Hydrodynamics & Magnetohydrodynamics

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Topics covered:

Fluid approximation, Kinetic theory, Boltzmann eq., relativistic kinetic theory, hydrodynamic eqs, perfect fluids, Bernoulli's theorem, irrotational flow, hyperbolic eqs, non linear wave, Riemann problem, introduction to plasmas, orbit theory, Debye shielding, MHD waves, magnetic reconnection.

References:

- LR, O. Zanotti: "Relativistic Hydrodynamics", OUP 2013
- Choudhuri: "The physics of fluids and plasmas", CUP 1998
- Goedbloed and Poedts: "Principles of MHD", CUP 2004

On the fluid approximation

Take a system of N identical particles, interacting though some coupling, but otherwise "free", ie not part of a rigid structure, eg a lattice.

There are two different ways of describing the system.

Let λ_{DB} be the De Broglie wavelength, ie

$$\lambda_{DB} = \frac{hc}{pc} = \frac{h}{p} = \frac{\text{Planck const.}}{\text{linear momentum}} \cong 10^{-11} \text{ cm for electron at 1eV}$$

Then the dimensionless quantity

$$R := \frac{\lambda_{DB}}{\ell} = \frac{\lambda_{DB}}{(\text{inter-particle seperation})}$$

$$= \begin{cases} \gtrsim 1 : & \text{wave functions overlap and quantum mechanical description is necessary (N particle wave function using Schrödinger eq.)} \\ \ll 1 : & \text{wave functions are widely seperated, quantum interference not important, classical point particles} \end{cases}$$

- This latter case is described by <u>Ehrenfest theorem</u>: the states of the system are prescribed by the <u>positions</u> and <u>velocities</u> of the particles and the evolution dictated by the laws of classical mechanics.

- Of course, if N is very large it is not reasonable to handle the dynamics of each particle separately, but it is more convenient to think in terms of <u>distribution functions</u>, $f(t, \vec{x}, \vec{u})$. The dynamics in this case is described by the <u>Boltzmann eq.</u> and the corresponding framework is referred to as "<u>kinetic theory</u>"

- There is also another regime, which is not tractable by kinetic theory and is instead the one which we will consider in this course

Consider the situation in which the number of particles N is so large that it is not convenient to treat it in statistical terms, especially because the particles occupy a volume L^3, with $L \gg \ell \gg \lambda_{DB}$.

In this case, which would correspond to $R \lll 1$, the dynamics of the system can be approximated by a <u>continuum</u>, ie a <u>fluid</u>. The validity of this approximation is contained in the <u>Knudsen number</u> $Kn := \frac{\ell}{L}$. Clearly, if $Kn \gtrsim 1$ one needs to resort to a kinetic-theory description or even a quantum-mechanical one. However, if $Kn \ll 1$ the system can be thought as composed of <u>"elements"</u> which are large enough to contain a large number of particles, but small enough to ensure homogeneity within the element and hence the concept of continuum: fluid.

The Boltzmann equation

Let's consider a single-component system with no internal degrees of freedom, ie a system of particles of the same species and indistinguishable; their rest mass is m.

$f(t, \vec{x}, \vec{u})$: distribution function $\Leftrightarrow$

"probability that a particle has at time t a velocity $\vec{u}$ (*) in the velocity space element d^3u around $\vec{u}$ and a position $\vec{x}$ in the coordinate volume element d^3x around $\vec{x}$"

In other words $f(t, \vec{x}, \vec{u})d^3x\,d^3u$ is the number of particles in the six-dimensional phase space volume $\underline{d^3x\,d^3u}$

(*) Similar considerations can be made in terms of the linear momentum $\vec{p} = m\vec{u}$

The total no. of particles N is therefore given by

$$N = \int_{-\infty}^{\infty} d^3x \int_{-\infty}^{\infty} d^3u \; f(t, \vec{x}, \vec{u}) = \int f(t, \vec{x}, \vec{u})\, d^3x\, d^3u$$

For this approach to be meaningful it is necessary that volume elements d^3x, d^3u are large enough to guarantee a small statistical variance (ie contain a sufficiently large no. of particles), but small enough to justify the use of a continuum.

Ex.

Ordinary gas at ordinary conditions : $n \sim 10^{19}$ molecules/cm^3

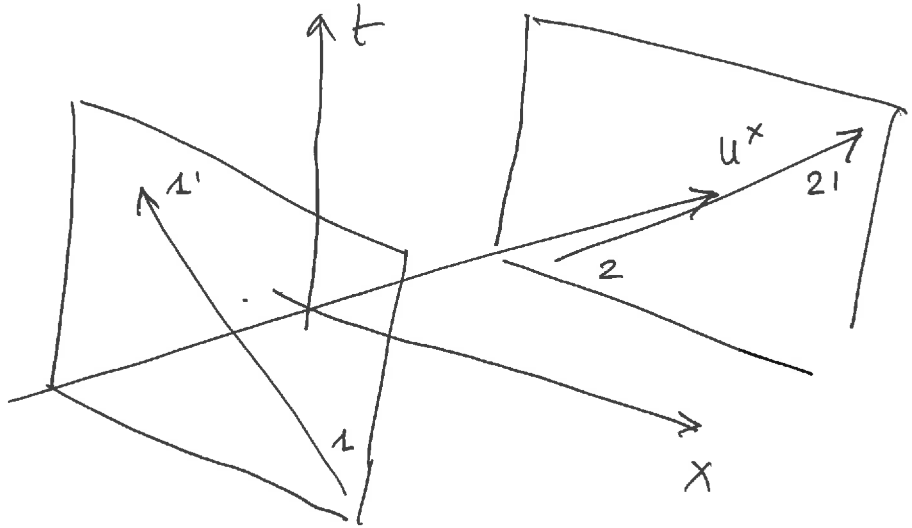
If $d^3x \sim 10^{-10}\,cm^3$ (ie $dx \sim 10^{-3}\,cm \sim 10\,\mu m$; $dx \gg \ell$)

$N_{d^3x} = n\, d^3x \sim 10^9$ molecules : this is a large enough number to think of a continuum: fluid.

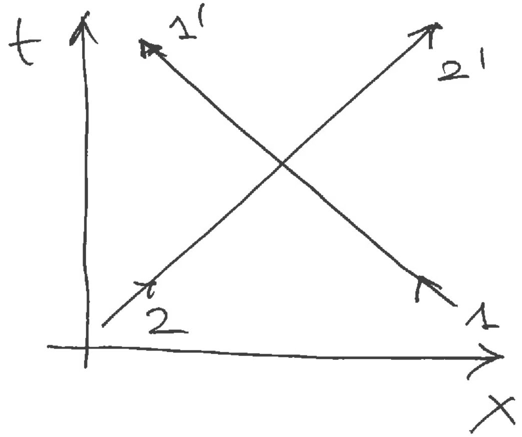
The goals of Kinetic theory are of obtaining the evolution laws of $f(t,\vec{x},\vec{u})$ and determine whether an equilibrium distribution exists which is time independent $f_0(\vec{x},\vec{u})$.

