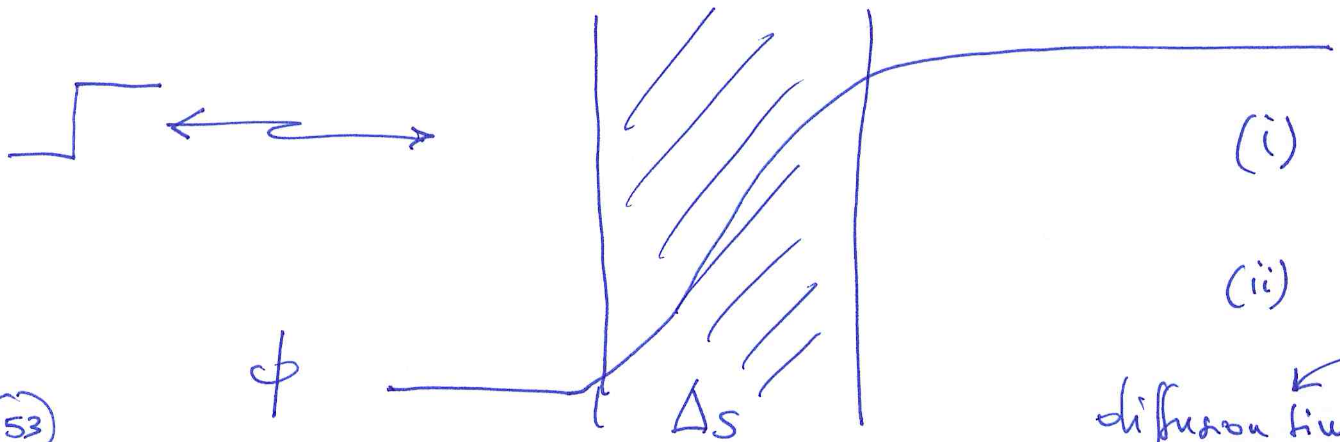




As discussed when considering the Burgers equation discontinuous waves can be produced from compressive motions having smooth initial data.

A discontinuous wave is a mathematical artefact to cope with flows in which the properties vary very rapidly on a very small lengthscale. No physics breaks down at a shock wave! Simply, complex and steep gradients are replaced by simple junction conditions.

The use of shock wave is reasonable when



(i) $\Delta s \approx l_{mfp} \ll \phi / \partial_x \phi$

(ii) $\tau \ll \underline{\phi / \partial_t \phi}$

τ: diffusion timescale

$\phi / \partial_t \phi$: timescale of front motion

Recap

- Sound waves: linear perturbation of hydrodynamics equations obey wave equation with propagation speed $c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s$

- Nonlinear waves

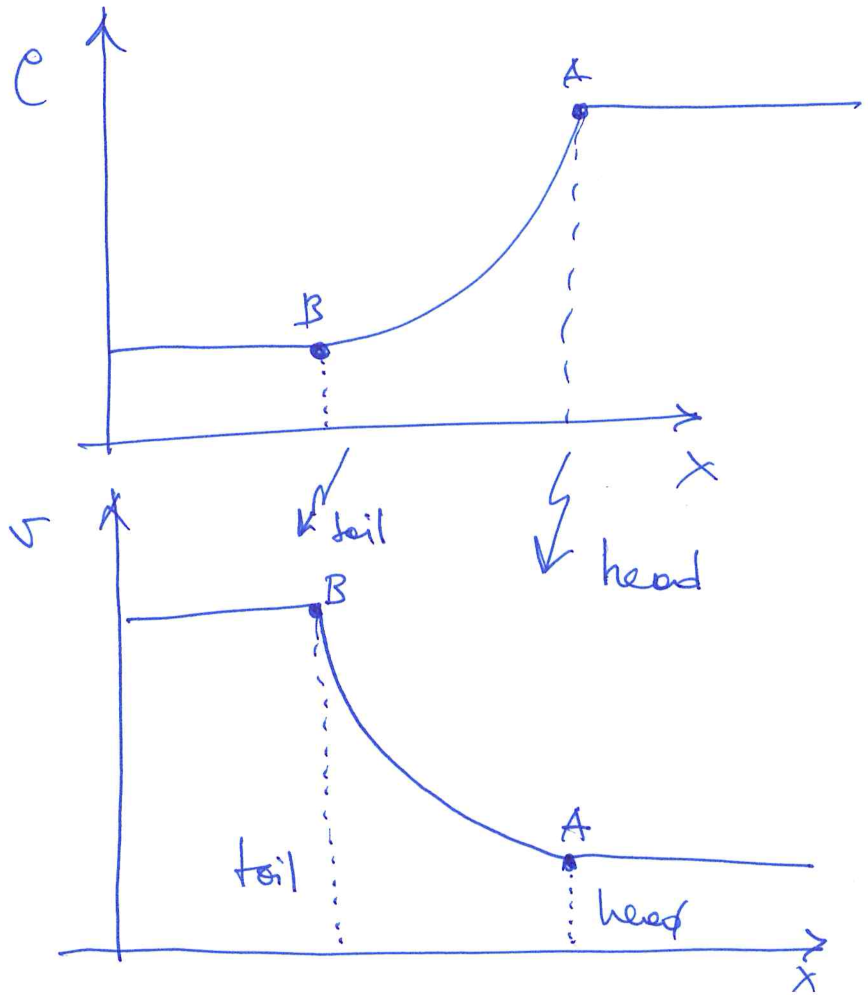
  - simple waves : rarefaction waves

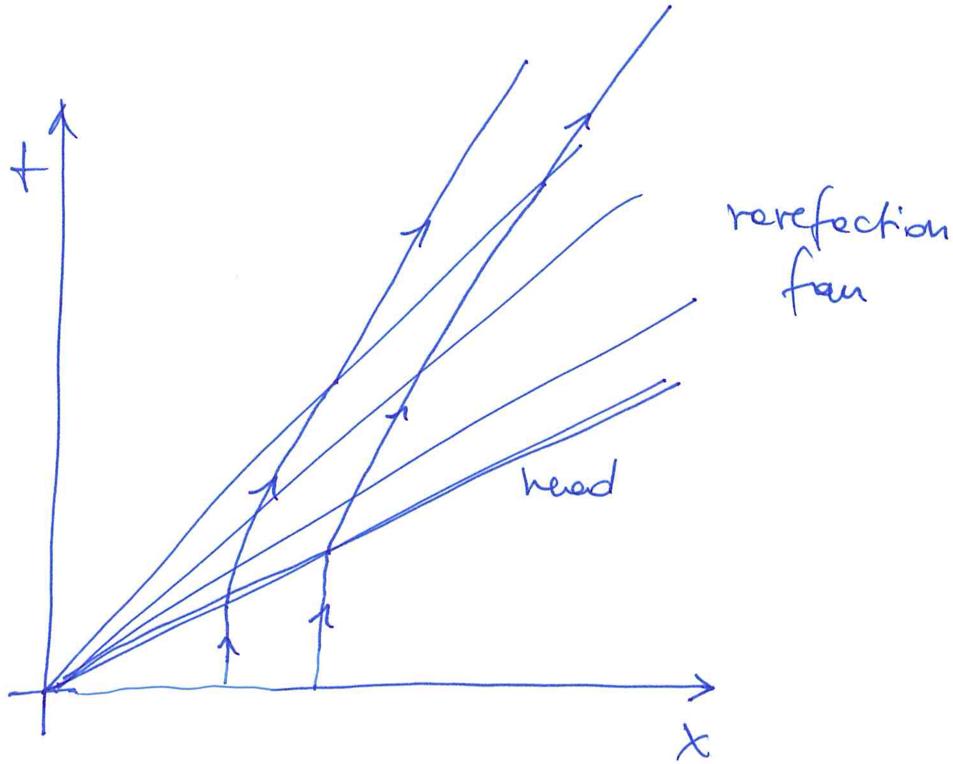
  - discontinuous waves

    * contact waves (no mass flux)
    * shock waves (mass flux; no chemical change)
    * reaction waves (mass flux; chemical change)
      - detonations
      - deflagrations

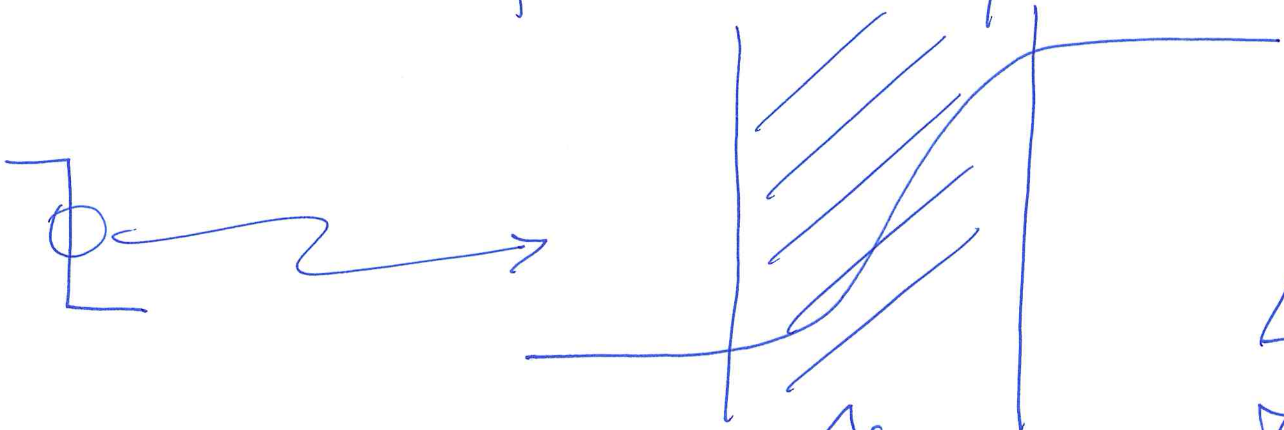
- rarefaction waves : isentropic non linear waves.

$$J^{\pm} = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) \pm \int \frac{c_s}{\rho}\,d\rho = \text{const.}$$ RIEMANN INVARIANT





- physics does not break down at discontinuity surfaces.
  Hydrodynamics is replaced by algebraic junction conditions
  At microscopic level all quantities are continuous



Δs

$$\Delta_s \sim l_{mfp} \ll \phi/\partial_x \phi$$

$$\tau \ll \underbrace{\phi/\partial_t \phi}_{\sim \text{ front speed}}$$

diffusion timescale

If these conditions are not met, then the discontinuous wave approximation is not valid and more sophisticated approaches are necessary, eg Boltzmann equation.

The junction conditions mentioned above are simple algebraic conditions that guarantee the conservation across the shock front of rest-mass, energy and momentum

To derive these conditions in a covariant form we start from the conservation equations

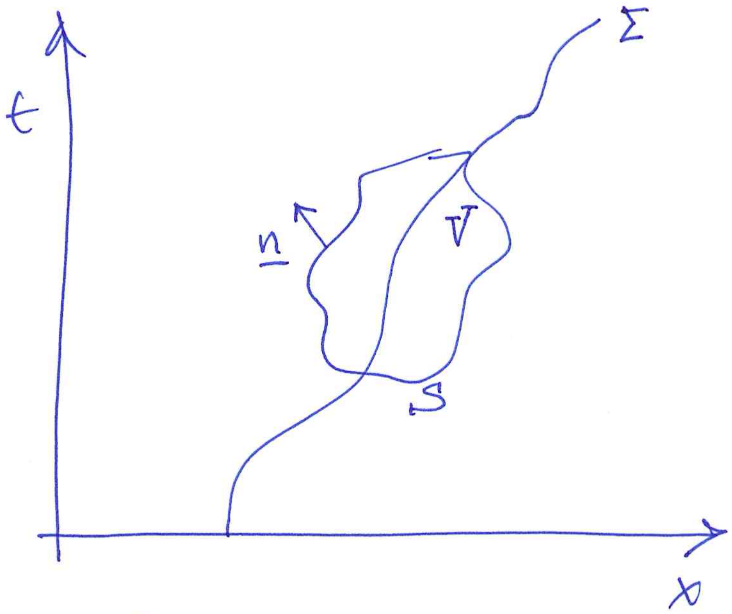
$$\begin{cases} \nabla_\mu (\rho u^\mu) = 0 \\ \nabla_\mu T^{\mu\nu} = 0 \end{cases}$$

which we rewrite as

$$\begin{cases} \nabla_\mu (\rho u^\mu f) = \rho u^\mu \nabla_\mu f \\ \nabla_\mu (T^{\mu\nu} \lambda_\nu) = T^{\mu\nu} \nabla_\mu \lambda_\nu \end{cases}$$

where f and λ_ν are two arbitrary scalar function and vector field, respectively

Let Σ be the history of a 2D spacelike surface representing the shock front. Let $\mathcal{V}$ be a 4D volume around Σ.



$$\int_{\mathcal{V}} \nabla_\mu (\rho u^\mu f) d^4x = \int_{\mathcal{V}} \rho u^\mu \nabla_\mu f \, d^4x$$

$$= \oint_{\mathcal{S}} \rho u^\mu f n_\mu \, d^3x$$

Stoke's theorem

Similarly

$$\int_{\mathcal{V}} \nabla_\mu (T^{\mu\nu} \lambda_\nu) d^4x = \int_{\mathcal{V}} T^{\mu\nu} \nabla_\mu \lambda_\nu \, d^4x = \oint_{\mathcal{S}} T^{\mu\nu} \lambda_\nu n_\mu \, d^3x$$

Consider now the limit in which $\mathcal{V} \to 0$. The first two integrals vanish, while the third ones reduce to the calculation of the integrand on both sides of Σ, ie

$$
(*)\quad \begin{cases} \displaystyle\int_{\Sigma'} f\, [\![\rho u^\mu]\!]\, n_\mu\, d^3x = 0 \\[2ex] \displaystyle\int_{\Sigma'} \lambda_\mu\, [\![T^{\mu\nu}]\!]\, n_\nu\, d^3x = 0 \end{cases}
$$

ahead ("1")

behind ("2")

where $[\![Q]\!] \equiv \underbrace{Q_a - Q_b}_{\text{jump of } Q \text{ across the shock front.}}$: "double bracket notation"

Given the arbitrariness in the choice of f and λ, the conditions $(*)$ can be satisfied iff

$$
(**)\quad \begin{cases} [\![\rho u^\mu]\!]\, n_\mu = 0 \\[2ex] [\![T^{\mu\nu}]\!]\, n_\nu = 0 \end{cases}
$$

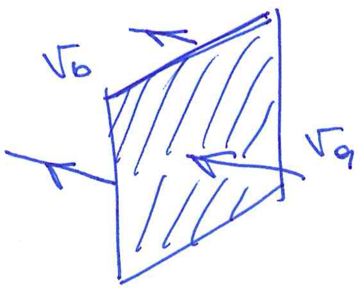
: (relativistic) Rankine Hugoniot conditions (junction / jump conditions)

Example: flat spacetime 1+1 flow, shock front moving in x direction. Here after use shock as rest frame to remove boost.

$u^\mu = W(1, v, 0, 0)$; $n_\mu = (0, 1, 0, 0)$ and $(**) \iff$

$$\rho_a u_a^x = \rho_b u_b^x \; ; \quad T_a^{xx} = T_b^{xx} \; ; \quad T_a^{tx} = T_b^{tx}$$

$$\iff$$



$$J := \rho_a W_a v_a = \rho_b W_b v_b \qquad : \text{rest-mass} \quad (1)$$

$$\rho_a h_a W_a^2 v_a^2 + p_a = \rho_b h_b W_b^2 v_b^2 + p_b \qquad : \text{energy} \quad (2)$$

$$\rho_a h_a W_a^2 v_a = \rho_b h_b W_b^2 v_b \qquad : \text{momentum} \quad (3)$$

These equations can also be written as

$$[\![J]\!] = 0 \qquad \text{rest-mass}$$

$$J^2 = - \frac{[\![p]\!]}{[\![h/\rho]\!]}$$

$$[\![hW]\!] = 0 \quad \text{momentum.}$$

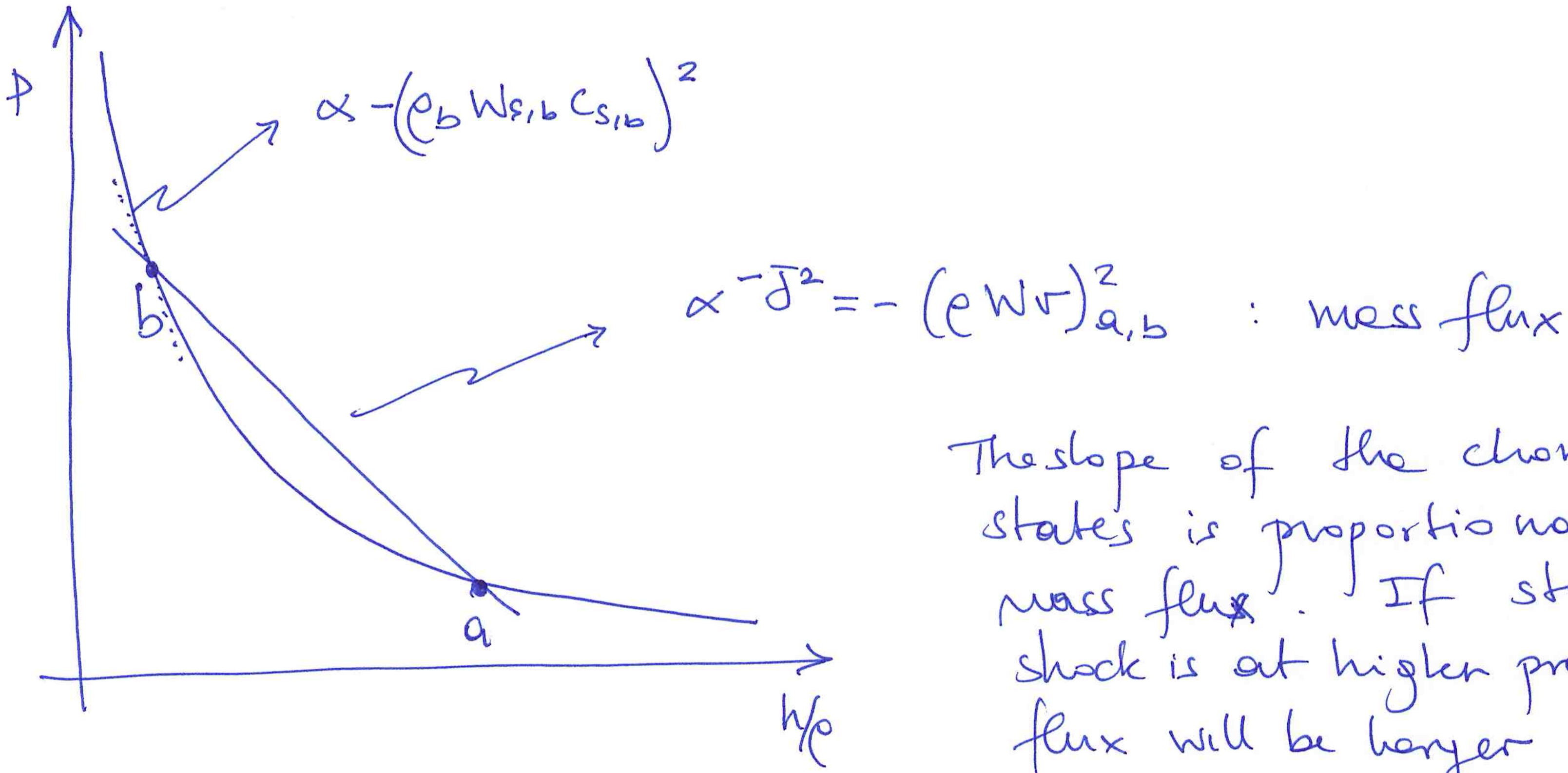
Their combination can be used to derive a very important equation:

$$[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b} \right) [\![p]\!]$$

: Taub adiabat (relativistic equivalent of the Hugoniot adiabat)

$$[\![\epsilon + \frac{p}{\rho}]\!] = \frac{1}{2} \left(\frac{1}{\rho_a} + \frac{1}{\rho_b} \right) [\![p]\!]$$

The importance of the Taub adiabat is that provides a simple and graphical description of the fluid changes across a shock wave.