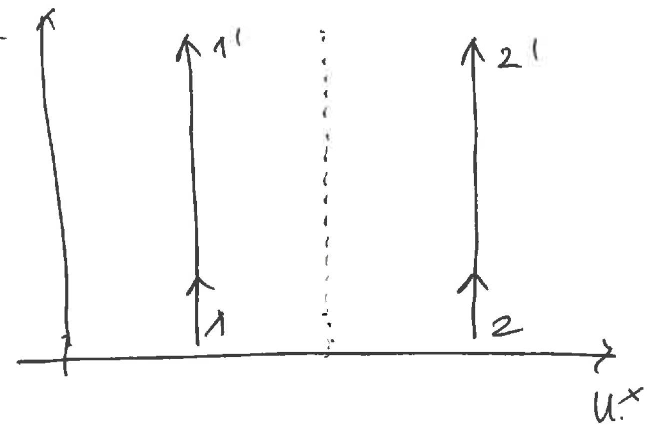
A fundamental difference in the evolution of $f(t,\vec{x},\vec{u})$ is introduced by the possibility that particles can interact (either in binary or multiple interactions).

These interactions are called collisions and the simplest scenario is the one in which they are absent: collisionless.



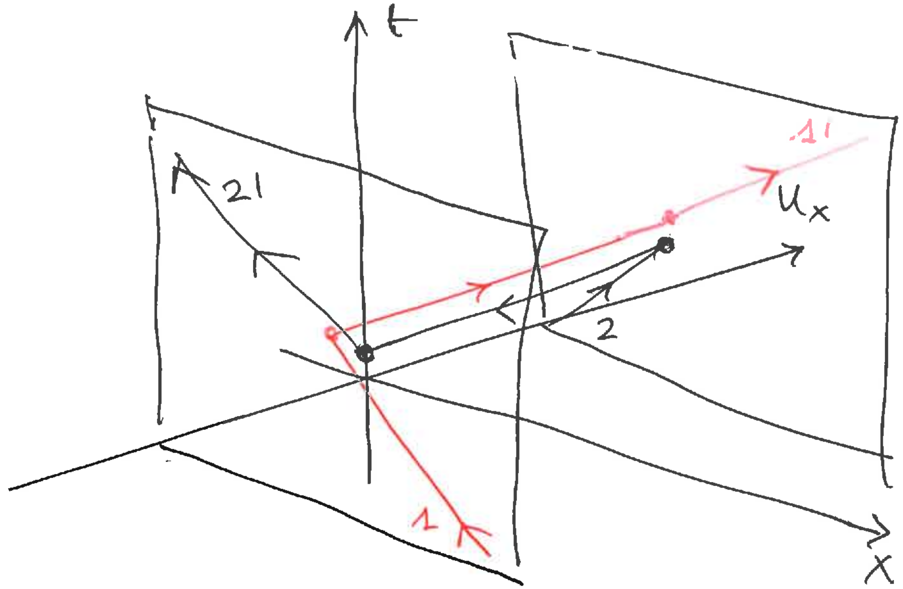
Particles can be at the same position at the same time but do not exchange momentum and hence do not change velocity

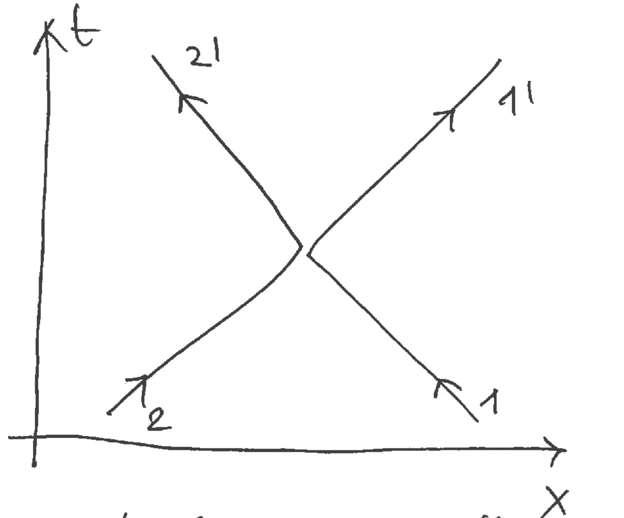




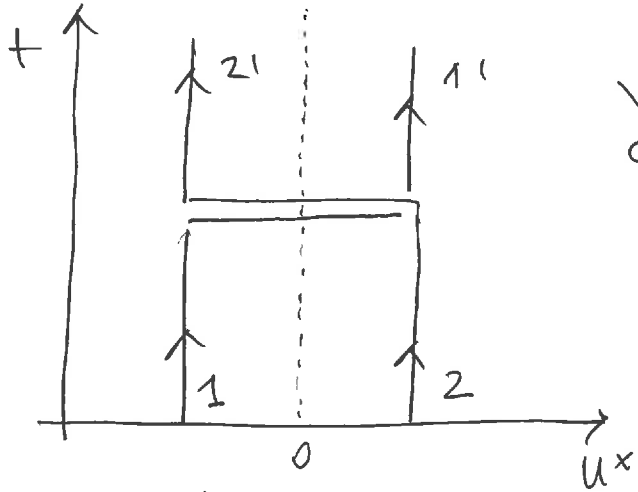
velocities do not change sign

More complex but also more realistic is the situation in which the particles do interact via <u>elastic collisions</u>, ie via collisions in which the momentum changes sign but not in modulus





particles are deflected at collision



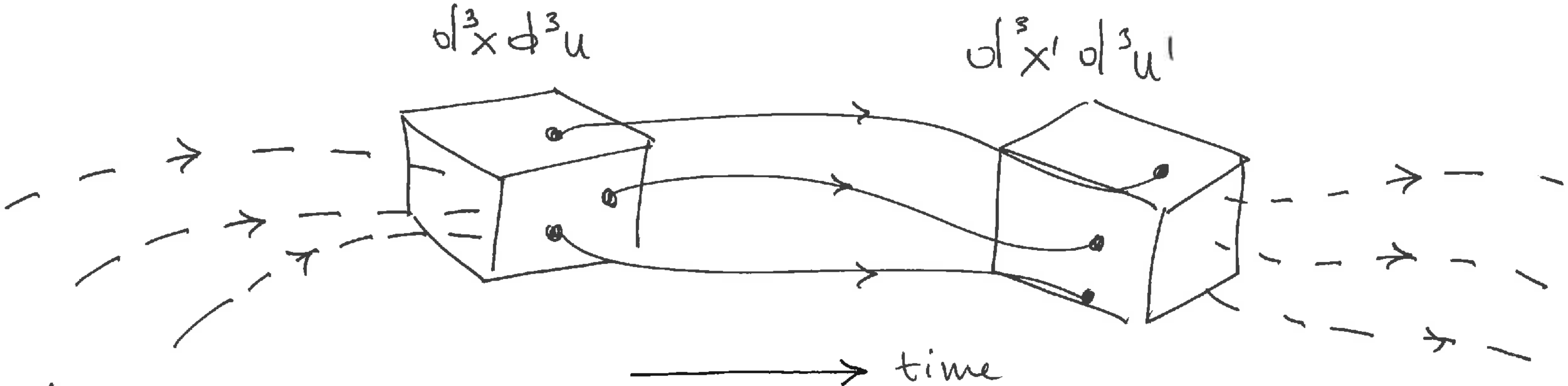
velocities change sign

particle with negative velocity acquires a positive one after collision and viceversa □

Let's consider first the case of a collisionless system of particles, but where an external force $\vec{F}$ (eg gravitational) is present. In the absence of collisions all particles in the phase-space cell at $(\vec{x}, \vec{u})$ will move to a new cell around $(\vec{x} + \vec{u}\,dt,\ \vec{u} + (\vec{F}/m)dt)$, with volume $d^3x'\,d^3u'$.

Hence, in the absence of collisions, the number of particles in a cell is an <u>invariant</u>

(*) $$f(t+dt, \vec{x}+\vec{u}\,dt, \vec{u}+(\vec{F}/m)dt)\, d^3x'\, d^3u' = f(t,\vec{x},\vec{u})\, d^3x\, d^3u$$



Each element is losing and gaining, maintaining the same number of particles. Mathematically we can see this as a coordinate transformation

$$\left.\begin{aligned} \vec{x} &\longrightarrow \vec{x}' = \vec{x} + \vec{u}\,dt \\ \vec{u} &\longrightarrow \vec{u}' = \vec{u} + (\vec{F}/m)\,dt \end{aligned}\right\} \text{advection in phase-space}$$

The transformation has Jacobian $J_{ik} = \partial x'^i / \partial x^k$ with determinant $J = 1 + O(dt^2)$, ie $J = 1$ is correct at 1st order in dt, so that

$$d^3x' \, d^3u' = J \, d^3x \, d^3u \cong d^3x \, d^3u + O(dt^2)$$

We can therefore Taylor expand the LHS of (*) to obtain

$$(**) \quad \boxed{\frac{\partial}{\partial t} f + \vec{u} \cdot \vec{\nabla} f + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{\vec{u}} f = 0}$$

Collisionless Boltzmann equation

$$\Longleftrightarrow \quad \frac{\partial f}{\partial t} + u^i \frac{\partial f}{\partial x^i} + \frac{F^i}{m} \frac{\partial f}{\partial u^i} = 0$$

$$\Longleftrightarrow \quad \partial_t f + u^i \partial_i f + \frac{F^i}{m} \partial_{u^i} f = 0$$

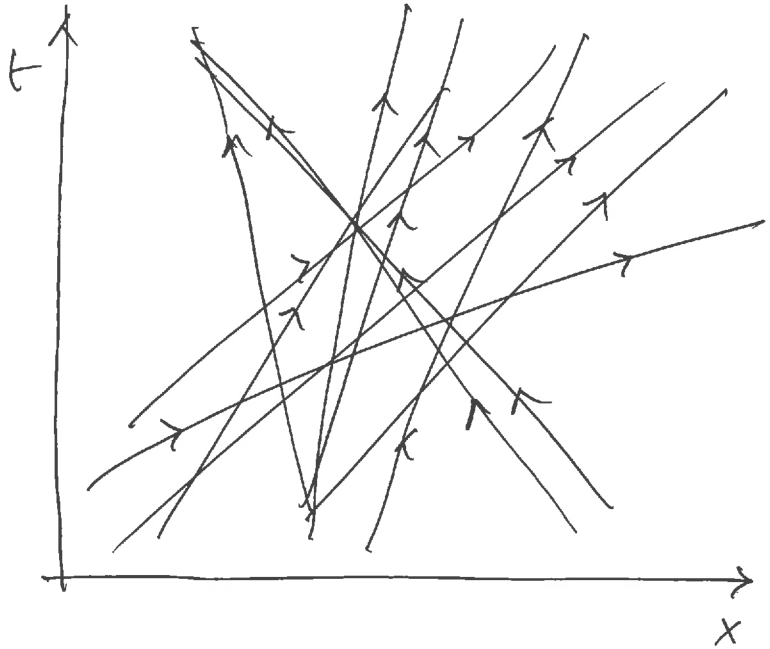
where I've been using the Einstein summation convention on repeated indices

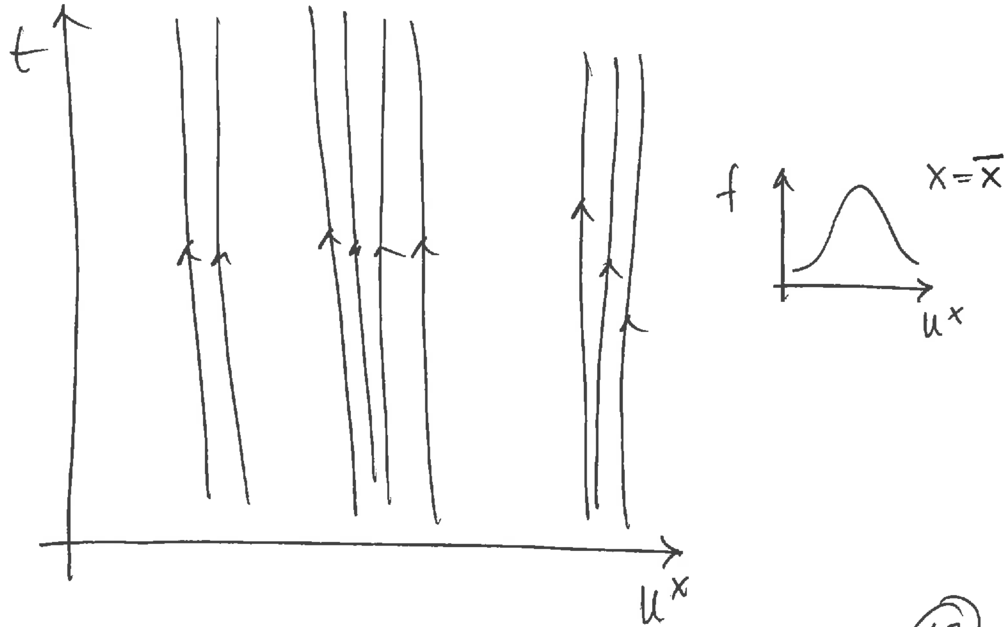
$$A^i B_i = \sum_{1}^{3} {}_i \, A^i B_i$$