The slope of the chord joining two states is proportional to the mass flux. If state behind shock is at higher pressure, the mass flux will be larger

Slope of Taub adiabat at any point is negative and proportional to the local sound speed

$$\frac{dp}{d(h/\rho)} = -\rho^2 W_s^2 c_s^2 \qquad W_s = (1-c_s^2)^{-1/2} \qquad c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s$$

$$= \frac{-\rho^2 c_s^2}{1-c_s^2} \propto -c_s^2$$

Exercise

1. Compute Taub adiabat for an ideal-fluid EOS

How do we know if a discontinuity front is physically realistic? The answer comes from considering <u>weak shocks</u>, ie discontinuities in which the states ahead and behind are not very different (derivatives are discontinuous but flow is continuous). In this case, the jumps in the specific entropy are given by

$$[\![s]\!] = \left[\frac{1}{12hT}\left(\frac{\partial^2(h/\rho)}{\partial p^2}\right)\right]_a [\![p]\!]^3 + O\left([\![p]\!]^4\right)$$

It is possible to show, using the Poisson adiabat, ie the equivalent of the Taub adiabat for states having the same entropy, that $\partial^2 p / \partial (h/\rho)^2 > 0$, ie the second derivative of the Taub adiabat is always positive

Second law of thermodynamics imposes that entropy must increase, ie

$$[\![s]\!] < 0 \Rightarrow \qquad s_a - s_b < 0$$

$$[\![p]\!] < 0$$

in other words, across a physically realistic shock

$$\boxed{s_b > s_a} \; ; \qquad \boxed{p_b > p_a}$$

so that the state behind the shock necessarily has higher pressure and entropy. A shock such that $[\![s]\!] > 0$ is unphysical. Since the new state must lie on the

Taub adiabat

$$p_b > p_a \Rightarrow \frac{h_b}{\rho_b} < \frac{h_a}{\rho_a}$$

but $\quad [\![h^2]\!] \propto [\![p]\!] \Rightarrow$

$$h_b > h_a \quad \Longrightarrow$$

$\boxed{\rho_b > \rho_a}$: ie the fluid is compressed by the shock.

What about the velocity?

: momentum conservation implies

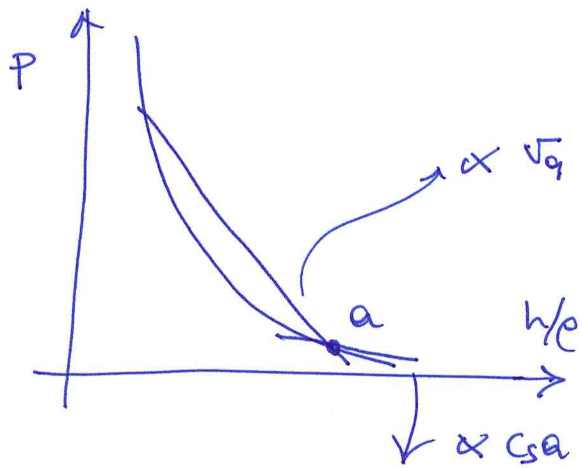
$$\rho_a h_a W_a^2 v_a^2 + p_a = \rho_b h_b W_b^2 v_b^2 + p_b$$

and since ρ, p, h increase across the shock, it must be that

$$W_a^2 v_a^2 < W_b^2 v_b^2 \quad \Rightarrow$$

$\boxed{|v_a| > |v_b|}$ ie the fluid is decelerated across the shock

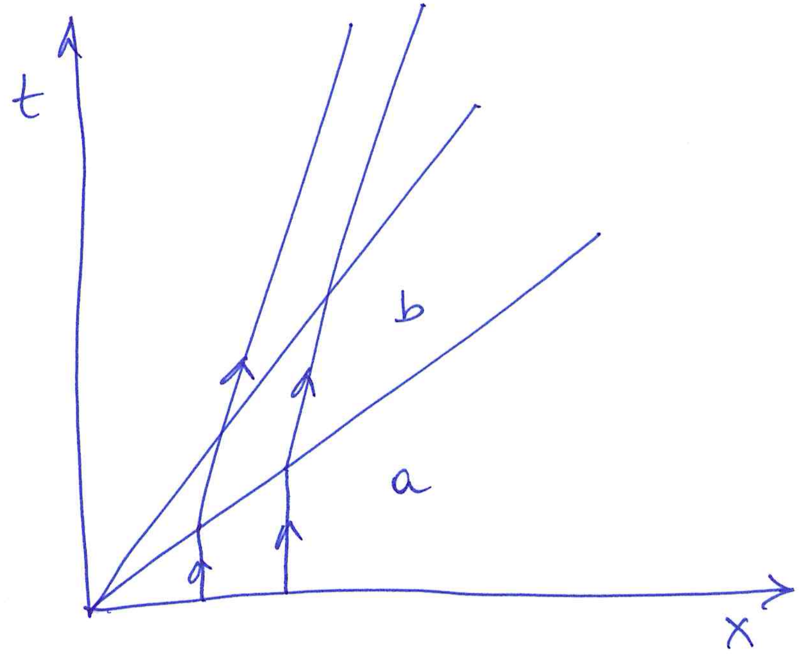
Furthermore it's easy to recongnize that



$v_a \geq c_{sa}$ ($\propto$ tg at a)

$v_b \leq c_{sb}$ ($\propto$ tg at b)

in other words: flow is <u>supersonic</u> when entering the shock and <u>subsonic</u> when leaving the shock

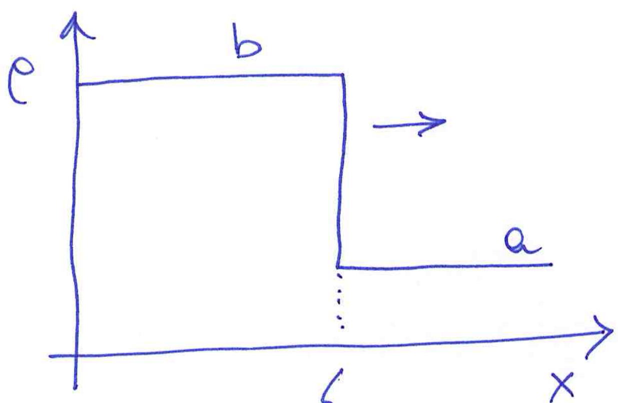


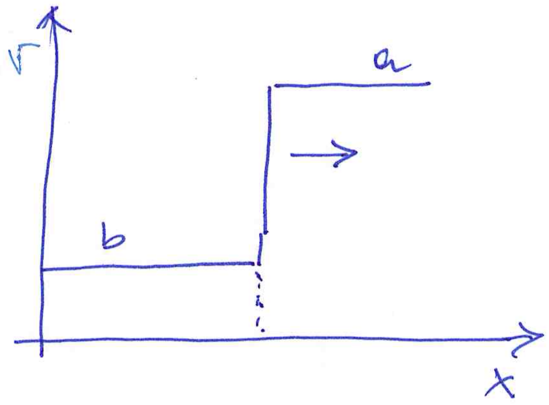
: spacetime diagram of shock moving to the positive x-direction

Recap: across a shock

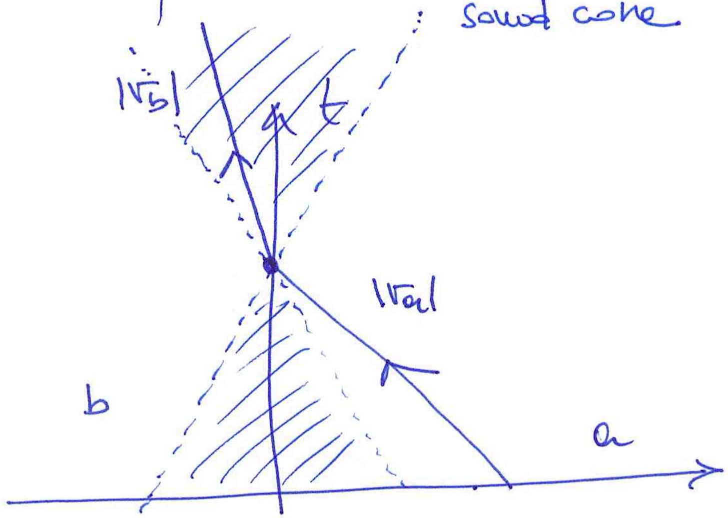
$$\left\{\begin{array}{l} S_b > S_a \\ p_b > p_a \\ h_b > h_a \\ \rho_b > \rho_a \\ \underbrace{|v_b| < |v_a|} \end{array}\right.$$

note that this is the velocity in a frame comoving with the shock!

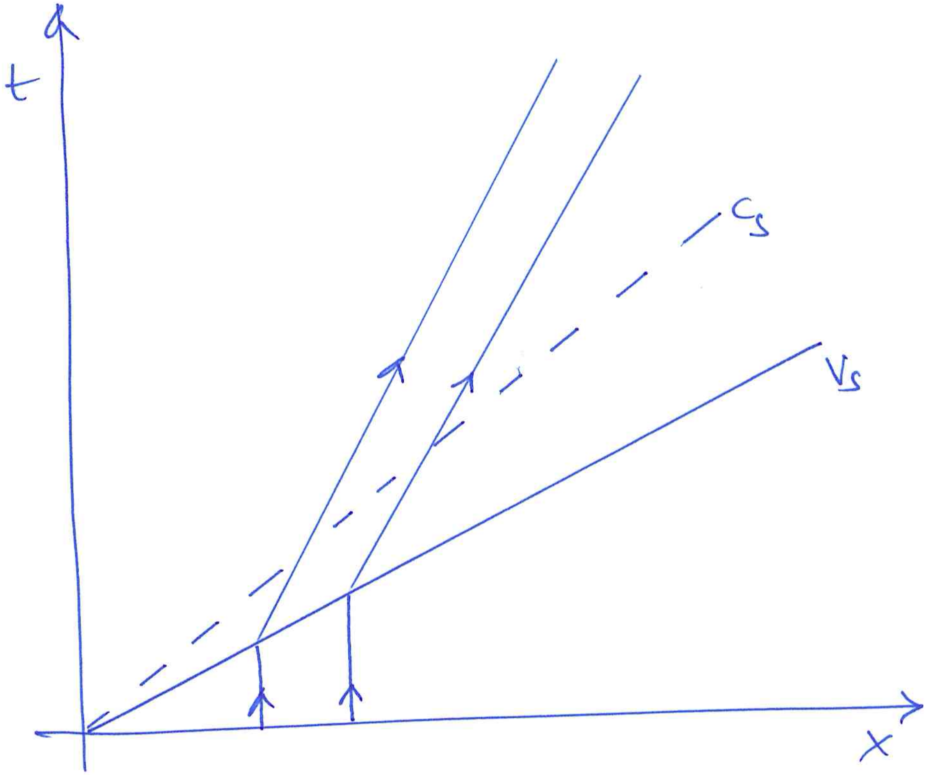




Shock position



The spacetime diagram showing representative fluidlines (ie worldlines of representative fluid elements) is shown below:



Note that in the Eulerian frame the shock is supersonic, ie moves with $V_s > c_s$ (cf the slope of the sound speed: this is the sound speed ahead of the shock even though it is marked in the shocked fluid

Note also that the fluid is compressed, $\rho_b > \rho_a$ as deduced from the separation between fluidlines.

The junction conditions can be manipulated to obtain expressions for the velocities on either side of the shock in terms of the quantities there, eg

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)}$$

$$v_b^2 = \frac{(p_a - p_b)(e_a + p_b)}{(e_a - e_b)(e_b + p_a)}$$

Contact discontinuity

Note that $v_a \to 0$ for $p_a \to p_b$

On the other hand $e_b > e_b$ for a physical shock

or the relative velocity

$$v_{ab} = \frac{v_a - v_b}{1 - v_a v_b} = \left(\frac{(p_a - p_b)(e_a - e_b)}{(e_a + p_b)(e_b + p_a)} \right)^{1/2}$$

Recap

- covariant formulation of jumps across discontinuity

$$[\![\rho a^\mu]\!] n_\mu = 0$$

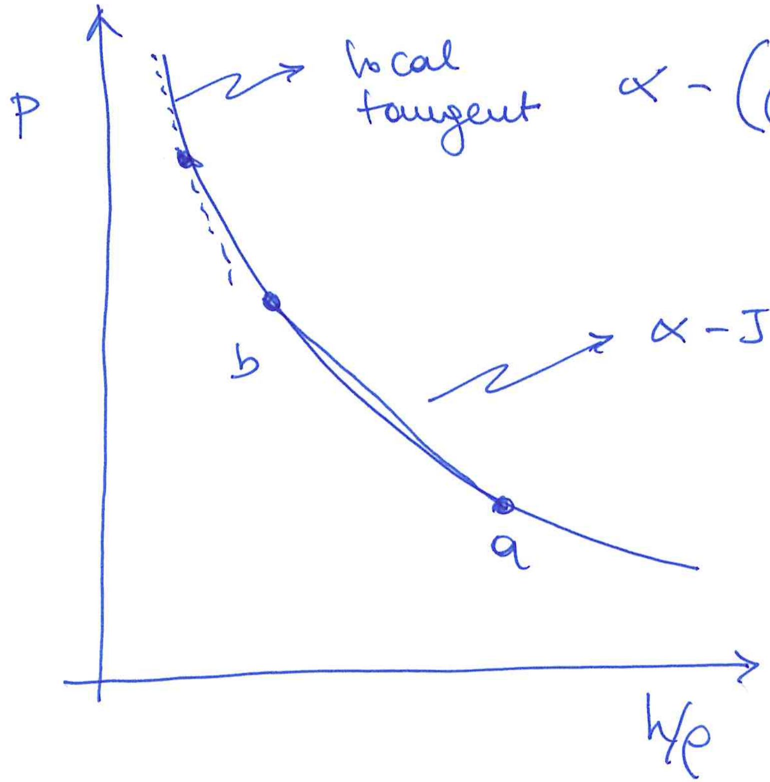
$$[\![T^{\mu\nu}]\!] n_\nu = 0$$

for 1-D flow in flat spacetime

$$\rho_a u_a^x = \rho_b u_b^x \; ; \quad \underbrace{T_a^{xx} = T_b^{xx}}_{\substack{\text{cons. of} \\ \text{momentum} \\ \text{(flux).}}} \; ; \quad \underbrace{T_a^{0x} = T_b^{0x}}_{\substack{\text{cons. of energy} \\ \text{(flux).}}}$$

- using these equations one obtains Taub adiabat

$$[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right)[\![p]\!]$$



$\propto -(\rho W_s c_s)^2$ local tangent proportional to local sound speed.

$\propto -J^2 = (\rho W v)^2_{a,b}$: slope of chord is proportional to mass flux

Across a physical shock entropy must increase

$s_b > s_a \implies$

$p_b > p_a$

$\Rightarrow \quad \rho_b > \rho_a$

$\Rightarrow \quad |v_a| < |v_b|$: relative to shock front

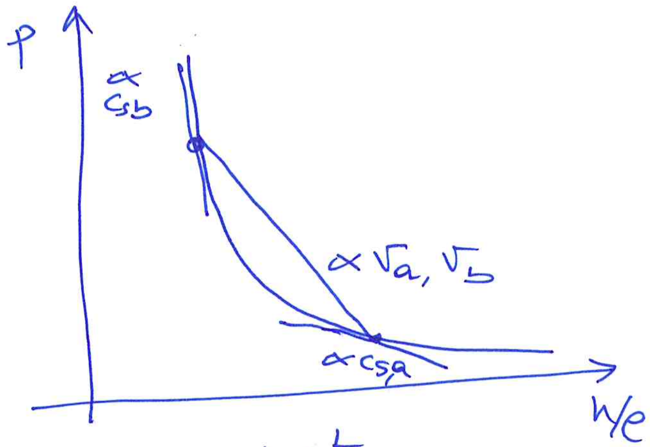
In other words, across a shock the fluid is compressed, heated up and decelerated.

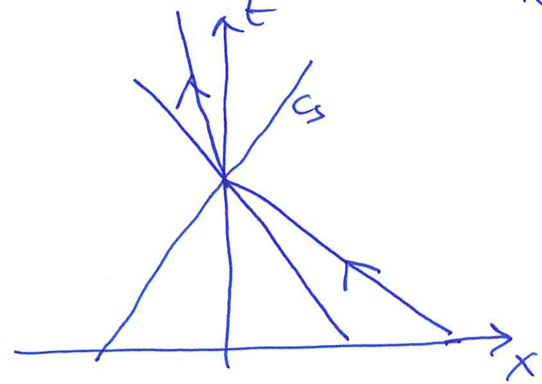
The Taub adiabat also reveals that

$|v_a| > c_{sa}$

$|v_b| < c_{sb}$

The flow entering the shock is supersonic and is subsonic when leaving the shock (relative to the shock)





The junction conditions can be manipulated to obtain expressions for the velocities on either side of the shock in terms of the quantities there, eg

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)}$$

$$v_b^2 = \frac{(p_a - p_b)(e_a + p_b)}{(e_a - e_b)(e_b + p_a)}$$

Contact discontinuity

Note that $v_a \to 0$ for $p_a \to p$
On the other hand $e_b > e$
for a physical shock

or the relative velocity

$$v_{ab} = \frac{v_a - v_b}{1 - v_a v_b} = \left(\frac{(p_a - p_b)(e_a - e_b)}{(e_a + p_b)(e_b + p_a)} \right)^{1/2}$$

or the ratio / product of velocities

$$\frac{v_a}{v_b} = \frac{e_b + p_a}{e_a + p_b} \quad ; \quad v_a v_b = \frac{p_a - p_b}{e_a - e_b}$$

These equations are particularly telling in the case of an ultrarelativistic fluid, ie $p = \frac{1}{3} e$ and $c_s = \frac{1}{\sqrt{3}}$, in which case

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b} \right) v_b \quad ; \quad v_a = \frac{1}{3 v_b}$$

while the corresponding Lorentz factors are

$$W^2_{a(b)} = \frac{3}{8} \left(\frac{3 e_{a(b)} + e_{b(a)}}{e_{a(b)}} \right)$$

$$W^2_{ab} = \frac{(3e_a + e_b)(3e_b + e_a)}{16 e_a e_b} = \frac{4}{9} W_a^2 W_b^2$$

with the additional property that

$$W_a^2 - 2W_{ab}^2 W + W_b^2 = 1.$$

All of the relations described so far are measured in the shock rest frame, but in such a frame one cannot measure the shock velocity. To derive it, it is sufficient to bear in mind that the mass flux is invariant under Lorentz boosts in the x-direction, ie

$$J = \rho_a W_a v_a = \rho_a \tilde{W}_a \tilde{W}_s (V_s - \tilde{v}_a) = \rho_b \tilde{W}_b \tilde{W}_s (V_s - \tilde{v}_b)$$

where the tilde is meant to refer to velocities in the Eulerian frame.