Note that (**) is effectively stating that <u>f is conserved in the absence of collisions</u>, ie

$$\frac{d}{dt} f(t, \vec{x}, \vec{u}) = \frac{\partial f}{\partial t} + \dot{\vec{x}} \frac{\partial f}{\partial \vec{x}} + \dot{\vec{p}} \frac{\partial f}{\partial \vec{p}} = 0$$

where $\dot{\vec{x}} = \frac{\partial \vec{x}}{\partial t} = \vec{u}$; $\dot{\vec{p}} = \frac{\partial \vec{p}}{\partial t} = \frac{\vec{F}}{M}$.





Let's next consider a collisional system still subject to an external force. For simplicity we will consider the collisions to be only binary ones, although this is also the most realistic scenario unless densities are extremely large.

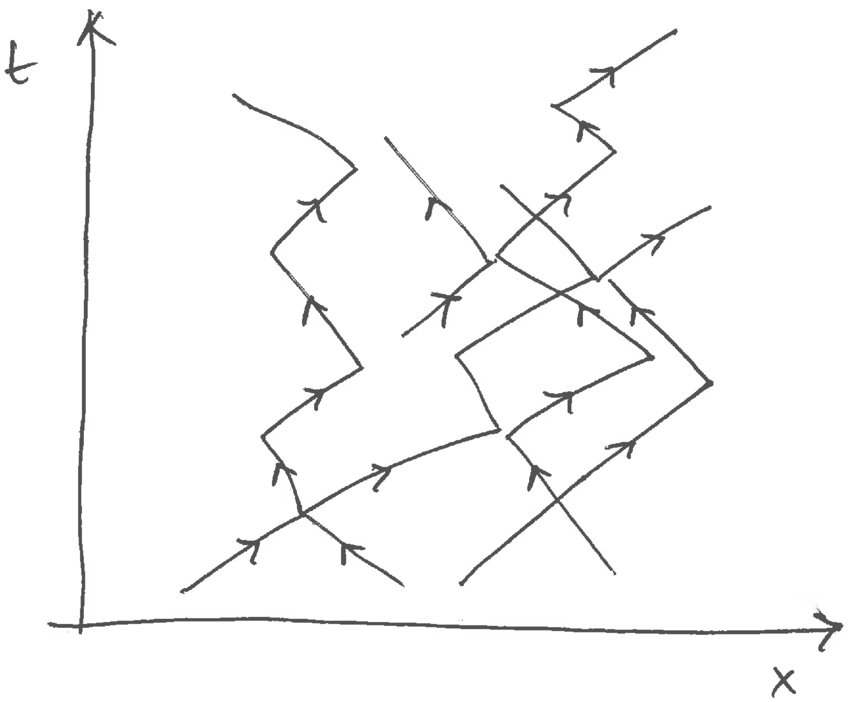
The main difference in this case is that particles are not simply advected in phase space, but they can move from one cell to the next because of the result of collisions. Conservation of the number of particles then requires that

$$f(t+dt, \vec{x}+\vec{u}\,dt, \vec{u}+(\vec{F}/m)dt)\,d^3x'\,d^3u' = f(t,\vec{x},\vec{u})\,d^3x\,d^3u + \Gamma(f)\,d^3x\,d^3u\,dt$$

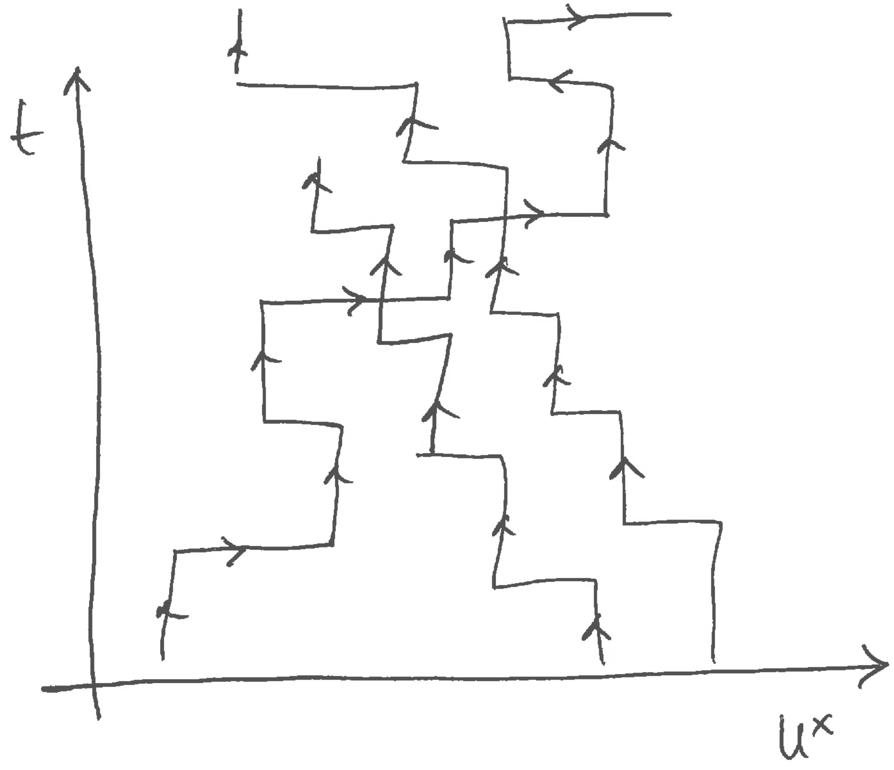
where

$\Gamma(f)$: change in dt of the number of particles in the cell at $(\vec{x},\vec{u})$ caused by collisions.

In analogy with what shown in the collisionless case, the evolution of the distribution function in phase space can be represented as



velocities change sign at (elastic) collision sites



single component needs not change sign.

Proceeding in the same manner as before we find

$$\boxed{\frac{\partial f}{\partial t} + \vec{u}\cdot\vec{\nabla} f + \frac{\vec{F}}{m}\cdot\vec{\nabla}_{\vec{u}} f = \Gamma(f) = \left(\frac{\partial f}{\partial t}\right)_{coll}} \qquad (***)$$

Collisional (full) Boltzmann equation.

collision integral

In the simplest case of binary collisions at $\vec{x}$ between particles with velocities $\vec{u}_1$ and $\vec{u}_2$, and in the absence of an external force, the collision integral is given by

$$(\square) \quad \left(\frac{\partial f}{\partial t}\right)_{coll} = \int d^3u_2 \int d\Omega\, \sigma(\Omega)\, |\vec{u}_1 - \vec{u}_2|\, (f_2' f_1' - f_2 f_1) = \Gamma(f)$$

where

$f_{1,2} = f(t, \vec{x}, \vec{u}_{1,2})$: distributions before collision

$f'_{12} = f(t, \vec{x}, \vec{u}'_{1,2})$: " after collision

$\sigma(\Omega)$: cross section of solid angle Ω ; $d\Omega$: solid angle element

Replacing ($\square$) in (***) we appreciate that the full Boltzmann eq. is an integro-differential equation for the distribution function f.

Notes

- Collisionless Boltzmann
  ↳ - Vlasov-Maxwell if external force is Coulomb force
  - Einstein-Vlasov if gravity is treated via general relativity

- Collisional Boltzmann
  ↳ - Fokker-Planck eq. if gravity is Newtonian $\vec{F} = -m\vec{\nabla}\phi$
  In this case the collisional integral is replaced by diffusion coefficients and Boltzmann eq. is a PDE (ODE in 1D). Used in dense stellar systems, eg globular clusters.

Given a distribution function f, we can use it as basis for various "moments"

- For any quantity Ψ, its averaged value is given by

$$\langle \Psi \rangle := \frac{1}{n} \int \Psi f \, d^3u$$

$[\langle \Psi \rangle] = [\Psi]$: the dimensions of the average are the same as of Ψ.

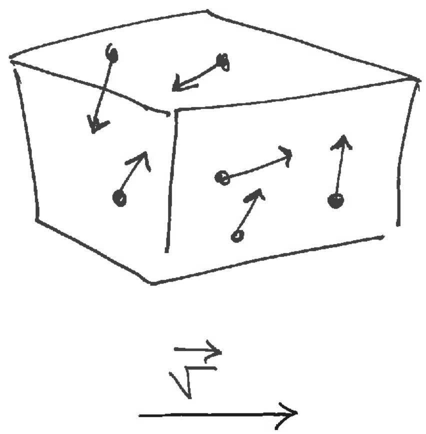
- $N = \int d^3x \int d^3u \, f = \int d^3x \, n$ where

$\boxed{n := \int f \, d^3u}$ is the particle number density

- If $\vec{u}$ is the velocity at the phase-space cell, then the <u>mean macroscopic velocity</u> is defined as

$$\vec{v} := \langle \vec{u} \rangle = \frac{1}{n} \int f \, \vec{u} \, d^3u$$

$\vec{v}$ represents the "fluid velocity" and if $\vec{v} \neq 0$ it highlights the fact that although the particles have a given velocity distribution with local value $\vec{u}$ the system also has a global velocity $\vec{v} = \langle \vec{u} \rangle$



$\vec{v} = \langle \vec{u} \rangle = 0$: the box does not move

$\vec{v} \neq 0$: the whole box moves

The H-theorem (Boltzmann 1872)

I will not give a proof of the theorem, which is instead suggested as an exercise. I will however use the result of the theorem that says the equilibrium distribution function, ie f_0 such that

$$\frac{d}{dt} f_0 = \Gamma(f_0) = 0$$

is equivalent to the condition that (this is a trivial result following the definition of the collision integral)

$$f_0(\vec{u}_1') f_0(\vec{u}_2') - f_0(\vec{u}_1) f_0(\vec{u}_2) = 0 \qquad (\Delta)$$

or, equivalently, that (this is not a trivial result and follows from the algebra in the exercise no. 1)

$$\frac{dH}{dt} = \frac{d}{dt} \int f \ln f \, d^3u \leq 0 \qquad (\Delta\Delta)$$

Expression (1) states that the product of the dist. functs. before and after the collision are the same in a stationary (equilibrium) system, ie

$$\ln(f_0(\vec{u}_1')) + \ln(f_0(\vec{u}_2')) = \ln(f_0(\vec{u}_1)) + \ln(f_0(\vec{u}_2))$$

In other words, the distribution function expresses the conservation of some quantity before and after the collision. We will see that this condition can be exploited to derive the explicit expression of f_0 in the case of a monoatomic gas.

The importance of the H theorem comes also from the fact that it provides a microscopical definition of the

<u>entropy</u> of the system in a volume V:

$$S := -k_B V H(t) = -k_B V \int f \ln f \, d^3u$$

and since $dH/dt \leq 0$, this provides a microscopic description of the second law of thermodynamics

$\frac{dS}{dt} \geq 0$. Note that the irreversibility is to be meant only in a statistical sense and that the system can be temporarily with $\frac{dH}{dt} > 0$, but it will later on move towards an equilibrium where $\frac{dH}{dt} \leq 0$.

Recap

- $f(t, \vec{x}, \vec{u})$: distribution function: probability particle has position $\vec{x}$, velocity $\vec{u}$ at time t.

$$N = \int d^3x \int d^3u \, f = \int d^3x \, n \quad \Leftrightarrow \quad n = \int f \, d^3\vec{u}$$: no. density

- Particle conservation in 6D phase space $\Leftrightarrow$

$$\partial_t f + \vec{u} \cdot \vec{\nabla} f + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{\vec{u}} f = 0$$ collisionless system

$$\partial_t f + \vec{u} \cdot \vec{\nabla} f + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{\vec{u}} f = \Gamma(f) = (\partial_t f)_{coll}$$ collisional Boltzmann eq.