This expression can be inverted to obtain:

$$V_s^{\pm} = \frac{\rho_a^2 W_a^2 v_a \pm |J| \sqrt{J^2 + \rho_a^2}}{\rho_a^2 W_a^2 + J^2}$$

so that the shock velocity is fully determined by the properties of the flow ahead.

$$J^2 = - \frac{[\![p]\!]}{[\![h/\rho]\!]}$$

Eg.

$v_a = 0$; $W_a = 1$

$$V_s^{\pm} = \frac{\rho_b W_b v_b}{\rho_b W_b \mp \rho_a}$$

Let's consider some interesting limits (still for an ultrarelativistic fluid with $p = e/3$)

$\boxed{\text{weak shocks}}$: $e_b \sim e_a$

then

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b}\right) v_b \sim v_b \quad ; \quad v_a \simeq \frac{1}{3 v_b} \Rightarrow$$

$$v_a \sim \frac{1}{\sqrt{3}} = c_{sa}$$

similarly $v_b \to c_{sb}$: weak shock in ultrarelativistic fluid tends to a sound wave.

$\boxed{\text{strong shocks}}$: $e_b \to \infty$

$v_a \to 1$; $v_b \to \frac{1}{3}$; note that $v_b / c_{sb} = 1/\sqrt{3} \leq 1$

ie the shocked fluid is still subsonic despite $v_s \to 1$!

Contact discontinuities:

Discontinuous nonlinear waves with $[\![j]\!] = 0 \Rightarrow$

$[\![p]\!] = 0 \underset{\text{Hugoniot}}{\Rightarrow} [\![h]\!] = 0$ but $[\![e]\!] \neq 0$

ie

$$[\![p]\!] = 0 = [\![h]\!] = [\![v]\!] \quad ; \quad [\![e]\!] \neq 0$$

pressure, specific enthalpy, (normal)⊙ velocities are continuous across CDs ; density is discontinuous across CDs

⊙ Tangent velocities can indeed be discontinuous across a CD. This is relevant for multi-dimensional Riemann problems.

Riemann problem

Determine the flow pattern (ie the number and type) of nonlinear waves that develops from constant and discontinuous initial data.

Riemann worked on this more than 150 years ago and the solution of this problem is the basis for many advanced numerical methods in (relativistic) hydrodynamics

Mathematically, this is defined as

$$\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x<0 \\ \underline{U}_R & \text{if } x>0 \end{cases}$$

$\underline{U}_L \neq \underline{U}_R$

$\underline{U}_L = \text{const};$

$\underline{U}_R = \text{const};$

Physically you can think of a tube containing a membrane and in which you can specify the properties of the fluid on either side of the membrane

The problem consists then in determining the evolution of the system if the membrane is removed istantaneously.

This solution is schematically represented as

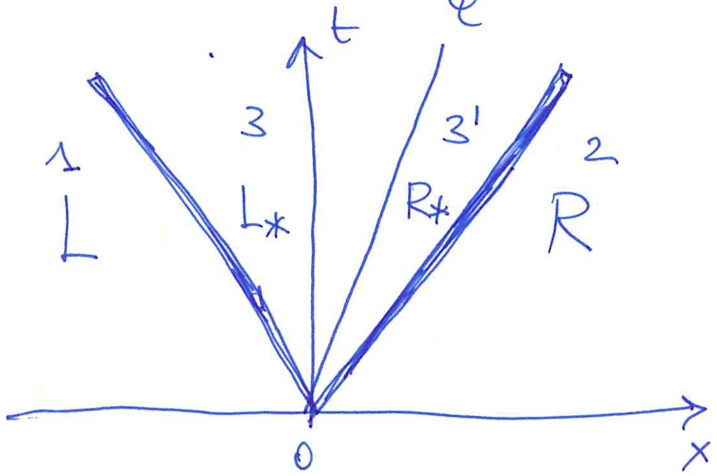
$L \; R$ at $t=0$

$L \; \underset{\leftarrow}{W} \; L_* \; \mathcal{C} \; R_* \; \underset{\rightarrow}{W} \; R$ at $t>0$

left propagating nonlinear wave

contact discontinuity

right-propagating nonlin. wave



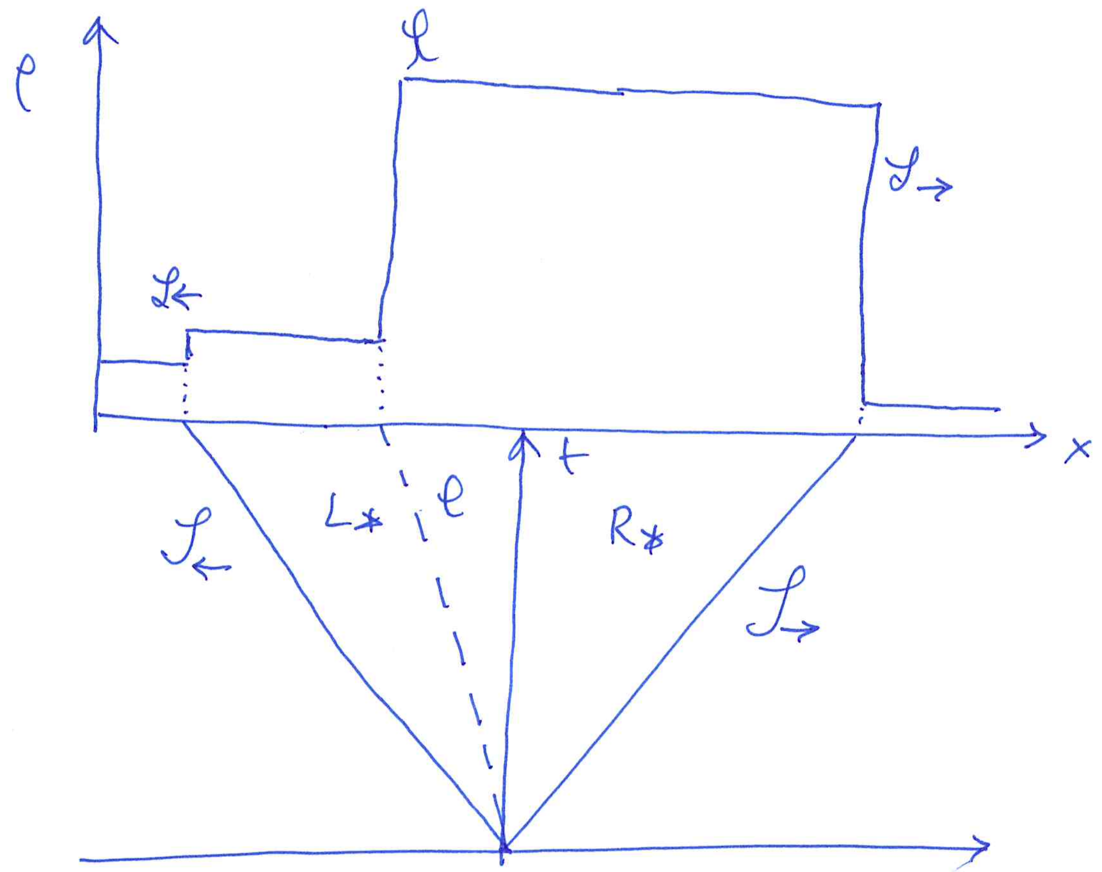
Note :

- the states L and R are the original ones as the waves have not yet reached them
- the regions $L*$ and $R*$ are separated by a contact discontinuity and hence $p_{L*} = p_{R*} = p_*$
- no assumption is made on the waves $W_{\rightarrow}$ and $W_{\leftarrow}$: these can be shocks or rarefaction waves.

In Newtonian hydrodynamics Riemann concluded that the one-dimensional flow resulting from the initial data ($\triangledown$) will lead to four different solutions, or equivalently, three wave patterns.

(i) two shock waves moving to the right and to the left

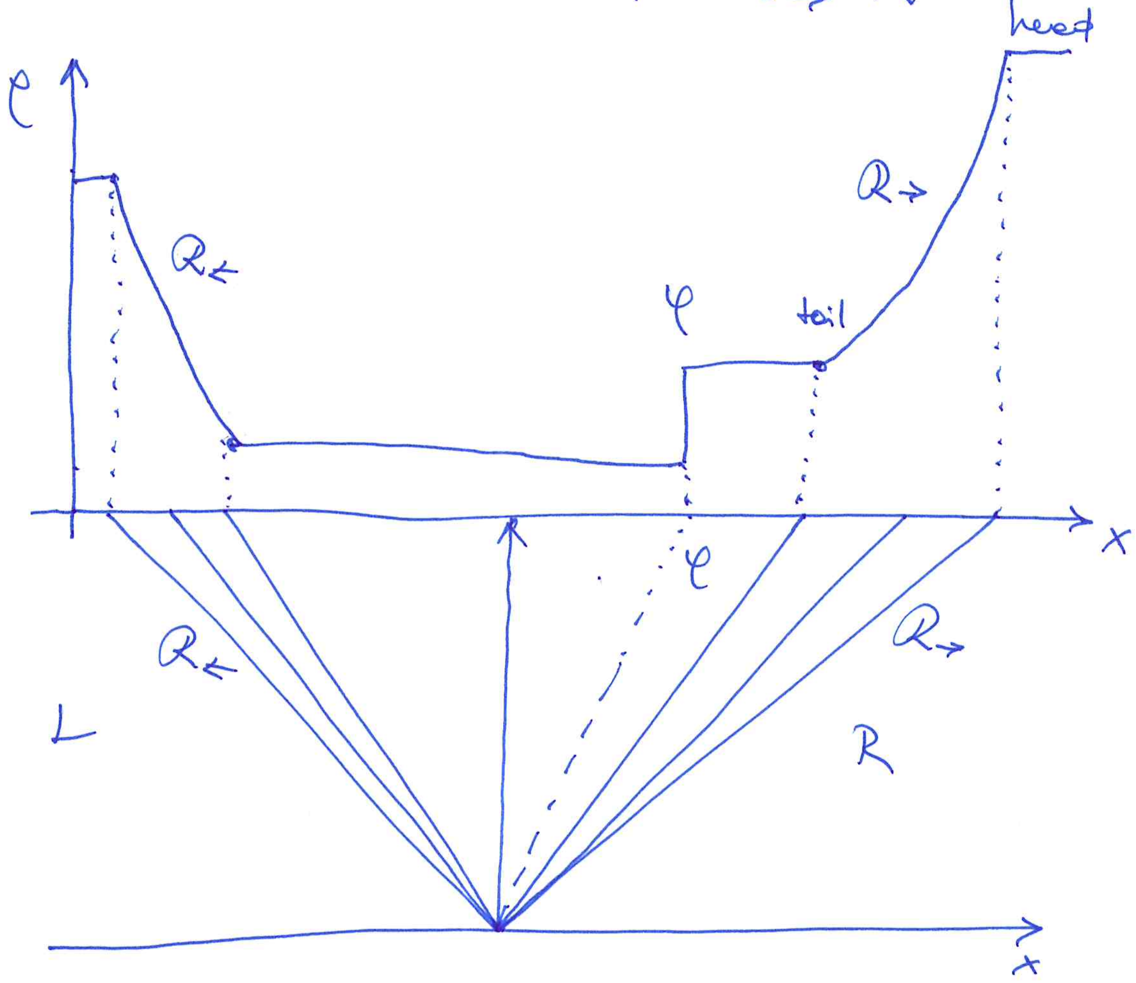
$$L \; \mathcal{S}_{\leftarrow} \; L_* \; \mathcal{C} \; R_* \; \mathcal{S}_{\rightarrow} \; R$$



Note that the shocked fluid has larger density.

(ii) two rarefaction waves: moving to the right and to the left

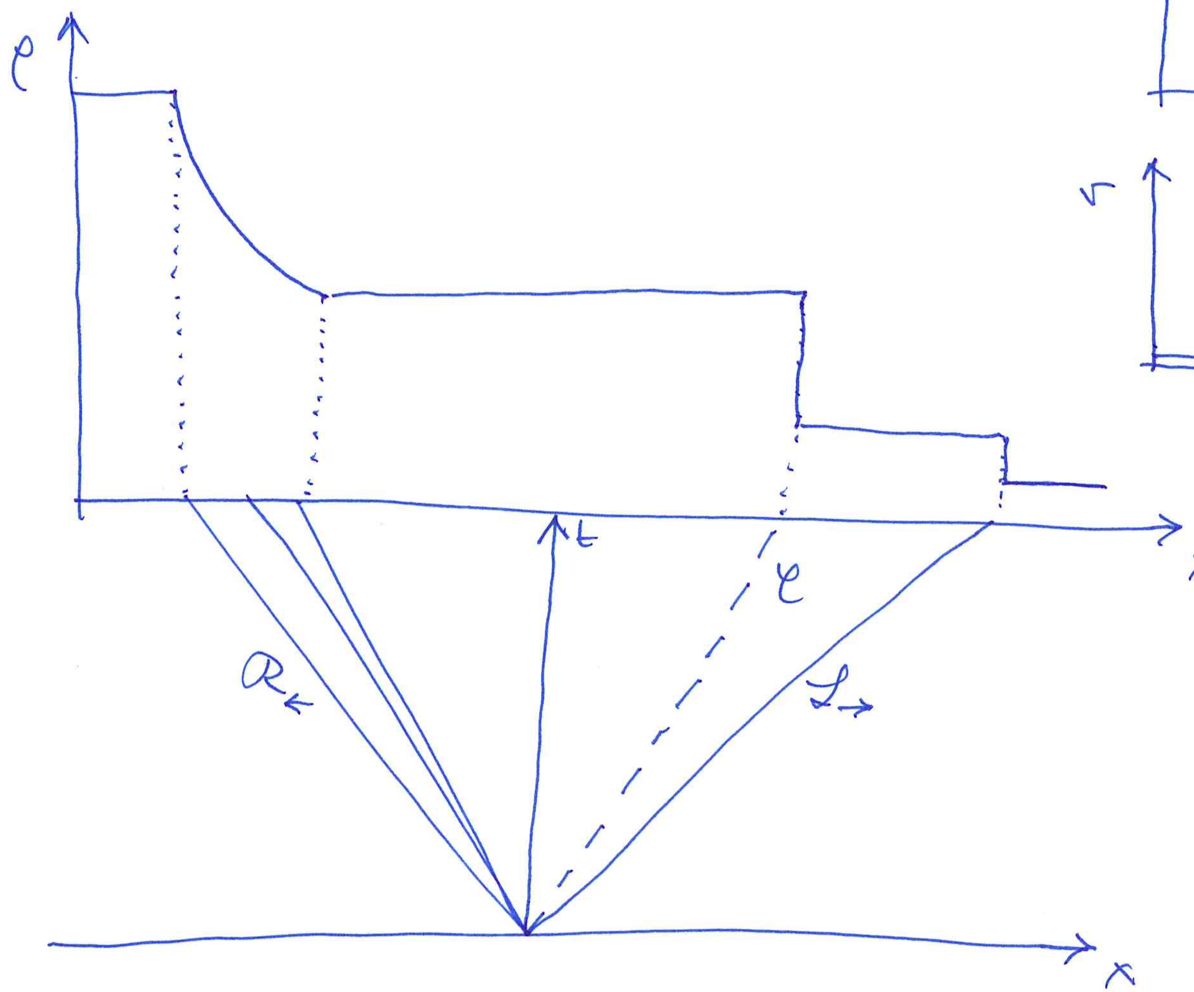
$$L \ \mathcal{R}_{\leftarrow} \ L_* \ \mathcal{C} \ R_* \ \mathcal{R}_{\rightarrow} \ R$$

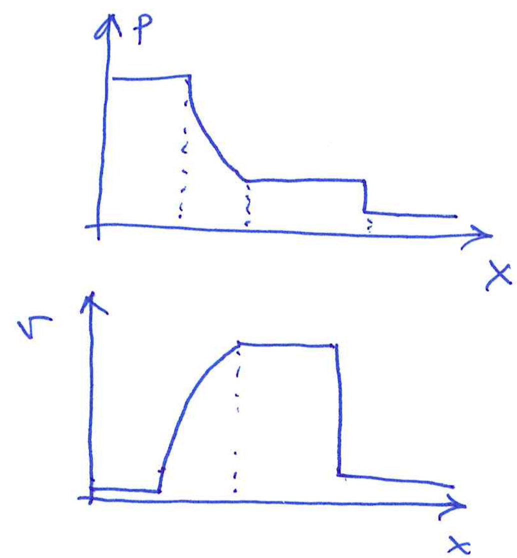


(iii) one shock and one rarefaction wave: moving to the right/left

$$L \;\; \mathcal{R}_{\leftarrow} \;\; L_* \;\; \mathcal{C} \;\; R_* \;\; \mathcal{S}_{\rightarrow} \;\; R$$

$$L \;\; \mathcal{S}_{\leftarrow} \;\; L_* \;\; \mathcal{C} \;\; R_* \;\; \mathcal{R}_{\rightarrow} \;\; R$$





Solution of the Newtonian Riemann problem (RP)

- no general solution of the RP is known in closed analytic form, not even in Newtonian hydrodynamics. Only exception is for an ultrarelativistic fluid with EOS $p = we$ $\quad 0 \le w \le 1$

- in practice numerical solution can be obtained to arbitrary accuracy and for this reason one talks of "exact" solution of the RP even if the solution is numerical (ie with a given truncation error).