$\Gamma(f)$ → collisional integral

- For any quantity ψ, $\langle \psi \rangle \doteq \frac{1}{n}\int \psi f \, d^3u$: (velocity) average of ψ

$\vec{V} := \langle \vec{U} \rangle = \int u f \, d^3u$: average macroscopic velocity (fluid velocity)

- f_0 : equilibrium distribution function, ie

$\Gamma(f_0) = \left(\frac{\partial f_0}{\partial t}\right)_{coll} = 0$: this dist. fct. does not depend on time;

- H theorem then proves that

$$\underbrace{\ln(f_0(\vec{u}_1')) + \ln(f_0(\vec{u}_2'))}_{\text{after collision}} = \underbrace{\ln(f_0(\vec{u}_1)) + \ln(f_0(\vec{u}_2))}_{\text{before collision}}$$

equilibrium dist. fuct. expresses conservation law among quantities before and after collision

The importance of the H theorem comes from the fact that it provides a microscopical definition of the entropy

$$S := -k_B V H(t) = -k_B V \int f \ln f \, d^3u$$

Since $\frac{dH}{dt} \leq 0 \implies \frac{dS}{dt} \geq 0$: this is a microscopic description of the 2nd law of thermodynamics

$\square$

- We can now move from kinetic theory to the theory describing the dynamics of a fluid and we do this via the moment equations.

The moment equations

The power of the Boltzmann equation and hence of kinetic theory can be extended also to a fluid, ie to a continuum collection of particles. To do this we need to introduce the concept of <u>collisional invariant</u>, ie a quantity that are conserved in binary collisions.

In particular, it is possible to show that a generic quantity ψ is a <u>collision invariant</u> if

$$\boxed{\int \Gamma \psi \, d^3u = 0}$$

Γ: collisional integral

A little bit of algebra then allows one to obtain from the full Boltzmann equation the following <u>moment eqs.</u>

$$\boxed{\frac{\partial}{\partial t}(n\langle\psi\rangle) + \frac{\partial}{\partial x^i}(n\langle u_i\psi\rangle) - n\left\langle u_i \frac{\partial \psi}{\partial x^i}\right\rangle - \frac{n}{m}\left\langle F^i \frac{\partial \psi}{\partial u^i}\right\rangle - \frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i}\psi\right\rangle = 0} \quad (1)$$

where ψ is a collisionally invariant quantity and

$$n\langle u_i\psi\rangle = \phi_i = \int u_i \psi f \, d^3u$$

represents the "transport flux" of ψ, ie the amount of ψ transported per unit time and unit area along the direction i.

Proof of the moment eqs. (1)

Let's start from the Boltzmann eq

$$\boxed{\partial_t f + \vec{u}\cdot\vec{\nabla} f + \frac{\vec{F}}{m}\cdot\vec{\nabla}_{\vec{u}} f = T(f)}$$

Multiply both sides by ψ and integrate in velocity space

$$\underbrace{\int \psi\, \partial_t f\, d^3u}_{(a)} + \underbrace{\int \psi u^i \partial_i f\, d^3u}_{(b)} + \underbrace{\int \frac{F^i}{m} \psi \frac{\partial f}{\partial u^i}\, d^3u}_{(c)} = \underbrace{\int \psi \Gamma\, d^3u}_{\psi \text{ is collisional invariant}} = 0 \qquad (\Delta)$$

(a) : $\psi = \psi(\vec{x}, \vec{u})$ but not of time $\Rightarrow$

$$\int \psi\, \partial_t f\, d^3u = \partial_t \int \psi f\, d^3u = \partial_t (n \langle \psi \rangle)$$

Liebniz rule

(b): $$\int \psi u^i \partial_i f\, d^3u = \int \partial_i(\psi u^i f)\, d^3u - \int (\partial_i \psi) u^i f\, d^3u - \underbrace{\int \psi \cancel{\partial_i u^i} f\, d^3u}_{\text{space and velocity are independent canonical variables } \partial_i u^i = 0}$$

$$= \partial_i \int \psi u^i f\, d^3u - n \langle u^i \partial_i \psi \rangle$$

$$= \partial_i (n \langle \psi u^i \rangle) - n \langle u^i \partial_i \psi \rangle$$

(c) $\int \psi \frac{F^i}{m} \frac{\partial}{\partial u^i} f \, d^3u = \int \frac{\partial}{\partial u^i}\left(\psi \frac{F^i}{m} f\right) d^3u - \int \frac{F^i}{m}\left(\frac{\partial \psi}{\partial u^i}\right) f \, d^3u - \int \frac{\psi}{m}\left(\frac{\partial F^i}{\partial u^i}\right) f \, d^3u$

$= \int \frac{\partial}{\partial u^i} A^i \, d^3u$: volume integral of the divergence of a vector. We can use Gauss theorem and rewrite the integral as the flux across a surface Σ in velocity space

$= \int_\Sigma A^i n_i \, d^2u$

$= 0$ $\quad$ Σ can be taken to be at ∞ where $A^i \to 0$ because $f \to 0$ for a system which is compact in velocity space.

$= -\frac{n}{m}\left\langle F^i \frac{\partial \psi}{\partial u^i}\right\rangle - \frac{n}{m}\left\langle \psi \frac{\partial F^i}{\partial u^i}\right\rangle$

collecting terms, (A) yields (Spieces)

$$\partial_t (n \langle \psi \rangle) + \partial_i (n \langle \psi u^i \rangle) - n \langle u^i \partial_i \psi \rangle - \frac{n}{m} \langle F^i \frac{\partial \psi}{\partial u^i} \rangle - \frac{n}{m} \langle \psi \frac{\partial F^i}{\partial u^i} \rangle = 0$$

qed.

ψ is a collisional invariant, hence some quantity that is expected to be conserved in the collision. The first-moment equation is given by $\psi = m \Rightarrow \partial_i \psi = 0 = \frac{\partial \psi}{\partial u^i}$

(A) $\Leftrightarrow$

$$\partial_t (nm) + \partial_i (n \langle m u^i \rangle) = 0$$ where we have assumed F^i is independent of velocity

$nm = \rho$: rest-mass density

$n \langle m u^i \rangle \overset{m = \text{const.}}{=} nm \langle u^i \rangle \overset{\text{def. of } v^i}{=} \rho v^i$, so that the 1st moment equation is

① $$\boxed{\partial_t \rho + \partial_i (\rho v^i) = 0}$$ continuity equation (mass conservation)

Let's rewrite the continuity equation in "conservative form", ie as

$$\underbrace{\partial_t \varrho = - \partial_i \phi^i}$$

$\phi^i = \varrho v^i$: rest-mass flux

the variation is more simply appreciated after a volume integral (space)

$$\int_V \partial_t \varrho \, d^3x = \partial_t M = - \int_V \vec{\nabla} \cdot \vec{\phi} \, d^3x = \underbrace{- \int_\Sigma \phi^i n_i \, d^2x}_{\text{flux across surface } \Sigma}$$

$M = \int \varrho \, d^3x$

In other words, the variation of M in a volume is equal to the flux of matter entering or leaving the volume. $\square$

- 2nd - moment equation

$\psi = m u^i \Rightarrow$

$n \langle \psi \rangle = n \langle m u^i \rangle = \varrho v^i$

$n \langle \psi u^j \rangle = n \langle m u^i u^j \rangle = \varrho \langle u^i u^j \rangle + \varrho v^i v^j$

Let's consider the identity

$$\langle (u^i - v^i)(u^j - v^j) \rangle = \langle u^i u^j - v^j u^i - v^i u^j + v^i v^j \rangle$$

$$= \langle u^i u^j \rangle - \langle v^j u^i \rangle - \langle v^i u^j \rangle + \langle v^i v^j \rangle$$

$$= \langle u^i u^j \rangle - v^j \langle u^i \rangle - v^i \langle u^j \rangle + v^i v^j$$

$$= \langle u^i u^j \rangle - v^j v^i - \cancel{v^i v^j} + \cancel{v^i v^j} \Rightarrow$$

$\psi = m u^i$

$$\langle u^i u^j \rangle = \langle (u^i - v^i)(u^j - v^j) \rangle + v^i v^j \Rightarrow \partial_i (n \langle \psi u^j \rangle) = \partial_i (\rho \langle u^i u^j \rangle)$$

$$= \partial_i (\rho v_i v_j) + \partial_i (\rho \langle (u^i - v^i)(u^j - v^j) \rangle)$$

$$\langle u^i \partial_i \psi \rangle = \langle u^i \partial_i m u^j \rangle = m \langle u^i \partial_i u^j \rangle = 0$$

$$\langle F^i \frac{\partial \psi}{\partial u^i} \rangle = m \langle F^i \frac{\partial u^j}{\partial u^i} \rangle = m \langle F^i \delta^j_i \rangle = m \langle F^j \rangle = m F^j$$

constant force

$$\langle \psi \frac{\partial F^i}{\partial u^i} \rangle = \langle m u^j \frac{\partial F^i}{\partial u^i} \rangle = 0$$

$\vec{F} = \vec{F}(\vec{X})$ function of space only collecting terms, the 2nd-moment equation is given by

$$\boxed{\partial_t (\rho v^i) + \partial_j (\rho v^i v^j) - \frac{\rho}{m} F^i + \partial_j P_{ij} = 0} \qquad (2)$$

where $P_{ij} := \rho \langle (u^i - v^i)(u^j - v^j) \rangle = \rho \left(\langle u^i u^j \rangle - v^i v^j \right)$

Eq. (2) expresses the conservation of linear momentum by accounting for the evolution of the momentum flux ρv^i, ie the amount of linear momentum per unit time.(*)

The tensor P_{ij} is called the pressure tensor and is a 3x3 matrix which measures velocity dispersions wrt the averages in a given direction.

Eg $P_{xx} = \langle (u^x - v^x)(u^x - v^x) \rangle = \langle (u^x - v^x)^2 \rangle$; $P_{xy} = \langle (u^x - v^x)(u^y - v^y) \rangle$; ...

(*) $$-\frac{Mv}{\Delta t} = -\int \frac{d^3x \, \rho v}{\Delta t} \overset{v = dx/dt}{=} -\int \rho v^2 d^2x \overset{\rho = \text{const.}}{=} -\rho v^2 \Sigma = -\overbrace{\rho v \overbrace{\frac{\Delta x}{\Delta t}}^{v} \Sigma}^{M} = -\frac{Mv}{\Delta t}$$

Note that $P_{ij} \neq 0$ even if the fluid is at rest, ie $v^i = 0$

$P_{ij} = \langle u^i u^j \rangle$ (think of fluid in a box which is not moving).

- 3rd-moment equation

Take $\psi = \frac{1}{2} m |\vec{u} - \vec{v}|^2$: kinetic energy of particle around $\vec{u}$

$$n\langle \psi \rangle = \frac{1}{2} \rho \langle |\vec{u} - \vec{v}|^2 \rangle$$

$$n\langle \psi u^i \rangle = \frac{1}{2} \rho \langle u^i |\vec{u} - \vec{v}|^2 \rangle$$

$$n\langle u^i \partial_i \psi \rangle = n\langle u^i \partial_i \left(\frac{1}{2} m |\vec{u} - \vec{v}|^2\right) \rangle$$

$$= \frac{1}{2} \rho \langle u^i \partial_i (|\vec{u} - \vec{v}|^2) \rangle$$

$$\langle F^i \frac{\partial}{\partial u^i} \psi \rangle = \langle F^i \frac{\partial}{\partial u^i} \left(\frac{1}{2} m |\vec{u} - \vec{v}|^2\right) \rangle = 0$$

kinetic energy function of space only $\frac{\partial}{\partial u^i} = 0$

$$\langle \psi \frac{\partial F^i}{\partial u^i} \rangle = \frac{1}{2} m \langle |\vec{u}-\vec{v}|^2 \frac{\partial F^i}{\partial u^i} \rangle = 0 \qquad F^i \text{ function of space only}$$

Putting things together

$$\partial_t \left(\frac{1}{2} \rho \langle |\vec{u}-\vec{v}|^2 \rangle \right) + \frac{1}{2} \partial_i \left(\rho \langle u^i |\vec{u}-\vec{v}|^2 \rangle \right) - \frac{1}{2} \rho \langle u^i \partial_i (|\vec{u}-\vec{v}|^2) \rangle = 0 \qquad (\nabla)$$

Define now

$$\epsilon := \frac{1}{2} \langle |\vec{u}-\vec{v}|^2 \rangle = \frac{1}{2n} \int |\vec{u}-\vec{v}|^2 f \, d^3u$$

: specific internal energy; it measures internal motions and is different from kinetic energy despite being quadratic in velocities. Kinetic energy (specific) $\propto \frac{v^i v_i}{2}$ and is associated with bulk motions.

so that the first term above is

$\partial_t (\rho \epsilon)$;

Introduce

$\vec{q} := \frac{1}{2} \rho \langle (\vec{u}-\vec{v}) |\vec{u}-\vec{v}|^2 \rangle$ or in components

$$q_i := \frac{1}{2} \rho \langle (u_i - v_i) |\vec{u}-\vec{v}|^2 \rangle$$

so that

add and subtract

$$\frac{1}{2}\partial_i\left(\rho\langle u_i|\vec{u}-\vec{v}|^2\rangle\right)=\frac{1}{2}\partial_i\left(\rho\langle(u_i-v_i)|\vec{u}-\vec{v}|^2+v_i|\vec{u}-\vec{v}|^2\rangle\right)=$$

$$=\frac{1}{2}\partial_i\left(\rho\left[\langle(u_i-v_i)|\vec{u}-\vec{v}|^2\rangle+\langle v_i|\vec{u}-\vec{v}|^2\rangle\right]\right)$$

$$=\frac{1}{2}\partial_i\left(2q^i+\rho v^i\langle|\vec{u}-\vec{v}|^2\rangle\right)$$

$$=\frac{1}{2}\partial_i\left(2q^i+2\rho v^i\epsilon\right)$$

$$=\frac{2}{2}\partial_i(q^i)+\frac{2}{2}\partial_i(\rho v^i\epsilon)=\partial_i q^i+\partial_i(\rho v^i\epsilon)$$