- there are at least two different approaches to the solution of the RP and I will discuss the first proposed by Rezzolla and Zanotti (2001)

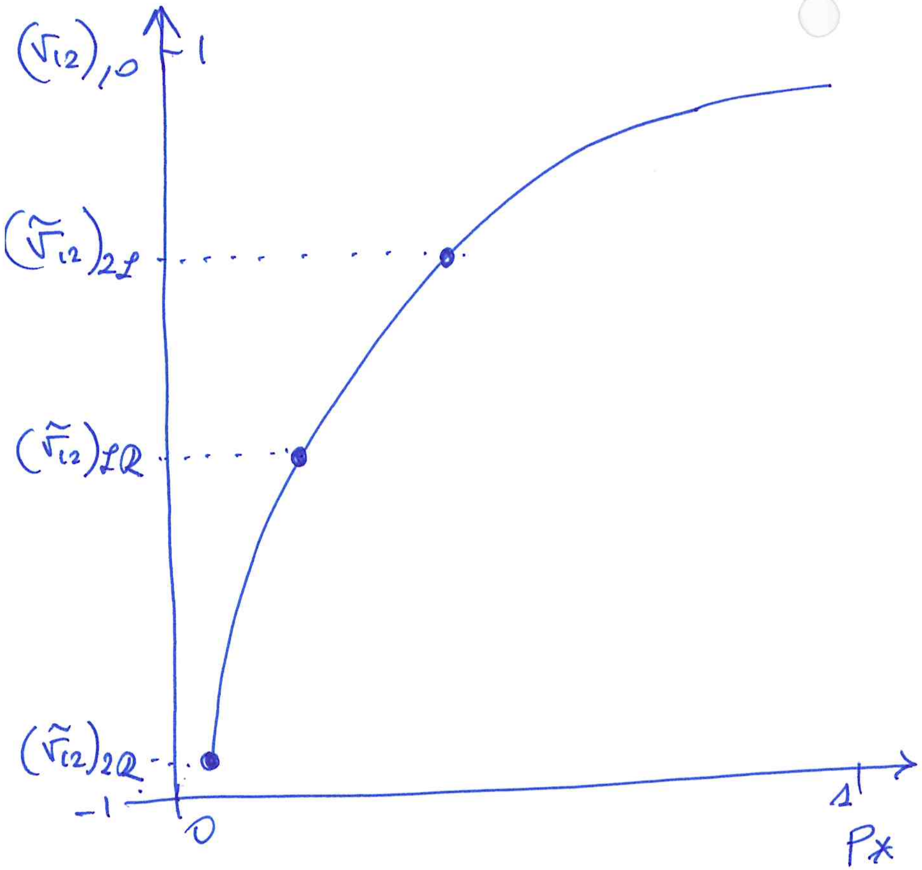
1) Using the initial states, ie $(v, p, e)_{1,2}$ ⊙ determine the wave pattern that will develop. This can be done after comparing the relative velocity of the initial states with the limiting values for the three different wave patterns.

$$(v_{12})_0 \equiv \frac{v_1 - v_2}{1 - v_1 v_2}$$: relativistic invariant

Compare $(v_{12})_0$ with $(\tilde{v}_{12})_{2S}$, $(\tilde{v}_{12})_{SR}$, $(\tilde{v}_{12})_{2R}$, which is possible because $(v_{12})_0$ is a monotone function of p_* and the three branches smoothly join at the limiting values.

⊙ Note that I'm using the indices 1 and 2 to refer to the states on the L and R.

(178)



In other words: if

- $(\bar v_{12})_0 > (\tilde v_{12})_{2\mathcal{L}}$: $2\mathcal{L}$ wave pattern
- $(\tilde v_{12})_{\mathcal{LR}} < (\bar v_{12})_0 \le (\tilde v_{12})_{2\mathcal{L}}$: $\mathcal{LR}$ wave pattern
- $(\tilde v_{12})_{2\mathcal{R}} < (\bar v_{12})_0 \le (\tilde v_{12})_{\mathcal{LR}}$: $2\mathcal{R}$ wave pattern

The limiting values $(\tilde v_{12})_{2\mathcal{L}}$, $(\tilde v_{12})_{\mathcal{LR}}$ and $(\tilde v_{12})_{2\mathcal{R}}$ are complex functions of the states in the RP, but they are analytic! For explicit expression see book.

□

2) Compute the pressure in the * (3-3') region as the root of the nonlinear equation

$$\underbrace{v_{12}(p*)} - (v_{12})_0 = 0$$

different functional form for the three different wave patterns.

$v_{12}(p*)$ is an invariant and can be calculated in any frame, but there is a frame, the one comoving with the contact discontinuity, which is particularly useful, because in this frame there is no jump in the velocities! $v_{3,e} = 0 = v_{3',e}$

As a result, $(v_{13})_{,e} = \dfrac{v_{1,e} - v_{3,e}}{1 - v_{1,e}\, v_{3,e}} = v_{1,e} = v_{13} = \dfrac{v_1 - v_3}{1 - v_1 v_3}$

invariant

Similarly

$$(v_{23'})_{,\ell} = v_{2,\ell} = v_{23'} = \frac{v_2 - v_{3'}}{1 - v_2 v_{3'}}$$

so that

$$v_{12}(p_*) = (v_{12})_{,\ell} = \frac{v_{1,\ell} - v_{2,\ell}}{1 - v_{1,\ell} v_{2,\ell}}$$

where $v_{1,\ell}$ and $v_{2,\ell}$ involve only quantities in states 1,2 and then in $p_3 = p_{3'} = p_*$

$\square$

3) Once $p_3 = p_{3*} = p_*$ is computed, one can derive all quantities in the state. For example, if the wave is a shock, the Taub adiabat allows to compute h_b ⊙, from which one can compute the density ⊙⊙

(181) ⊙ $[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right)[\![p]\!]$ ⊙⊙ $\rho = \frac{\gamma p}{(\gamma-1)(h-1)}$ for ideal fluid EOS

$\boxed{\text{Recap}}$

- Riemann problem deals with evolution of a discontinuous initial problem

$$\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x<0 \\ U_R & \text{if } x>0 \end{cases}$$

$\underline{U}_L = \text{const.}$

$\underline{U}_R = \text{const.}$

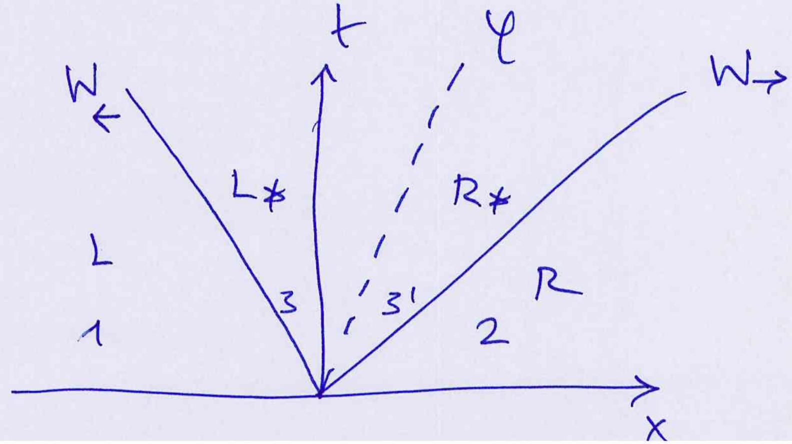
$\underline{U}_L \neq \underline{U}_R$

ie $\quad L \; R \qquad t=0$

$L \; W_{\leftarrow} \; L_* \; \mathcal{C} \; R_* \; W_{\rightarrow} \; R \qquad t>0$

left prop. wave

contact discontinuity



- regions $L*$ and $R*$ are seperated by a contact discontinuity, hence with

$$p_{L*} = p_{R*} = p_*$$
$$v_{L*} = v_{R*} = v_*$$
$$\rho_{L*} \neq \rho_{R*}$$

- $W_\leftarrow, W_\rightarrow$ can be any nonlinear wave

- Riemann solution to the problem: four different solutions corresponding to three different wave patterns.

$L\, \mathcal{S}_\leftarrow\, L_*\, \mathcal{C}\, R_*\, \mathcal{S}_\rightarrow\, R$: 2 shocks

$L\, \mathcal{R}_\leftarrow\, L_*\, \mathcal{C}\, R_*\, \mathcal{R}_\rightarrow\, R$: 2 rarefaction waves

$$\left.\begin{array}{l} L\, \mathcal{R}_\leftarrow\, L_*\, \mathcal{C}\, R_*\, \mathcal{S}_\rightarrow\, R \\ L\, \mathcal{S}_\leftarrow\, L_*\, \mathcal{C}\, R_*\, \mathcal{R}_\rightarrow\, R \end{array}\right\} : \text{1 shock - 1 rarefaction wave}$$

- no general analytic solution of the RP is known, but solution can be obtained numerically to arbitrary precision, hence the name of "exact" RP.

- simplest solution of RP in relativistic hydrodynamics consists in comparing the initial relative velocity with limiting values for the different wave patterns

$$(v_{12})_0 = \frac{v_1 - v_2}{1 - v_1 v_2}$$

$(v_{12})_0 > (\tilde{v}_{12})_{2\mathcal{S}}$: 2 shocks

$(\tilde{v}_{12})_{\mathcal{S}\mathcal{R}} < (v_{12})_0 \leq (\tilde{v}_{12})_{2\mathcal{S}}$: 1 shock, 1 RW

$(\tilde{v}_{12})_{2\mathcal{R}} < (v_{12})_0 \leq (\tilde{v}_{12})_{\mathcal{S}\mathcal{R}}$: 2 RWs

- once the wave pattern is determined, compute the pressure p_* via the nonlinear algebraic equation

$$\underbrace{v_{12}(p_*)}_{} - (v_{12})_0 = 0$$

evaluated across contact discontinuity where expressions for relative velocities are simpler

$$v_{12}(p_*) = (v_{12})_e = \frac{v_{1,e} - v_{2,e}}{1 - \underbrace{v_{1,e}\, v_{2,e}}_{}}$$

involve only quantities in state 1,2 and whose expressions change for the three wave patterns.

- once p_* is known, compute all other quantities.

The importance of the RP stems from the fact that the solution of such a problem is at the hearth of advanced numerical scheme, named high-resolution shock-capturing schemes (HRSC). We will see these methods in the future lectures, but before then we need to discuss two important concepts: 1) weak formulation
2) conservative formulation.

Let's go back to the fact that the eqs. of relativistic hydrodynamics can be written in the generic first-order form

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = \underline{S} \qquad (*)$$

which is said to be hyperbolic if the matrix A is diagonalizable with a set of real eigenvalues and linearly independent right eigenvector $\underline{R}^{(i)}$

If the matrix A is the Jacobian of a flux vector $F(\underline{u})$, ie if

$$A = \frac{\partial F(\underline{u})}{\partial \underline{u}}$$

then the <u>homogeneous</u> system (*) can be written as

$$\partial_t \underline{u} + \nabla F(\underline{u}) = 0 \qquad (\square)$$

which represents the <u>conservative formulation</u> of (*) with $\underline{u}$ the vector of <u>conserved variables.</u>

The importance of a conservative formulation is contained in two important theorems, which however require the definition of "weak solution" and of "weak formulation". The latter is a way of handling solutions that are discontinuous and that would otherwise be impossible to handle mathematically (functions are no longer differentiable at the discontinuity). The basic idea is that of resorting to an integral formulation that would allow also for non-smooth solutions.

Let's consider the conservative formulation (□) in 1D

$$\partial_t \underline{U} + \partial_x F(\underline{U}) = 0 \qquad (\Delta)$$

and let's multiply it by a smooth function of compact support $\phi(x,t)$ with $x \in (-\infty, \infty)$, $t \in [0, \infty)$ and such that

$$\phi(x, t=\infty) = 0 = \phi(x=-\infty, t) = \phi(x=\infty, t)$$

A double integration then yields

$$\int_0^\infty \int_{-\infty}^\infty \left(\phi \, \partial_t \underline{u} + \phi \, \partial_x F(\underline{u}) \right) dx\, dt = 0$$

Integrating by parts

$$\int_{-\infty}^\infty \phi \, \underline{u} \, dx \Big|_{t=0}^{t=\infty} + \int_0^\infty \phi F(\underline{u}) \, dt \Big|_{x=-\infty}^{x=\infty} = \int_0^\infty \int_{-\infty}^\infty \left(\underline{u} \, \partial_t \phi + F \, \partial_x \phi \right) dx\, dt$$

$$= 0$$

The second term is zero because of the compact support of ϕ, while the first term reduces to $\int_{-\infty}^{\infty} \phi \underline{U} \, dx \Big|^{t=0}$

As a result eq. (Δ) is equivalent to

$$\int_0^{\infty} \int_{-\infty}^{\infty} \left(\underline{U} \, \partial_t \phi + \underline{F} \, \partial_x \phi \right) dx \, dt = - \int_{-\infty}^{\infty} \phi(x,0) \, \underline{U}(x,0) \, dx \qquad (\nabla)$$

$\underline{U}$ is said to be a "weak solution" of (Δ) if it is solution of (∇). Note that $\underline{U}$ can be discontinuous since all derivatives are moved to ϕ, which is smooth by definition.

(∇) is then said to be the "weak formulation" of (Δ).

With these definitions in hand we can now discuss two theorems.

- <u>Theorem I</u> (Lax-Wendroff, 1960)

  "Conservative numerical schemes[⊙], if convergent, do converge to the week solution of the problem"

- <u>Theorem II</u> (Hou - Le Floch, 1994)

  "Numerical schemes not written in a conservative form, do not converge to the correct solution if a shock ~~were~~ is present in the flow."

[⊙] A numerical scheme based on the conservative formulation of the equations.

In other words:
- if CF is used, then we are guaranteed to converge to the correct solution;
- if CF is not used, then we are guaranteed to converge to the incorrect solution.

Example: take Burgers equation with discontinuous initial data

$$\partial_t u + u\, \partial_x u = 0$$

Any numerical solution will lead to incorrect propagation speed. However if Burgers eq. is rewritten in conservative form, ie

$$\partial_t u + \frac{1}{2}\, \partial_x (u^2) = 0$$

then the correct solution is obtained with any scheme.

A <u>conservative formulation</u> can be derived also for the equations of relativistic hydrodynamics. To see how this is possible, let's start from the continuity eq. (conservation of rest mass)

$$0 = \nabla_\mu (\rho u^\mu) = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, \rho u^\mu \right)$$

$$= \frac{1}{\sqrt{-g}} \left[\partial_t \left(\sqrt{-g}\, \rho u^t \right) + \partial_i \left(\sqrt{-g}\, \rho u^i \right) \right]$$

$\sqrt{-g} = \alpha \sqrt{\gamma}$

$$= \frac{1}{\sqrt{-g}} \left[\partial_t \left(\sqrt{\gamma}\, \alpha \rho u^t \right) + \partial_i \left(\sqrt{\gamma}\, \alpha \rho u^i \right) \right]$$

$W = \alpha u^t$

$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i \right)$

$$= \frac{1}{\sqrt{-g}} \left[\partial_t \left(\sqrt{\gamma}\, \rho W \right) + \partial_i \left(\sqrt{\gamma}\, \alpha \rho \left(W v^i - u^t \beta^i \right) \right) \right]$$

Define $D := \rho W = \rho \alpha u^t$

$$\alpha\rho(W v^i - u^t \beta^i) = \alpha D v^i - \alpha D \beta^i = D(\alpha v^i - \beta^i)$$

$\Longrightarrow$

$$\boxed{0 = \partial_t(\sqrt{\gamma} D) + \partial_i[\sqrt{\gamma} D(\alpha v^i - \beta^i)]} \qquad (1)$$

D is a conserved quantity.

We can proceed in a similar way also for the other equations.

Recall:

$E := n_\mu n_\nu T^{\mu\nu}$: Eulerian energy density

$S^\mu := -\gamma^\mu{}_\alpha n_\beta T^{\alpha\beta}$: momentum density

$S^{\mu\nu} := \gamma^\mu{}_\alpha \gamma^\nu{}_\beta T^{\alpha\beta}$: spatial part of $T^{\mu\nu}$

$$T^{\mu\nu} = (e+p)\, u^\mu u^\nu + p\, g^{\mu\nu} = \rho h u^\mu u^\nu + p\, g^{\mu\nu}$$

$$= E n^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu} \qquad (*)$$

$u^\mu = W(n^\mu + v^\mu)$

$g^{\mu\nu} = \gamma^{\mu\nu} - n^\mu n^\nu$

$$= \rho h W^2 (n^\mu + v^\mu)(n^\nu + v^\nu) + p(\gamma^{\mu\nu} - n^\mu n^\nu)$$

$$= \rho h W^2 \left[n^\mu n^\nu + n^\mu v^\nu + v^\mu n^\nu + v^\mu v^\nu \right] + p(\gamma^{\mu\nu} - n^\mu n^\nu)$$

$\Longrightarrow$

$$S^{\mu\nu} = \rho h W^2 v^\mu v^\nu + p\, \gamma^{\mu\nu}$$

$$S^\mu = \rho h W^2 v^\mu$$

$$E = \rho h W^2 - p$$

Next, we write the conservation of energy and momentum as

$$0 = \nabla_\mu T^{\mu\nu} = g^{\nu\lambda}\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^\mu{}_\lambda\right) - \frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}\right] \Rightarrow$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^\mu{}_\nu\right) = \frac{1}{2}T^{\mu\lambda}\partial_\nu g_{\mu\lambda} \qquad (**)$$

We can now use (*) in (**) and restrict to a spatial index: $\nu \to j$ to obtain (Exercise)

$$(2) \quad \partial_t\left(\sqrt{\gamma}\,S_j\right) + \partial_i\left(\sqrt{\gamma}\left(\alpha S^i{}_j - \beta^i S_j\right)\right) = \frac{1}{2}T^{\mu\nu}\partial_j g_{\mu\nu}$$

$$= \sqrt{-g}\left(\frac{1}{2}S^{ik}\partial_j \gamma_{ik} + \frac{1}{\alpha}S_i\partial_j\beta^i - E\,\partial_j \ln\alpha\right)$$

Similarly, $n_\nu \nabla_\mu T^{\mu\nu} = 0 = \nabla_\mu\left(n_\nu T^{\mu\nu}\right) - T^{\mu\nu}\nabla_\mu n_\nu$

from which we obtain (Exercise)

$$(3) \qquad \partial_t\left(\sqrt{\gamma}E\right) + \partial_i\left(\sqrt{\gamma}\left(\alpha S^i - \beta^i E\right)\right) = -\sqrt{-g}\,T^{\mu\nu}\nabla_\mu n_\nu$$

$$= \sqrt{-g}\left(K_{ij}S^{ij} - S^i\partial_i \ln\alpha\right)$$

Eqs (1)-(3) can be written in matrix form as

$$(4) \qquad \boxed{\partial_t\left(\sqrt{\gamma}\,\underline{U}\right) + \partial_i\left(\sqrt{\gamma}\,\underline{F}^i\right) = \underline{S}}$$

: flux balanced form

where

$$\underline{U} = \underbrace{\begin{pmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - p \end{pmatrix}}_{\text{CONSERVED VARIABLES}} \; ; \quad \underline{F}^i = \begin{pmatrix} \alpha v^i D - \beta^i D \\ \alpha S^i_j - \beta^i S_j \\ \alpha S^i - \beta^i E \end{pmatrix}$$

$$\underline{U} = \left(D, S_j, E\right)^T$$

$$\underline{S} = \begin{pmatrix} 0 \\ \frac{1}{2}\alpha S^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - E\partial_j \alpha \\ \alpha S^{ij} K_{ij} - S^j \partial_j \alpha \end{pmatrix}$$

(4) is called the "Valencia" formulation of the relativistic hydrodynamics equations. Valid in any 3+1 spacetime.

<u>Notes</u>

- E, D conserved variables $\Longrightarrow$

  $\tau := E - D = \rho W(hW-1) - p$: also conserved variable

- ρ, v^i, ϵ: PRIMITIVE VARIABLES

$$
\begin{tikzcd}
\text{Primitive} \arrow[r, bend left, "\text{analytic}"] & \text{Conserved} \arrow[l, bend left, "\substack{\text{numerical} \\ \text{(root finding)}}"]
\end{tikzcd}
$$

The conversion of the conserved to primitive variables is a rather expensive operation that needs to be performed at each grid point via a nonlinear root finding algorithm. Optimizing this process is important and delicate.

Writing the hydrodynamic eqs in the conservative form (4) is important in those numerical methods that exploit the characteristic structure of the equations.

In particular, it is straightforward to derive the

Jacobians $\quad A^{(i)} = \dfrac{\partial(\sqrt{\gamma}\,\underline{F}^{i})}{\partial(\sqrt{\gamma}\,\underline{U})} = \dfrac{\partial \underline{F}^{i}}{\partial \underline{U}}$

$A^{(i)}$ are 3 different $\underbrace{5 \times 5}_{\text{5 conserved quantities}}$ matrix; as an example

$i = 1$, then the eigenvalues of $A^{(x)}$ are

$\lambda_0 = \alpha v^x - \beta^x$ (triple eigenvalue; advective or entropy or matter wave)

$$\lambda_\pm = \frac{\alpha}{1 - v^2 c_s^2}\left\{ v^x (1 - c_s^2) \pm c_s \sqrt{(1 - v^2)\left[\gamma^{xx}(1 - v^2 c_s^2) - v^x v^x (1 - c_s^2)\right]}\right\} - \beta^x$$

(acoustic waves).

In the limit of flat spacetime they reduce to

$\lambda_0 = v^x$

$\lambda_0 = \dfrac{v^x \pm c_s}{1 \pm v^x c_s} \sim v^x \pm c_s$ qed. $\quad$ Newtonian

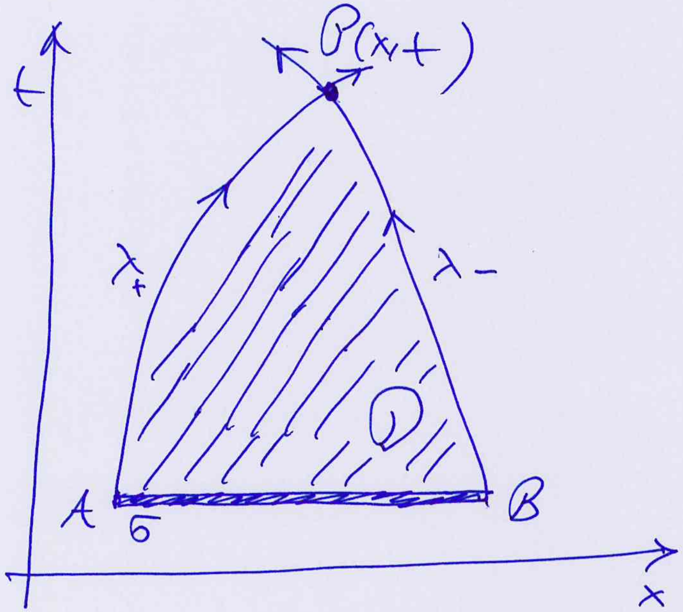
See book for more info on

Before entering the details of the numerical solution of the relativistic hydrodynamic equations it is useful to make a few more definitions.

Given a non linear set of hyperbolic equations and an event $\mathcal{P}(x,t)$ in the spacetime (x,t), it is possible to define the domain of dependence of $\mathcal{P}(x,t)$ as the portion $\mathcal{G}$ of the $t=0$ hypersurface Σ delimited by the event $\mathcal{A}$ and $\mathcal{B}$. On this domain (line in 1D) it is possible to specify the Cauchy data for $\mathcal{P}(x,t)$, that is, the set of possible initial conditions from which $\mathcal{P}(x,t)$ depends.

Note : Cauchy data is the actual initial data while the domain of dependence is a geometrical object

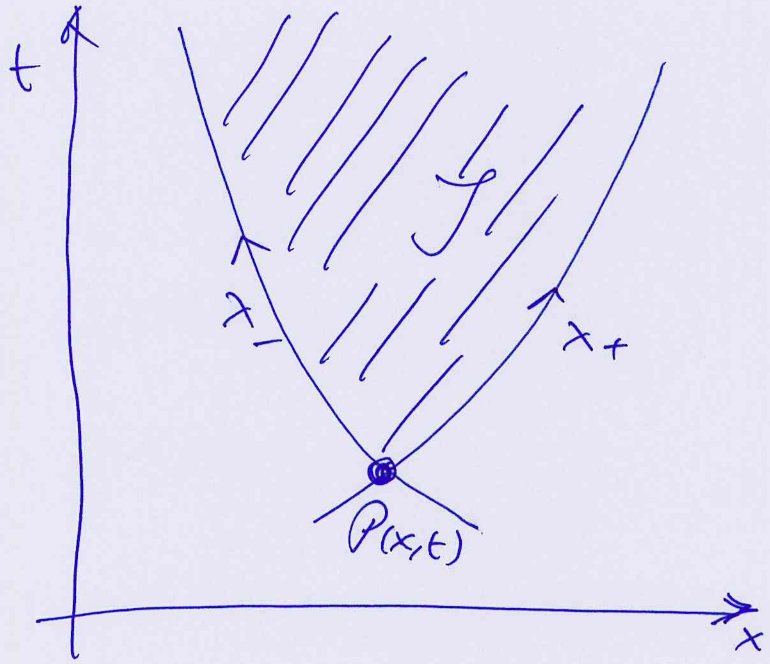
Associated with the domain of dependence there is the domain of determinacy (or future domain of dependence). This is the region of spacetime D limited by the domain σ and the forward/backward characteristics λ_+/λ_- issued from A/B, respectively



The solution at any point in D is fully determined by the Cauchy data assigned on σ.

The domain of determinacy brings a similar definition, i.e., that of the region of influence.

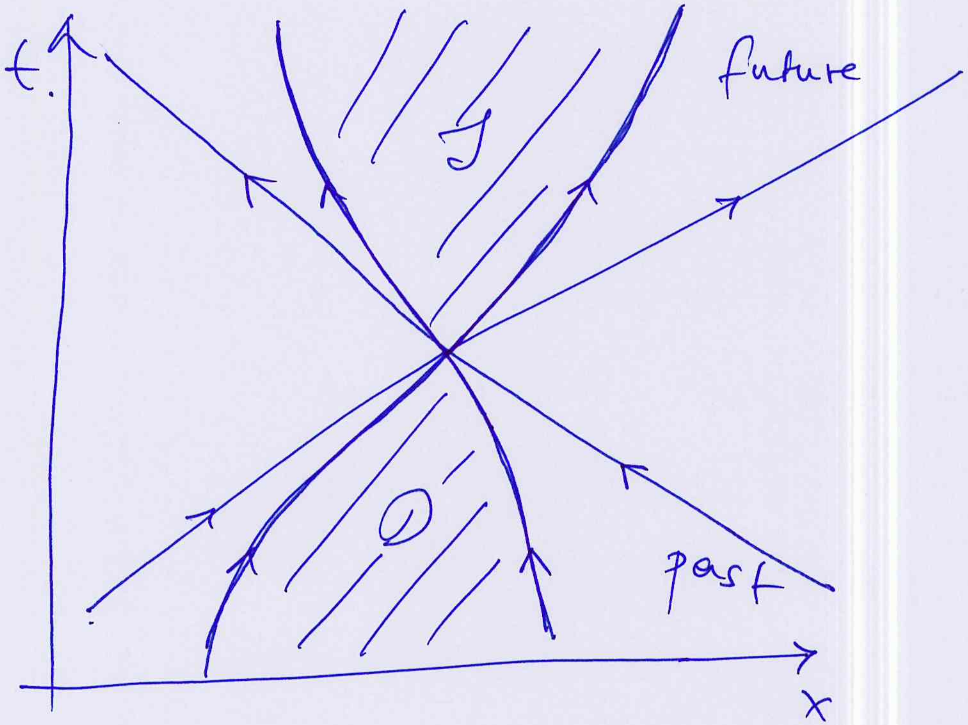
Given a spacetime event $P(x,t)$, the region of influence is defined as the region of spacetime $\mathcal{I}$ included between the $\lambda_\pm$ characteristics emanating from $P(x,t)$



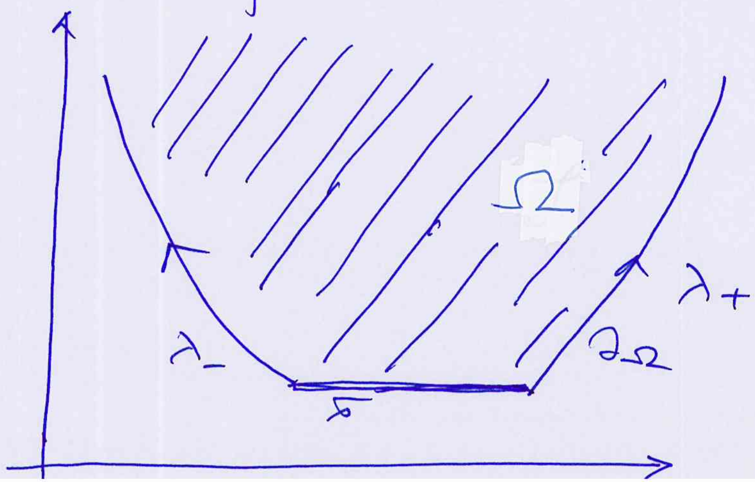
Clearly, any point in $\mathcal{I}$ will be influenced by the solution in $P(x,t)$.

It will be quite natural to recognize in the domain of determinacy and in the region of influence the well-known concepts of past and future light cones in special and general relativity.

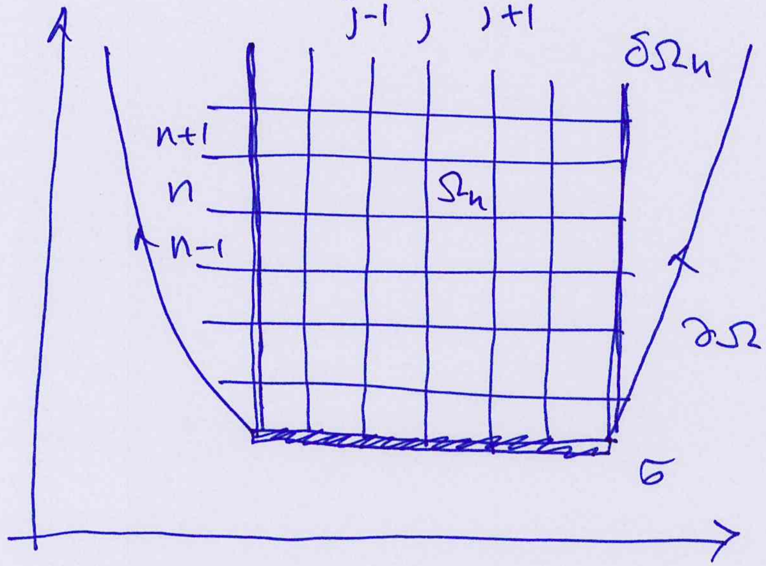
The main differences here are that $\lambda_\pm$ are not necessarily straight lines. Furthermore, if the hyperbolic eqs. are represented by the hydrodynamic eqs, the domains $\mathcal{D}$ and $\mathcal{I}$ are always contained in the light cones at $P(x,t)$.



The region of influence can also be considered when relative to a whole domain of dependence.



$\mathcal{S}$ is the region of influence for which a numerical solution needs to be found. The easiest and inevitable approach in the calculation of this solution requires the discretization of $\mathcal{S}$



ie <u>discretization</u> of <u>spacetime</u>

$$t \longrightarrow t^n = t^0 + n\,\Delta t \qquad n = 0, 1, \dots N$$

$$x \longrightarrow x_j = x_0 + j\,\Delta x \qquad j = 0, 1, \dots J$$

Note that in principle

$$x^n_j \longrightarrow x_j$$

and the set of $\{x^n_j\}$: gridpoints.

Consider now a generic initial-value problem (IVP)

$$\mathcal{L}(u) - F = 0$$

$u = u(x,t)$: smooth function

$\mathcal{L}$: differential operator acting on u

$F = F(u)$ function of u but not of its derivatives.

Example

$\partial_t u + v\,\partial_x u = 0$ $\qquad \mathcal{L}(u) : \partial_t + v\,\partial_x \;;\; F = 0.$

We can then proceed with a variable discretization :

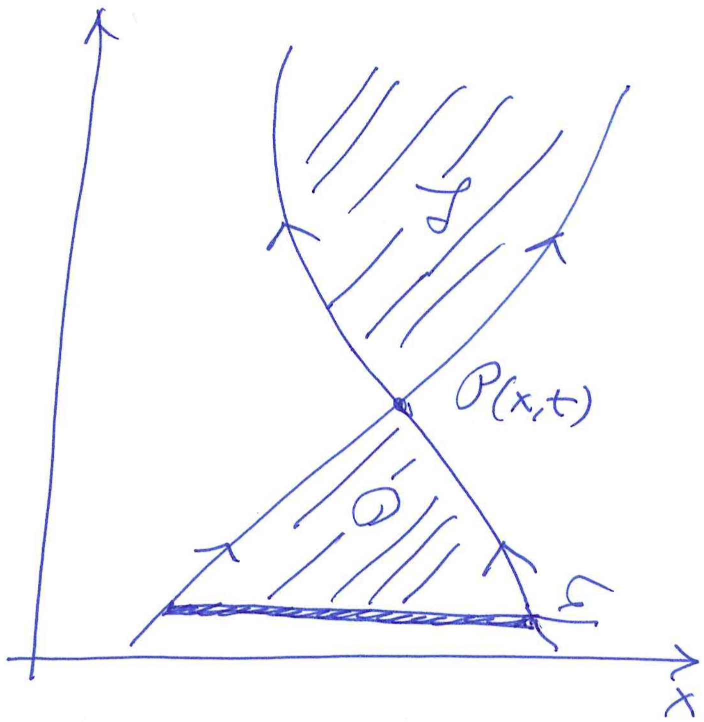
$$u(x,t) \longrightarrow \{u^n_j\} \approx u(\{x^n_j\}) = u(x = x_j, t = t^n) = \{U^n_j\}$$

approximate value of u

exact values of $u(x,t)$ at $\{x^n_j\}$

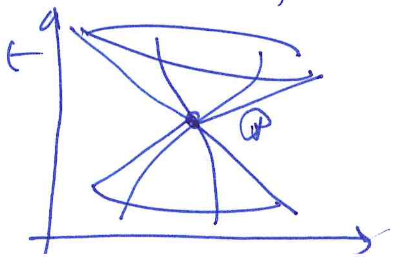
Recap

- Given an event in spacetime $P(x,t)$ it is possible to define
  - domain of dependence : σ
  - domain of determinacy : D
  - region of influence : I



- Cauchy data (ID) is assigned on σ

- I and D are contained in the local future and past light cones at P



- all concepts about local errors can be made "global" by using norms.

- Consistency condition $\lim_{h \to 0} \epsilon^{(n)} = 0$

  truncation error should decrease

- Convergence condition $\lim_{h \to 0} E^{(n)} = Ch^p = 0$

  measured error should decrease at the rate expected for the truncation error.

  In other words $\lim_{h \to 0} \hat{p} = p$

- Numerical solution of a problem implies its discretization.
  - discretization of spacetime : $(x,t) \longrightarrow \{x_j^n\}$
  - discretization of variables : $u(x,t) \longrightarrow U(x_j^n) \approx u_j^n$
  - discretization of operators : $\mathcal{L}(u) - F = 0 \longrightarrow L_h(u_j^n) - F_h = 0$
- Several types of numerical errors:
  - machine prec. error ϵ_M
  - round-off error $\epsilon_{RO} \sim \sqrt{N_{fp}}\, \epsilon_M$ N_{fp}: no of fp operations
  - truncation error $(\epsilon^{(h)})_j^n \equiv L_h(U_j^n) - F_h \neq 0$

  In general $(\epsilon^{(h)})_j^n = \mathcal{O}(c_1 \Delta t^q + c_2 \Delta x^p)$ and the local order of accuracy is $r = \min(q,p)$ if $c_1 \sim c_2$.
- Measuring the error is essential. This can be done via

The measure of the local error $E_j^{(h)} = U_j - u_j^{(h)}$

$$= C h^{\hat{p}_j} + \mathcal{O}(h^{\hat{p}_j+1})$$

- From local error we can compute the error ratio

$$R_j(h,k) = \frac{E_j^{(h)}}{E_j^{(k)}} = \frac{h^{\hat{p}_j}}{k^{\hat{p}_j}} + \mathcal{O}(h^{\tilde{p}_j})$$

$$\Rightarrow \quad \tilde{p}_j = \frac{\log |R_j(h,k)|}{\log(h/k)}$$

- If U_j not known, we can take another resolution ℓ because

$$u_j^{(h)} - u_j^{(k)} = E_j^{(k)} - E_j^{(h)}$$: difference of num. solutions does not depend on exact solution

then the error ratio is

$$R_j(h,k;\ell) = \frac{u_j^{(h)} - u_j^{(\ell)}}{u_j^{(k)} - u_j^{(\ell)}} = \frac{h^{\tilde{p}_j} - \ell^{\tilde{p}_j}}{k^{\tilde{p}_j} - \ell^{\tilde{p}_j}} \quad \Rightarrow \quad \tilde{p}_j = \frac{\log |R_j(h, h/\gamma, h/\gamma^2) - 1|}{\log(\gamma)}$$

if $k = h/\gamma$; $\ell = h/\gamma^2$

(212c)

- if no exact solution is known, a self-convergence test is always possible
- all of the global concepts illustrated here can be specialized locally

$\square$

When considering the definition of hyperbolic equations we have introduced the concept of <u>well posedness</u> as the property of a given IVP of leading to a bounded solution. This concept can be extended also to a discretized operator. Let L_h be the discretized version of the differential operator $\mathcal{L}(u)$. Applying this operator n times to a set of discretized variables $\{u_j\}$ can be seen as introducing an associated truncation error $\epsilon^{(n)}$ and the requirement is then that this error, if growing with time, is always bounded.

Hence L_h is said to be <u>numerically stable</u> if

$$|||L_h^n|||_1 \leq C_s \qquad \forall \; nh \leq T \quad \text{and} \quad h < h_0$$

Because L_h^n can be seen as the application of L_h n times, clearly the operator is stable if $|||L_h|||_1 \leq 1$. [The proof is trivial $|||L_h^n|||_1 \leq |||L_h|||_1^n \leq 1$]. In practice one allows for a moderate growth and so stability is enforced by requiring

$$|||L_h|||_1 \leq 1 + \gamma h$$

so that

$$|||L_h^n|||_1 \leq (1+\gamma h)^n \leq e^{\gamma h n} \leq e^{\gamma T}$$

ie a stable discretized operator produces an error that
(numerically) at most grows exponentially.

Having introduced the concepts of consistency, convergence and stability, we are ready to introduce an important theorem[a]:

(Lax, 1956): "Given a well-posed linear IVP and a finite-difference approximation that satisfies the consistency condition, stability is a sufficient and necessary condition for convergence".

In other words, for a consistent discretisation, convergence and stability are interchangeable: When measuring convergence, we also measure stability.

This powerful theorem unfortunately applies only to linear problems, although it is often used to deduce stability in nonlinear problems.

Question: how do we know if a given numerical method is actually stable?

[a] This is called Lax's equivalence theorem.

To answer this question we can employ the von Neumann stability analysis. Let's consider our prototypical hyperbolic eq. $\mathcal{L}(u) - F = 0$ in its homogeneous form $\mathcal{L}(u) = 0 = \partial_t u - \mathcal{S}(u)$, where $\mathcal{S}(u)$ is some spatial differential operator (eg advection eq.).

In general, the EXPLICIT discretization of this equation can be written as

$$(\cdots) \qquad u_j^{n+1} = u_j^n + \Delta t \underbrace{S_n(u_j^n, u_{j\pm1}^n, \dots)}_{\text{discretization of spatial operator}}$$

so that the solution can be thought as the superposition of a set of eigenvalues that at each gridpoint is

$$(::) \qquad u_j^n = \xi^n e^{ikx_j} = \xi^n e^{ik(x_0 + j\Delta x)} \quad : \quad \begin{array}{l} k \text{ wavenumber} \\ \xi \in \mathbb{C} \end{array}$$

The application of $(\cdots)$ to $(\because)$ is equivalent to a change in the complex number ξ, which is therefore named: <u>amplification factor</u>

It follows that stability if $\boxed{|\xi|^2 = \xi\xi^* \leq 1}$. It is therefore possible, once a numerical method is defined, to check what's the norm of the associated amplification factor. $\square$

The study of the domain of dependence of the solution at a given point can also be used to derive a universal stability condition for a generic numerical method, that is explicit.

<u>Def.</u> <u>Explicit scheme</u>

$$u_j^{n+1} = f\left(u_j^n, u_{j\pm1}^n, u_{j\pm2}^n, \dots u_j^{n-m}, u_{j\pm1}^{n-m}, \dots\right) + O(\Delta x^p)$$

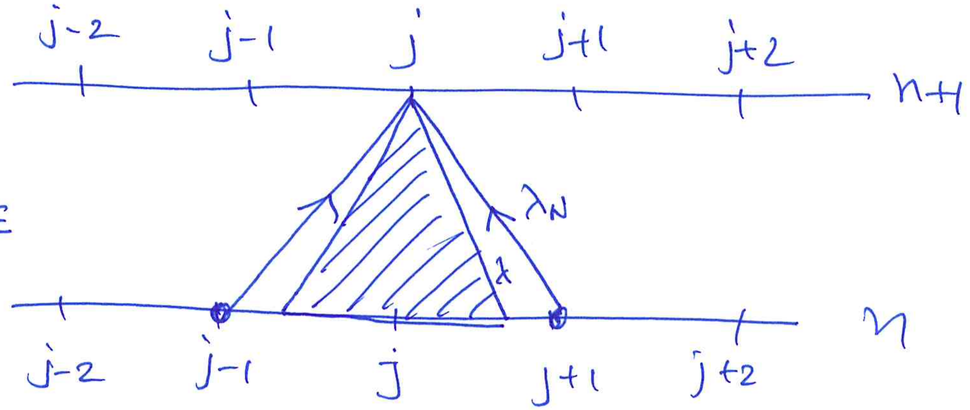
$$m < n$$

Def. <u>Implicit scheme</u>

$$u_j^{n+1} = f\left(u_j^n, u_{j\pm1}^{n+1}, u_{j\pm2}^{n+1}, \dots u_{j\pm1}^{n}, u_{j\pm2}^{n} \dots u_{j\pm1}^{n+m}, u_{j\pm2}^{n+m}, \dots\right) + \mathcal{O}(\Delta x^p)$$

This condition requires that the domain of dependence is always smaller than the numerical domain of dependence

CFL STABLE



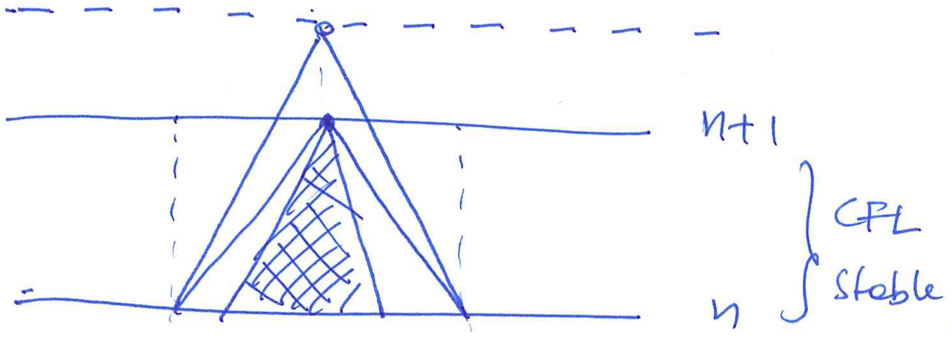
ie $\quad |\lambda| \leq \lambda_N := \dfrac{\Delta x}{\Delta t}$

This condition can be translated into a condition on the timestep

$$\boxed{\Delta t = C_{CFL} \frac{\Delta x}{|\lambda|}}$$

Courant-Friedrichs-Lewy (CFL) condition.

The CFL condition essentially imposes that any physical signal (eg sound, light) propagates at most for a fraction of a numerical cell in a timestep. Neighbouring points cannot influence each other.



Timesteps as large as this can lead to instabilities. Larger steps will inevitably lead to exponentially growing modes.

Notes:

- the CFL is only a necessary condition for stability and a numerical method can be unstable even if CFL stable
- If the system of eqs. contains a number of eigenvalues, the CFL condition is generalised as $\Delta t = \mathrm{CFL} \min_k \left(\frac{\Delta x}{|\lambda_k|} \right)$.

Finite-difference methods.

Here after I will illustrate a class of numerical methods called FD methods that represent a simple and powerful approach to the solution of hyperbolic eqs. For simplicity I will concentrate on the advection equation in 1+1 D

$$\boxed{\partial_t u + \lambda \partial_x u = 0}$$

Let $u(x,t)$ be evaluated at x_j^n and Taylor expanded in time around this point

$$u(x_j, t^n + \Delta t) = u(x_j, t^n) + \partial_t u(x_j, t^n)\Delta t + O(\Delta t^2) \quad \Longleftrightarrow$$

$$u_j^{n+1} = u_j^n + \partial_t u\big|_j^n \Delta t + O(\Delta t^2)$$

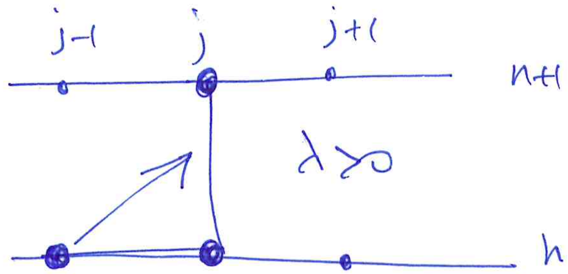
from which

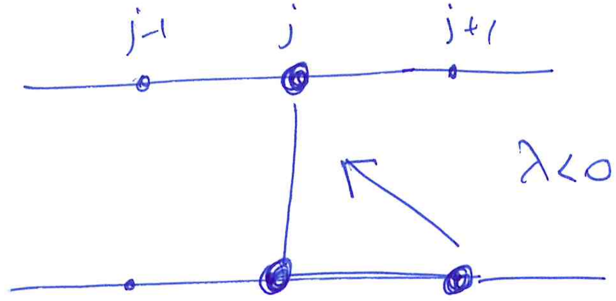
$$\boxed{\partial_t u\big|_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + O(\Delta t)}$$

can do the same thing for a Taylor expansion in space and obtain

$$\partial_x u\Big|_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + O(\Delta x) \; ; \; \partial_x u\Big|_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

the choice of the right stencil is simple in this case since we know that the causal structure of the equation is determined by the sign of the advection velocity





Putting things together

$$u_j^{n+1} = u_j^n - \alpha\left(u_j^n - u_{j-1}^n\right) + O(\Delta t^2, \Delta x \Delta t) \qquad \text{if } \lambda > 0$$

$$u_j^{n+1} = u_j^n - \alpha\left(u_{j+1}^n - u_j^n\right) + \mathcal{O}(\Delta t^2, \Delta x \Delta t) \qquad \text{if } \lambda < 0$$

where $\alpha := \lambda \Delta t / \Delta x$ UPWIND scheme
(one-sided stencil)

This is a first-order explicit scheme, which we will use also later on.

What about stability?

Von Neumann analysis shows

$$\xi = 1 - \alpha\left[1 - \cos(k\Delta x)\right] - i\alpha \sin(k\Delta x)$$

so that

$$|\xi|^2 = \xi\xi^* = 1 - 2\alpha(1-\alpha)\left(1 - \cos(k\Delta x)\right)$$

$|\xi|^2 \leq 1$ if $\alpha \leq 1$: this is indeed the CFL condition.

So the upwind method is stable if CFL stable.

We can improve on the upwind method by using a higher order approximation to the spatial derivative

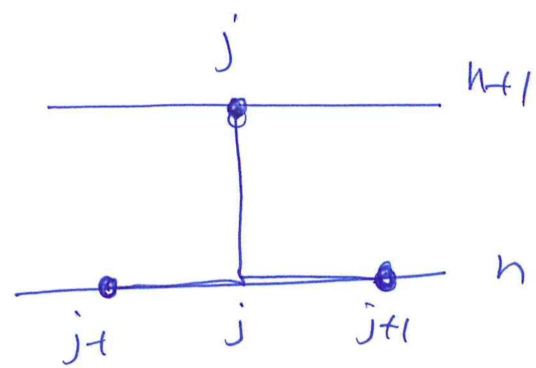
$$u^n_{j+1} = u^n_j + \partial_x u\Big|^n_j \Delta x + \frac{1}{2}\partial^2_x u\Big|^n_j \Delta x^2 + \mathcal{O}(\Delta x^3)$$

$$u^n_{j-1} = u^n_j - \partial_x u\Big|^n_j \Delta x + \frac{1}{2}\partial^2_x u\Big|^n_j \Delta x^2 + \mathcal{O}(\Delta x^3)$$

Adding leads

$$\partial_x u\Big|^n_j = \frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x} + \mathcal{O}(\Delta x^2):$$ 2nd-order spatial derivative

$$\frac{u^{n+1}_j - u^n_j}{\Delta t} = -\lambda\left(\frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x}\right) + \mathcal{O}(\Delta t, \Delta x^2)$$



so that

$$u^{n+1}_j = u^n_j - \frac{\alpha}{2}\left(u^n_{j+1} - u^n_{j-1}\right) + \mathcal{O}(\Delta t^2, \Delta x^2 \Delta t)$$

FTCS

(forward in time, centered in space)

The amplification factor is $\xi = 1 - i\alpha \sin(k\Delta x)$ so that

$$|\xi|^2 = 1 + (\alpha \sin(k\Delta x))^2 > 1 \quad \forall \alpha$$

In other words, FTCS is <u>unconditionally unstable.</u>

This teaches an important lesson: increase of accuracy (1st → 2nd order) does not guarantee an improved solution.

The instability of the FTCS scheme can be cured rather easily by replacing u^n_j with its spatial average, ie

$u^n_j \to \frac{1}{2}(u^n_{j+1} + u^n_{j-1})$, so that the resulting method is

$$u^{n+1}_j = \frac{1}{2}\left(u^n_{j+1} + u^n_{j-1}\right) - \frac{\alpha}{2}\left(u^n_{j+1} - u^n_{j-1}\right) + O(\Delta x^2) \qquad (*)$$

$$\hookrightarrow \Delta t = O(\Delta x)$$

This is called the Lax-Friedrichs scheme and has amplification factor $|\xi|^2 = 1 - \sin^2(k\Delta x)(1-\alpha^2) \leq 1$ for $|\alpha| \leq 1$.

How can such a simple substitution transform an unstable method into a stable one?

The answer is that we are effectively solving a different equation, ie an advection diffusion equation

$$\partial_t u + \lambda \partial_x u = \frac{1}{2} \varepsilon \partial_x^2 u \qquad (**)$$

To verify this it is sufficient to rewrite (*) as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\lambda \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right) + \frac{1}{2} \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta t} \right) \qquad (***)$$

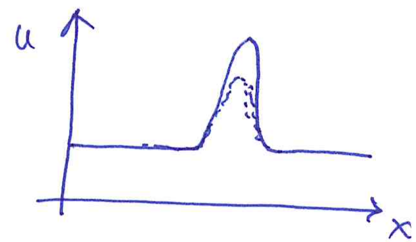
and realize that the second spatial derivative can be written as (Exercise)

$$\partial_x^2 u \Big|_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + O(\Delta x^2)$$

from which we can deduce that $\varepsilon = \frac{\Delta x^2}{\Delta t}$.

Stated differently, (***) is the FD representation of (**) and not of our advection eq. Is this acceptable? Yes! First the diffusive term has coefficient ε such that $\lim_{h \to 0} \varepsilon = 0$, ie in the continuum limit the diffusive term disappears. Second, the diffusive term is effective only on very small wavelengths. This can be deduced from the expression of the amplification factor

$$|\xi|^2 \sim 1 - \sin^2(k \Delta x)$$



Let $\bar{k} = 2\pi/L$ where L is the typical scale of the problem.

Then if $\bar{k}\Delta x \ll 1$ (ie small wavelengths wrt L), then $|\xi|^2 \sim 1$ ie: no diffusion. However, for $k\Delta x \sim \pi/2$, ie $\Delta x \sim \frac{\pi/2 L}{2\pi} \sim L/4$ then $|\xi|^2 \sim 0$: high dissipation.

Let's now increase also the order of the time derivative, ie

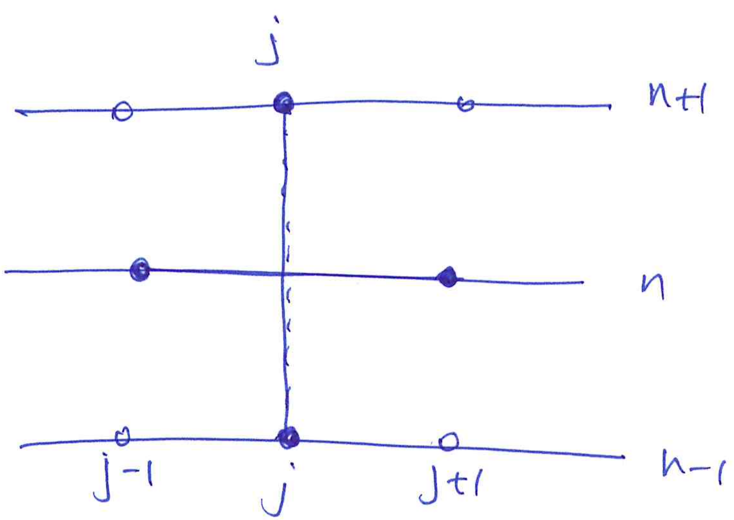
$$\partial_t u\Big|_j^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + \mathcal{O}(\Delta t^2)$$

so that the advection equation becomes

$$u_j^{n+1} = u_j^{n-1} - \alpha\left(u_{j+1}^n - u_{j-1}^n\right) + \mathcal{O}(\Delta x^2)$$

: 2nd-order scheme

<u>leapfrog</u>



This is a 2-level scheme

$$u_j^{n+1} = f(u_{j\pm1}^n, u_j^{n-1})$$

The amplification factor can be computed easily and is

$$\xi = -i\alpha \sin(k\Delta x) \pm \sqrt{1-[\alpha \sin(k\Delta x)]^2}$$

$$|\xi|^2 = \alpha^2 \sin^2(k\Delta x) + 1\{1-[\alpha \sin(k\Delta x)]^2\} = 1 \qquad \forall\, \alpha \leq 1$$

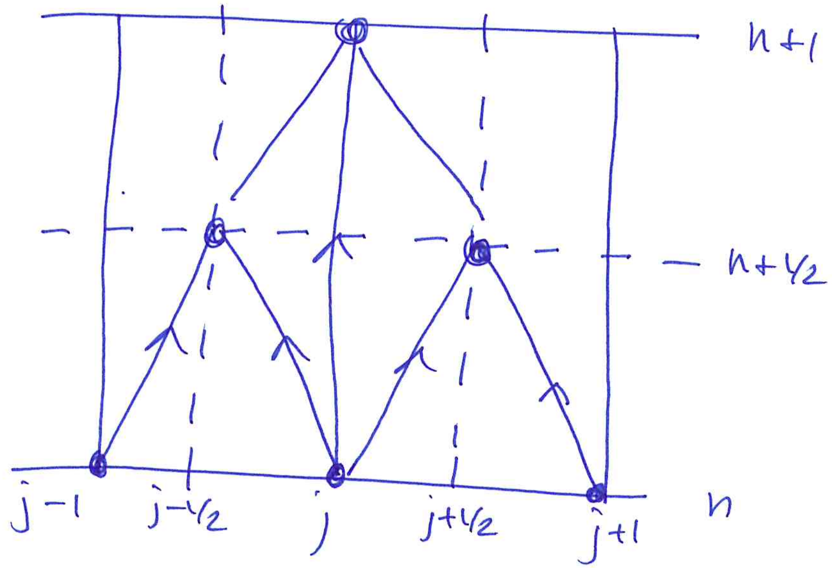
In other words, the leapfrog scheme has nominally zero dissipation.

Another 2nd-order one level scheme can be derived from the combination of the LF and leapfrog method

- take a LF step using a fictitious ½ step and ½ spacing

$$u^{n+1/2}_{j\pm1/2} = \frac{1}{2}\left(u^n_j + u^n_{j\pm1}\right) \mp \alpha\left(u^n_{j\pm1} - u^n_j\right) + O(\Delta x^2)$$

- take a leapfrog step using these fictitious levels, ie



$$u^{n+1}_j = u^n_j - \alpha\left(u^{n+1/2}_{j+1/2} - u^{n+1/2}_{j-1/2}\right) + O(\Delta x^2)$$

Putting things together we obtain

$$u_j^{n+1} = u_j^n - \alpha\left(u_{j+1}^n - u_{j-1}^n\right) + \alpha^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + O(\Delta x^2)$$

2nd-order one-level scheme: Lax-Wendroff.

All the information on the intermediate steps is removed analytically.

Amplification factor $\xi = 1 - \alpha \sin(k\Delta x) - \alpha^2[1 - \cos(k\Delta x)]$

$|\xi|^2 = 1 - \alpha^2(1-\alpha^2)[1 - \cos(k\Delta x)]^2 \sim 1 - \alpha^2 - \alpha^4 < 1$: some dissipation is present but is small.

As for LF also the LW scheme provides a finite-difference approximation to a different eq:

$$\partial_t u + \lambda \partial_x u = \varepsilon \partial_x^2 u + \beta \partial_x^3 u \quad (\square)$$

Exercise

where $\varepsilon = \dfrac{\alpha\lambda\Delta x}{2}$; $\beta = -\dfrac{\lambda \Delta x^2}{6}(1-\alpha^2)$

where, as expected $\lim_{h\to 0} \varepsilon = 0 = \beta$: the continuum limit is recovered.

Let's explore the consequences of ($\square$) by considering a solution consisting of a single Fourier mode moving in the x direction, ie $u(x,t) = e^{i(kx-\omega t)}$ with freq. ω. Then in the continuum limit

$$\partial_t u = -i\omega u \; ; \; \partial_x u = iku \; ; \; \partial_x^2 = -k^2 u \; ; \; \partial_x^3 u = -ik^3 u \, .$$

In the case in which the FD scheme provides an accurate approximation of the advection eq. (ie $\varepsilon = 0 = \beta$) we will have a single numerical mode

$$\tilde{u} = e^{ik(x-\lambda t)}$$

with phase velocity $v_p = \mathrm{Re}\left(\frac{\omega}{k}\right) = \lambda$ which is equal to the group velocity $v_{gr} = \partial_k \omega = \lambda$. However, when the FD scheme provides an approximation to an advection-diffusion equation, then the numerical solution will be

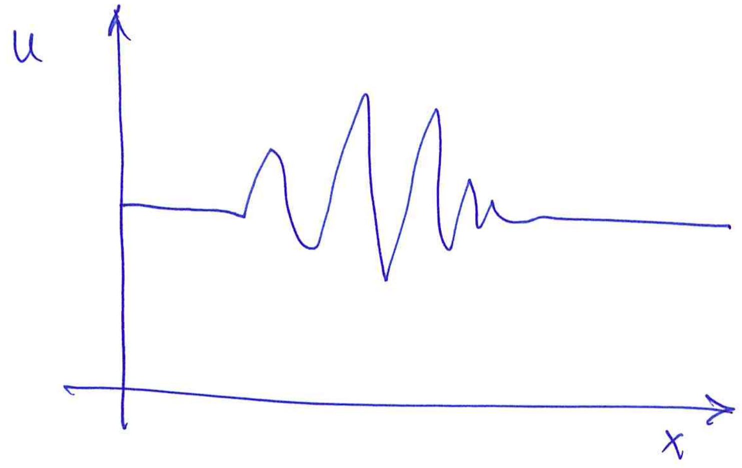
$$\tilde{u}(x,t) = e^{-\varepsilon k^2 t} \, e^{ik[x - \overbrace{(\lambda + \beta k^2)}^{\omega/k} t]}$$

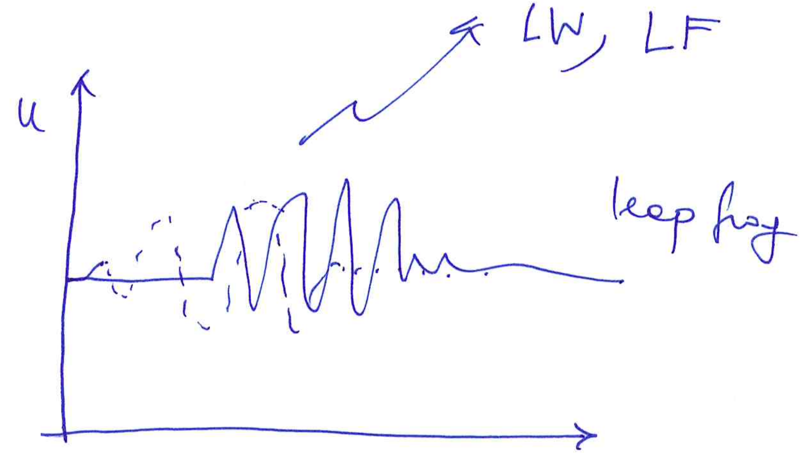
which clearly has a decaying term $\propto \varepsilon$ and a dispersive term $\propto \beta$

Indeed

$$v_{ph} = \lambda + \beta k^2 \quad ; \quad v_{gr} = \lambda + 3\beta k^2 .$$

Exercise





It is often useful when solving nonlinear evolution eqs with potential singularities to slightly modify the eqs. by introducing an <u>artificial-viscosity operator</u>, ie go from

$$\partial_t u = S(u) \longrightarrow \partial_t u = S(u) - \varepsilon (-1)^N \Delta x^{2N-1} \partial_x^{2N} u$$

Note that the spatial derivative is always even and when expressed in discretized form this amounts to

$$u_j^{n+1} = u_j^n + \Delta t \, S_h (u_j^n, u_{j\pm1}^n, \dots) - \varepsilon (-1)^N \Delta t \Delta x^{2N-1} D_h^{2N} (u_j^n, u_{j\pm1}^n, \dots)$$

Note that N cannot be arbitrary and it actually needs to go to zero faster or equal than the truncation error

Suppose that S_h is a fourth-order operator, ie $\epsilon^{(h)} = \mathcal{O}(\Delta x^4)$, then we want

$$\Delta t \Delta x^{2N-1} \leq \Delta t \Delta x^4 \Rightarrow N \geq 3$$

More generically, this condition is that $\boxed{N \geq \frac{p+1}{2}}$.

This type of artificial viscosity is named: Kreiss-Oliger dissipation and is routinely used in the solution of the Einstein equations.

For higher-order schemes, it is possible to use recurrence relation to express derivative of arbitrary order to given accuracy.

In the case of centred differences with uniform spacing Δx, these expressions are

$$\partial_x^N u\Big|_j = \frac{1}{\Delta x^N} \sum_{k=1}^{k=s} c_k \left(u_{k+j} - u_{j-k} \right) + O(\Delta x^p)$$

Ex

$$\partial_x f_j = \frac{1}{\Delta x} \left(\frac{1}{2} f_{j+1} - \frac{1}{2} f_{j-1} \right) + O(\Delta x^2) :$$ 2nd-order accurate 1st derivative

$$\partial_x f_j = \frac{1}{\Delta x} \left(-\frac{1}{12} f_{j+2} + \frac{2}{3} f_{j+1} - \frac{2}{3} f_{j-1} + \frac{1}{12} f_{j-2} \right) + O(\Delta x^4) :$$ 4th-order 1st derivative

Recap

- numerical stability of discretized operator

$$\mathcal{L}(u) - \mathcal{F} = 0 \longrightarrow L_h - F_h = 0$$

$$\| L_h^n \|_1 \leq (1+\gamma h)^n \leq e^{\gamma h n}$$: the solution at most grows exponentially

- Explicit numerical scheme

$$u_j^{n+1} = f(u_j^n, u_{j\pm 1}^n, \dots u_j^{n-m}, u_{j\pm 1}^{n-m} \dots)$$

- Implicit numerical scheme

$$u_j^{n+1} = f(u_j^n, u_{j\pm 1}^n, \dots u_{j\pm j}^{n+1} u_{j\pm 2}^{n+1}, \dots u_j^{n-m}, u_{j\pm 1}^n, \dots)$$

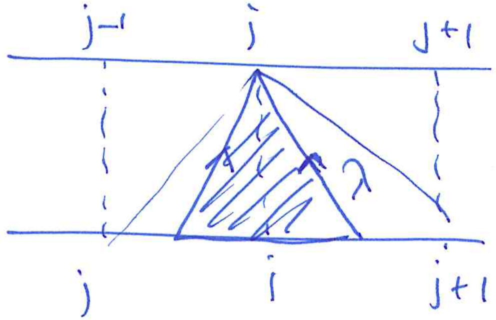
- von Neumann stability analysis

$$u_j^n = \xi^n e^{ikx_j} = \xi^n e^{ik(x_0 + j\Delta x)} \qquad \xi \in \mathbb{C}$$

ξ : amplification factor ; scheme is stable if $|\xi|^2 = \xi \xi^* \leq 1$

- CFL (Courant-Friedrichs-Lewy) condition

$$|\lambda| \leq \lambda_N \iff \Delta t = C_{CFL} \frac{\Delta x}{|\lambda|} \quad :$$



Any perturbation travels at most a fraction of a cell.

- Finite-difference method: use Taylor expansion at each grid point to evaluate approximation to spatial derivatives at arbitrary order

Ex

$$\partial_x u\Big|_j^n = \frac{u^n_{j+1} - u^n_j}{\Delta x} + O(\Delta x);$$

$$\partial_x u\Big|_j^n = \frac{u^n_{j+1} - u^n_{j-1}}{2\Delta x} + O(\Delta x^2);$$

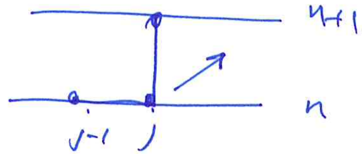
$$\partial^2_x u\Big|_j^n = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{\Delta x^2} + O(\Delta x^2).$$

$\vdots$

In this way it is possible to build scheme of various order with varying properties.

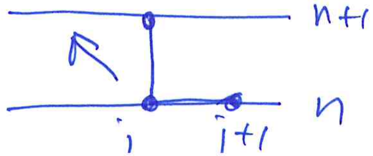
Let $\partial_t u + \partial_x u = 0$ be the representative eq.. Then

$$u^{n+1}_j = u^n_j - \alpha(u^n_j - u^n_{j-1}) \quad \text{if } \lambda > 0$$



; $\alpha = \lambda \Delta t / \Delta x$

$$u^{n+1}_j = u^n_j - \alpha(u^n_{j+1} - u^n_j) \quad \text{if } \lambda < 0$$



First-order explicit upwind scheme

- In a similar way it is possible to construct schemes that have various orders, amplification factors and dissipation terms. These appear in terms of higher-order spatial derivatives and are useful to damp growing instabilities or suppress error on small wavelengths.

Ex

$$\partial_t u + \lambda \partial_x u = \underbrace{\varepsilon \partial^2_x u}_{\text{dissipative term}} + \underbrace{\beta \partial^3_x u}_{\text{dispersive term}}$$

Lax-Wendroff scheme

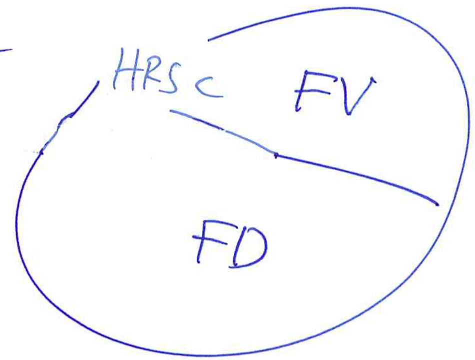
$$\partial_x^2 f = \frac{1}{\Delta x^2}\left(-\frac{1}{12}f_{j+2} + \frac{4}{3}f_{j+1} - \frac{5}{2}f_j + \frac{4}{3}f_{j-1} - \frac{1}{12}f_{j-2}\right) + \mathcal{O}(\Delta x^4)$$: 4th-order 2nd derivative

Tables exist for all the different coefficients c_k and for the different derivatives (see book). How are the c_k's calculated? A similar calculation can be made also for "one-sided" stencils.

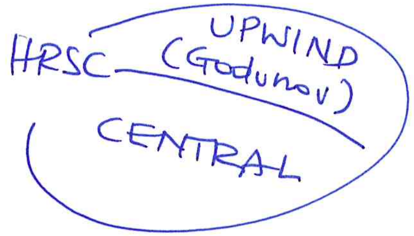
□

HRSC methods: methods that exploit the conservative formulation to accurately reproduce discontinuities in the solution

HRSC methods are then distinguished in finite-volume (FV) or finite difference methods according to whether the discretized operators use "cell-averages" values of the function or "pointwise" values.



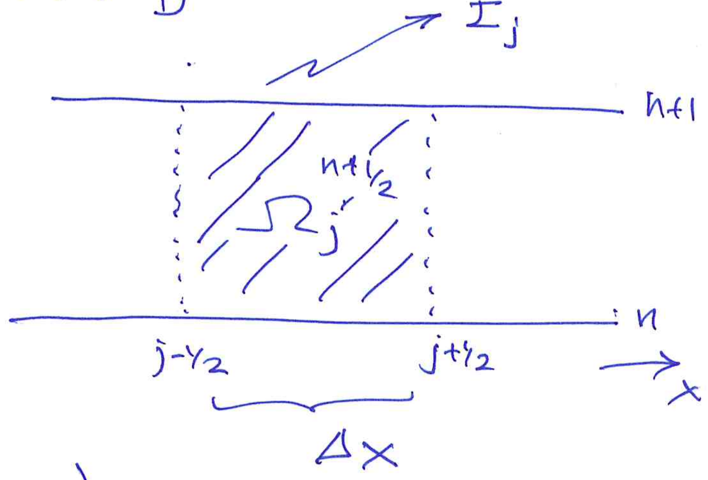
HRSC methods can be further distinguished in upwind methods (eg Godunov methods) if local RPs are introduced or central schemes if this information is not used.



Let's start from a generic CF eq. in 1+1 D

$$\partial_t \underline{U} + \partial_x \underline{F}(\underline{U}) = 0 \qquad (\circ)$$

and introduce the control volume $\Omega_j^{n+1/2}$



Integrate $(\circ)$ in space over I_j

$$\frac{d}{dt}\int_{x_{j-1/2}}^{x_{j+1/2}} \underline{U}(x,t)\,dx = \underline{F}(\underline{U}(x_{j-1/2},t)) - \underline{F}(\underline{U}(x_{j+1/2},t)) \qquad (\circ\circ)$$

Next integrate $(\circ\circ)$ in time between t^n and t^{n+1}

$$(\circ\circ\circ) \qquad \int_{x_{j-1/2}}^{x_{j+1/2}} \underline{U}(x,t^{n+1})\,dx = \int_{x_{j-1/2}}^{x_{j+1/2}} \underline{U}(x,t^n)\,dx + \int_{t^n}^{t^{n+1}} \underline{F}(\underline{U}(x_{j-1/2},t))\,dt - \int_{t^n}^{t^{n+1}} \underline{F}(\underline{U}(x_{j+1/2},t))\,dt$$

Define now the cell (volume) averages as

$$U_j^n := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \underline{U}(x,t^n)\,dx$$

numerical fluxes as

$$\underline{F}_{j\pm 1/2} := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \underline{F}\left(\underline{U}(x_{j\pm 1/2},t)\right)dt$$

then $(\circ\circ\circ) \Leftrightarrow$

$$\boxed{U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x}\left(\underline{F}_{j-1/2} - \underline{F}_{j+1/2}\right)} \qquad (\because\because)$$

Note that $(\because\because)$ is exact (!) as no approximations have been made yet.

Indeed the numerical approximations are all contained in the computation of the cell averages $\bar{U}_j$ and of the numerical fluxes $\underline{F}_{j\pm 1/2}$.

Ex.

Using

$$\underline{F}_{j\pm 1/2} = \frac{1}{2}\left[\underline{F}(\underline{u}_j^n) + \underline{F}(\bar{\underline{U}}_{j\pm 1}^n)\right] \mp \frac{\Delta x}{2\Delta t}\left(\bar{\underline{U}}_{j\pm 1}^n - \bar{\underline{U}}_j^n\right)$$

one recovers the expression for the Lax – Friedrichs scheme when

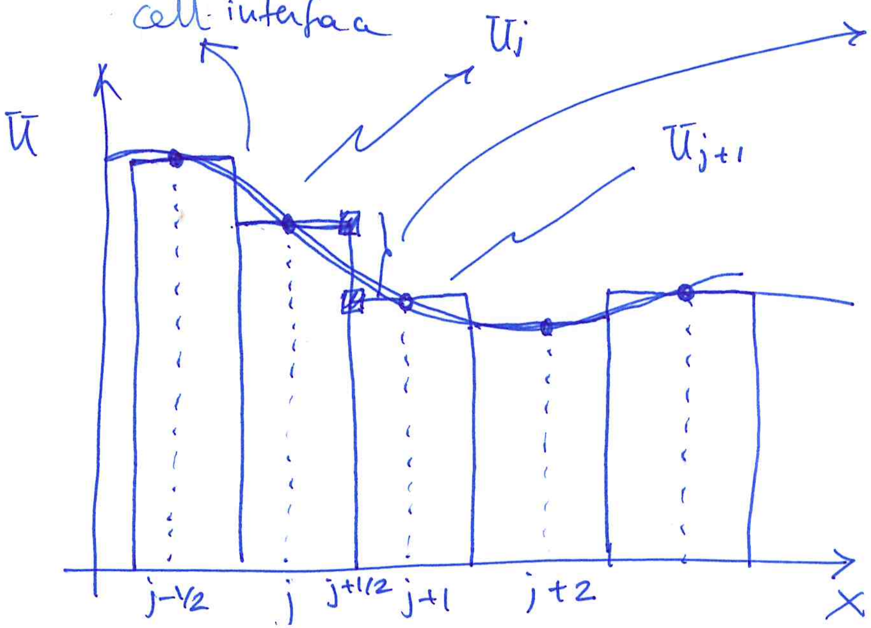
$\bar{U}_j^n \to u_j^n \quad ; \quad \underline{F}(\bar{U}_j^n) \to \lambda u_j^n$, ie for the advection eq.

In the following we will learn how to define the central averages and the numerical fluxes.

Exercise

Obtain the FV representation of the Lax-Wendroff scheme. $\square$

We have discussed that there is a class of upwind methods that goes under the name of Godunov methods as these originate around Godunov's original idea of considering a piecewise constant representation of a given quantity as a series of initial states of local Riemann problems.



Rieman problem at interface $x_{j+1/2}$ between states U_j, U_{j+1}

$$U(x,0) = \begin{cases} U_j^n & \text{if } x < x_{j+1/2} \\ U_{j+1}^n & \text{if } x > x_{j+1/2} \end{cases}$$

- The schemes are <u>monotone</u>, ie it does not introduce oscillations

<u>Def</u> A numerical scheme is monotone if after expressing the solution as

$$U_j^{n+1} = f\left(U_j^n, U_{j\pm1}^n, \dots U_j^{n-m}, U_{j\pm1}^{n-m}, \dots\right)$$

$$\frac{\partial f}{\partial U_j^n} \geq 0 \quad \forall j$$

: ie f must be a non-decreasing function of all its arguments

This condition is best understood if the method is <u>linear</u>

ie if

$$U_j^{n+1} = \sum_{k=-k_L}^{k_R} c_k U_{j+k}^n$$

then monotonicity $\iff$ $c_k \geq 0 \quad \forall k \in [k_L, k_R]$

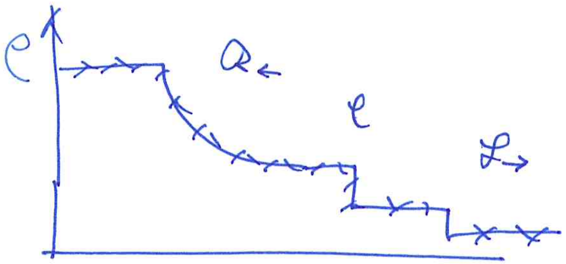
Another interesting property is that of <u>monotonicity preserving</u>. In particular, given a solution such that

$$U_j^n \geq U_{j+1}^n$$

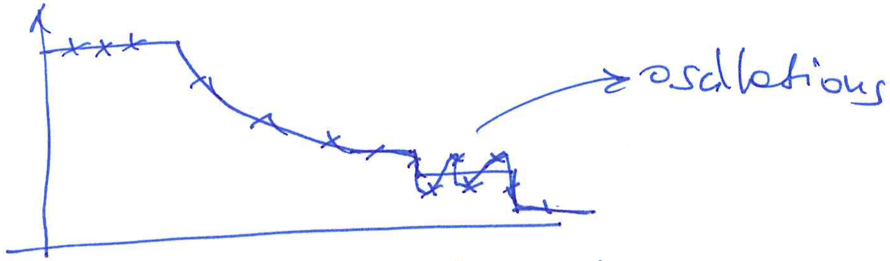
a method is said to be monotonicity preserving if

$$U_j^{n+1} \geq U_{j+1}^{n+1}$$

i.e. if oscillations in the solution do not appear



monotonicity preserving



non-monotonicity preserving.

Note : monotone scheme $\Longrightarrow$ monotonicity preserving scheme

Using these definition we can state an important theorem due to Godunov: "A linear and monotonicity preserving scheme is at most first-order accurate".

In other words, if the linear scheme is increased in order (eg to become more accurate) then oscillations will appear.

In this respect the original Godunov method (ie piecewise constant states) combines the simplest and least accurate strategy for the calculation of the left and right states with the most accurate strategy for the solution of the Riemann problem (ie exact solver).

Fortunately it is not difficult to overcome the limitation of Godunov's theorem: a linear monotone method cannot be more accurate than first order.

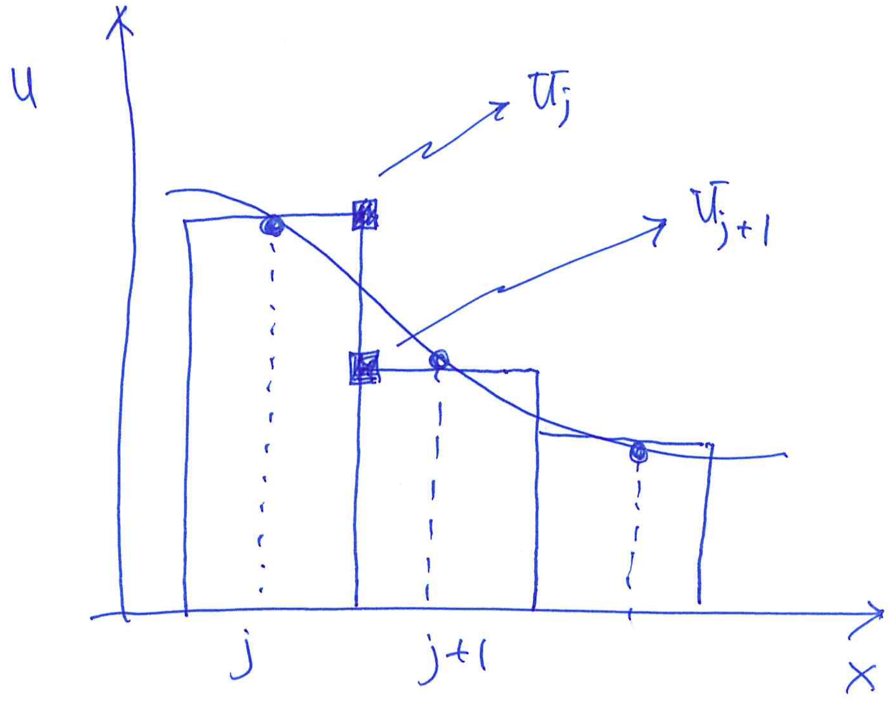
The solution is to make the method nonlinear, ie with nonlinear coefficients. This can be done by replacing the constant coefficients with variable ones $\left[U_j^{n+1} = \sum_k c_k U_j^n \right]$

$c_k(x)$

In turn, this is equivalent to replacing the piecewise constant states at each cell interface with suitably "reconstructed" representations of the solution, ie with some high-order polynomial representation of the solution with each cell.

More on reconstruction

In a piecewise-constant representation of a solution all information on the variation of the solution inside the cell is lost. However, this information can be recovered if a polynomial representation is made of the solution within the cell



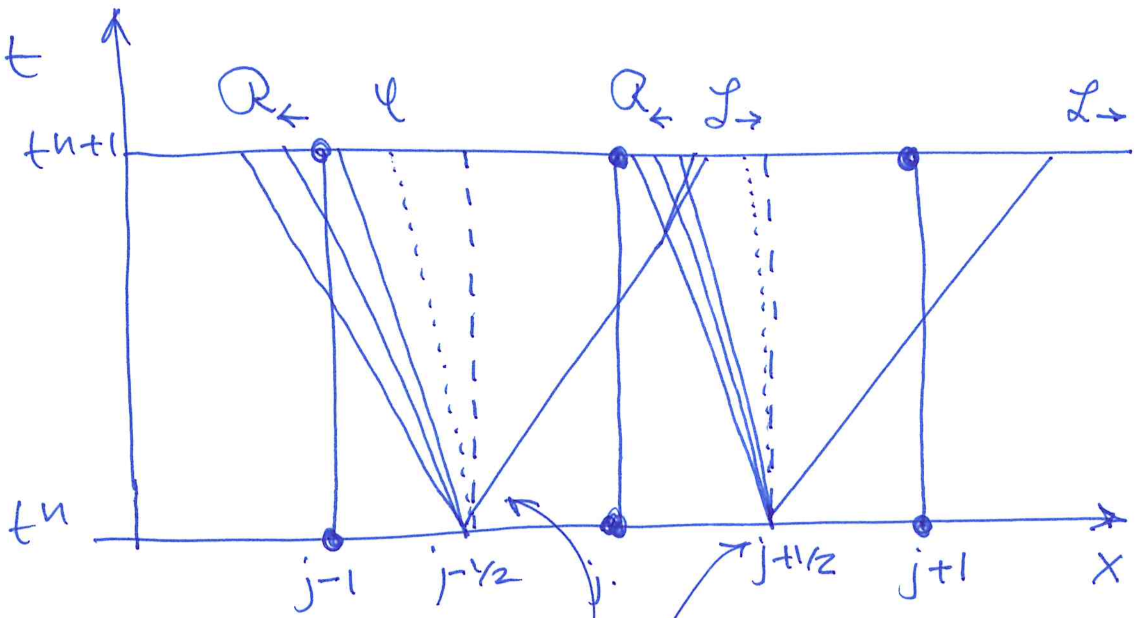
Clearly, the higher the order of the polynomial representation, the higher the accuracy.

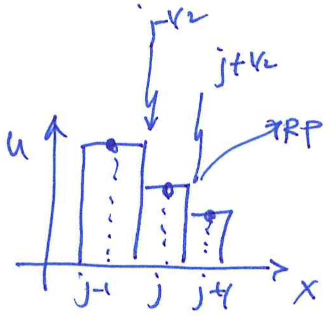
In HRSC methods, the spatial accuracy is determined by the accuracy of the reconstruction, which is done on the conserved variables for FV methods.

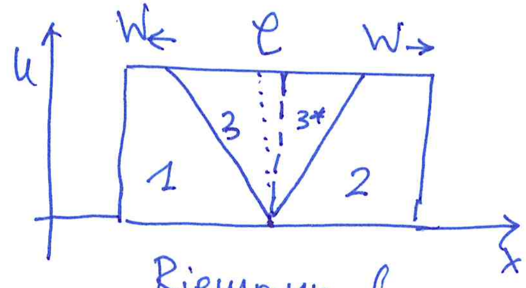
This is a Riemann problem, whose solution will provide the states at the interfaces $\bar{U}(x_{j\pm 1/2}, t)$ which can be used to compute the fluxes

$$F_{j\pm 1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F(\bar{U}(x_{j\pm 1/2}, t))\, dt$$

Note that the solution is <u>constant</u> along the direction $x/t = 0$ because the solution of the Riemann problem is self-similar. Hence, once the solution is known at $x_{j\pm 1/2}$, ie once the Riemann problem has provided the value of $\bar{U}_{j\pm 1/2}$, the integration in time can be done analytically.







Riemann fan

for example

a RP develops here ($R_\leftarrow$ C L) and its solution will provide the state 3* or $U_{j\pm 1/2}$

Note that the time step is chosen so as to satisfy the CFL conditions. This implies that waves can interact but this will generally lead to a decrease in speed. No wave from $x_{j-1/2}$ will reach the other interface $x_{j+1/2}$ and viceversa.

Some remarks on Godunov methods:

- they are conservative by construction where

$$F_{j+1/2} = F(\bar{U}_j, \bar{U}_{j+1})$$: the Riemann problem depends only only on the two (constant) states $\bar{U}_j$ and $\bar{U}_{j+1}$

- they are upwind in the sense that they exploit the full wave structure of the Riemann problem.

Ex.

Consider the advection eq. $\partial_t u + \lambda \partial_x u = 0$; in this case $\underline{F} = \lambda \bar{U}$ and the solution of the Riemann problem at $x_{j+1/2}$ is given by $\bar{U}^n_{j+1}$ if $\lambda < 0$ and by $\bar{U}^n_j$ if $\lambda > 0$.

The resulting scheme is therefore

$$U_j^{n+1} = U_j^n - \alpha \left(\underbrace{U_j}_{\frac{F(u_{j+1/2})}{\lambda}} - \underbrace{U_{j-1}}_{\frac{F(u_{j-1/2})}{\lambda}} \right) \quad \text{if } \lambda > 0$$

$$U_j^{n+1} = U_j^n - \alpha \left(U_{j+1} - U_j \right) \quad \text{if } \lambda < 0$$

- The schemes are first order accurate in space and time
- The schemes are CFL limited

$$\Delta t = C_{CFL} \min \left(\frac{\Delta x}{|\lambda_k^n|} \right) \qquad |\lambda_k^n| : \text{ all possible eigenvalues of the problem at } t^n$$

As a result, there are piecewise-linear or piecewise-parabolic reconstructions, which provide converge order $p \simeq 1-2$, respectively.

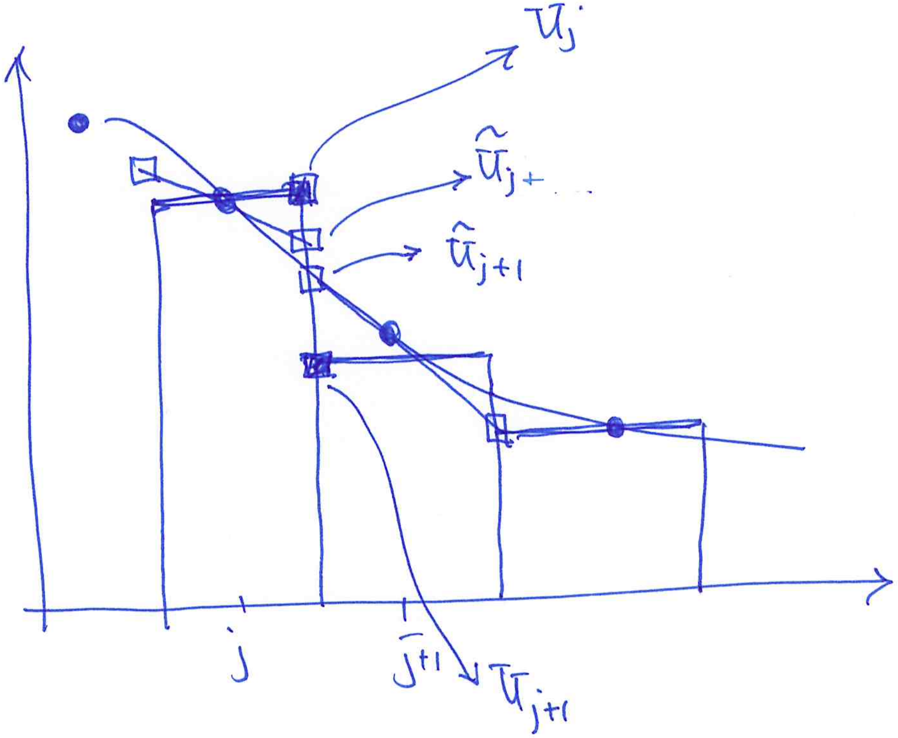
Piece-wise-linear reconstructions are usually expressed in terms of <u>slope limiters</u>, which provide an approximation of U_j^n within the cell,

$$U_j \longrightarrow \underbrace{U_j^n(x) = U_j^n}_{\text{piecewise constant}} + \sigma_j^n \,(x - x_j) \qquad x_{j-1/2} \leq x \leq x_{j+1/2}$$

where, for instance,

$$\sigma_j^n = \text{minmod}\left(\frac{U_j^n - U_{j-1}}{\Delta x}, \frac{U_{j+1}^n - U_j^n}{\Delta x}\right) \quad \text{and} \quad : \quad \text{MINMOD FLUX LIMITER}$$

$$\text{minmod}(\alpha, \beta) = \begin{cases} \alpha & \text{if } |\alpha| < |\beta| \quad \alpha\beta > 0 \\ \beta & \text{if } |\beta| < |\alpha| \\ 0 & \qquad\qquad\quad \alpha\beta \leq 0 \end{cases}$$



$\bar{u}_j$
$\hat{\tilde{u}}_{j+}$
$\hat{u}_{j+1}$
j
$j+1$
$\bar{u}_{j+1}$

Modern Godunov methods are nonlinear and differ in a number of ways. In all cases, however they consist of three basic steps: "reconstruct-solve-update"

1) Reconstruct the left and right states of a local Riemann problem at each cell interface with suitable polynomial

2) Use a Riemann solver at each interface (usually an approximate one)

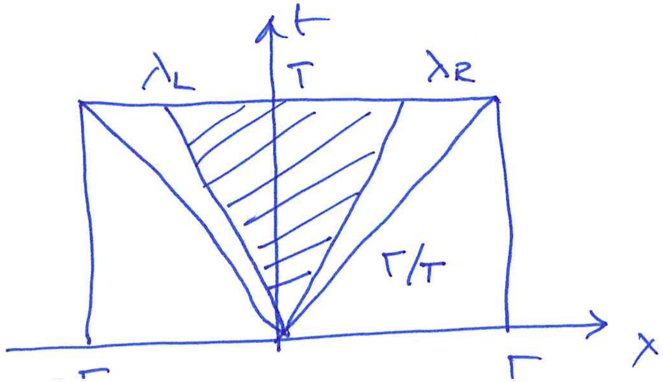
3) Update in time using at least second-order accuracy.

As an example of an approximate I will discuss the HLLE solver, which is the building block of many numerical codes. Its advantage is that it assumes only two waves propagating λ_L, λ_R and a single state between them

$$\bar{U}(x,t) = \begin{cases} U_L & \text{if } \quad x/t < \lambda_L \\ U^{HLLE} & \text{if } \lambda_L < x/t < \lambda_R \\ U_R & \text{if } \quad x/t > \lambda_R \end{cases} \qquad \begin{array}{l} \lambda_L \leq 0 \\ \lambda_R \geq 0 \end{array}$$

Let's consider the "control volume" $V = [-\Gamma, \Gamma] \times [0, T]$

$\Gamma > \max(|\lambda_L|, |\lambda_R|) \times T$



A generic conservative formulation $\partial_t U + \partial_x F(U) = 0$ applied to

$$\int_{-\Gamma}^{\Gamma} U(x,T)\,dx = \int_{-\Gamma}^{\Gamma} U(x,0)\,dx + \int_0^T F(U(-\Gamma,t))\,dt - \int_0^T F(U(\Gamma,t))\,dt$$

$$= \int_{-\Gamma}^{0} U_L\,dx + \int_0^{\Gamma} U_R\,dx + \int_0^T F(U(-\Gamma,t))\,dt - \int_0^T F(U(\Gamma,t))\,dt$$

$$= \Gamma(U_L + U_R) + T(F_L - F_R)$$

where $F_L = F(U_L)$; $F_R = F(U_R)$

The LHS is

$$\int_{-\Gamma}^{\Gamma} U(x,T)\,dx = \int_{-\Gamma}^{T\lambda_L} U_L\,dx + \int_{T\lambda_L}^{T\lambda_R} U^{HLLE}\,dx + \int_{T\lambda_R}^{\Gamma} U_R\,dx$$

$$= U_L (T\lambda_L + \Gamma) + U^{HLLE} (\lambda_R - \lambda_L) + U_R (\Gamma - T\lambda_R)$$

from which

$$U^{HLLE} = \frac{\lambda_R U_R - \lambda_L U_L + F_L - F_R}{\lambda_R - \lambda_L}$$

: state HLLE is computed using λ_L, λ_R

where $\lambda_L = \min\left(0, \lambda_-(U_L), \lambda_-(U_R)\right)$

$\lambda_R = \max\left(0, \lambda_+(U_L), \lambda_+(U_R)\right)$

$\lambda_\pm$: seen in previous lecture.

Using the Rankine-Hugoniot conditions we can express the fluxes across the left and right waves as

$$F_{L*} = F_L + \lambda_L \left(U^{HLLE} - U_L\right)$$

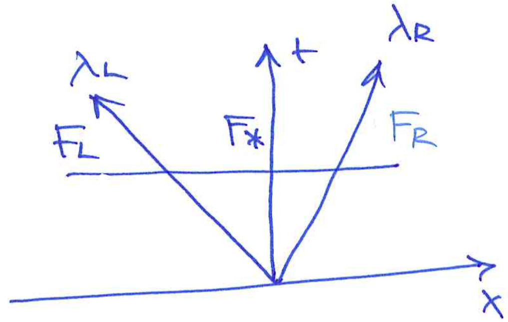
$$F_{R*} = F_R + \lambda_R \left(U^{HLLE} - U_R\right)$$

from which we obtain the HLLE flux

$$F_* = \frac{\lambda_R F_L - \lambda_L F_R + \lambda_L \lambda_R (U_R - U_L)}{\lambda_R - \lambda_L}$$

The logic of the Godunov method will then be

$$F^{HLLE} = \begin{cases} F_L & \text{if } x/t < \lambda_L \\ F_* & \text{if } \lambda_L < x/t < \lambda_R \\ F_R & \text{if } x/t > \lambda_R \end{cases}$$



$\square$

End of the course!