Finally, we are left with the term

$$-\frac{1}{2}\rho\langle u^i\partial_i|\vec{u}-\vec{v}|^2\rangle=2P_{ij}\Lambda^{ij}$$

where the last equality is found in a seperate exercise ⊙

Collecting terms (▽) $\Longleftrightarrow$

$$\boxed{\partial_t\rho\epsilon+\partial_i(\rho v^i\epsilon)+\partial_i q^i+P_{ij}\Lambda^{ij}=0}$$

conservation of internal energy

We will see later on that $q^i=0=P_{ij}\Lambda^{ij}$ for a perfect fluid

Exercise ⊙

Skip

Show that

$= \rho \langle u^i \partial_i |\vec{u} - \vec{v}|^2 \rangle = 2 P_{ij} \Lambda^{ij}$

$\vec{u} - \vec{v} = \vec{A}$; $|\vec{u} - \vec{v}|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{A} \cdot \vec{A}$

$A^j \partial_i A^j = A_j \partial_i A^j$

$= \rho \langle u^i \partial_i (A^j A^k \delta_{jk}) \rangle = \rho \langle u^i [(\partial_i A^j) A^k \delta_{jk} + A^j (\partial_i A^k) \delta_{jk}] \rangle$

$= 2\rho \langle u^i \partial_i A^j A_j \rangle$

$= 2\rho \langle u^i (\partial_i (u^j - v^j))(u^j - v^j) \rangle =$ $\partial_i u^j = 0$

$= -2\rho \langle u^i \partial_i v^j (u^j - v^j) \rangle = -2\rho \, \partial_i v^j \langle u^i (u^j - v^j) \rangle$

$\partial_i u^j = 0$ as they are indep. coords

Recall that

$P_{ij} = \rho \langle (u^i - v^i)(u^j - v^j) \rangle$

$= \rho \langle u^i u^j \rangle - \rho v^i v^j$

Skip

but also

$$P_{ij} = \rho \langle u^i (u^j - v^j) \rangle - \rho \langle v^i (u^j - v^j) \rangle$$

$$= \rho \langle u^i (u^j - v^j) \rangle - \rho [\langle v^i u^j \rangle - \langle v^i v^j \rangle]$$

$$= \rho \langle u^i (u^j - v^j) \rangle - \rho \underbrace{[v^i \underbrace{\langle u^j \rangle}_{v^j} - v^i v^j]}_{=0}$$

$$= \rho \langle u^i (u^j - v^j) \rangle \neq \rho v^i v^j - \rho v^i v^j \quad \text{because } \langle u^i u^j \rangle \neq v^i v^j$$

so

$$P_{ij} \underset{\text{definition}}{=} \rho \langle (u^i - v^i)(u^j - v^j) \rangle \underset{\text{1st identity}}{=} \rho \langle u^i u^j \rangle - \rho v^i v^j \underset{\text{2nd identity}}{=} \rho \langle u^i (u^j - v^j) \rangle$$

Skip

So we have shown

$$\rho \langle u^i \partial_i |\vec{u}-\vec{v}|^2 \rangle = -2 \underbrace{\rho\, \partial_i v^j \langle u^i (u^j - v^j)}_{P_{ij}} \rangle$$

$$= -2 P_{ij} \underbrace{\partial_i v^j}_{A_{ij}}$$

this is a generic tensor but P_{ij} is a symmetric tensor and hence A_{ij} must also be symmetric or its contraction with P_{ij} would be zero.

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$$

$$= \frac{1}{2}(\partial_i v^j + \partial_j v^i) = \Lambda_{ij}$$

$$= -2 P_{ij} \Lambda^{ij}$$

To conclude

$$-\frac{1}{2}\rho \langle u^i \partial_i |\vec{u}-\vec{v}|^2 \rangle = P_{ij} \Lambda^{ij}$$

qed End exercise ⊙

Summary: the moment equations can be rewritten as

1) $\partial_t \rho + \partial_i \rho v^i = 0$

2) $\partial_t v^j + v^i \partial_i v^j + \frac{1}{\rho} \partial_i P_{ij} - \frac{1}{m} F^j = 0$

3) $\partial_t \epsilon + v^i \partial_i \epsilon + \frac{1}{\rho} \partial_i q^i + \frac{1}{\rho} P_{ij} \Lambda^{ij} = 0$

where

$P_{ij} := \rho \langle (u^i - v^i)(u^j - v^j) \rangle = \rho \langle u^i u^j \rangle - v^i v^j$: pressure tensor

$\epsilon := \frac{1}{2} \langle |\vec{u} - \vec{v}|^2 \rangle$: specific internal energy

$q^i := \frac{1}{2} \rho \langle (u^i - v^i) |\vec{u} - \vec{v}|^2 \rangle$: energy flux

$\Lambda_{ij} := \frac{1}{2} (\partial_i v^j + \partial_j v^i)$: strain tensor

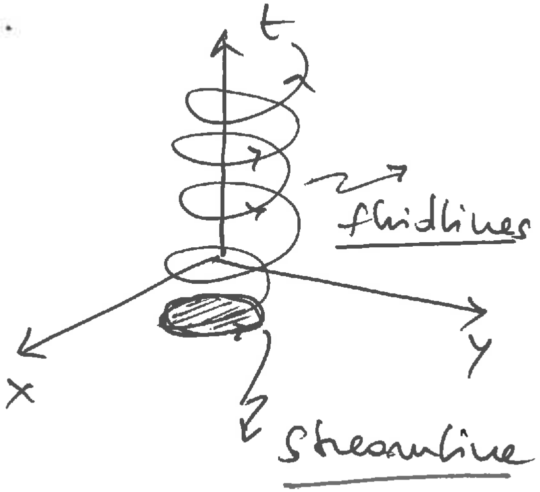
NOTES

- The conservation equations contain the differential operator:

$$\frac{D}{Dt} = \partial_t + \vec{v}\cdot\vec{\nabla}$$

: Lagrangian derivative (convective, material)

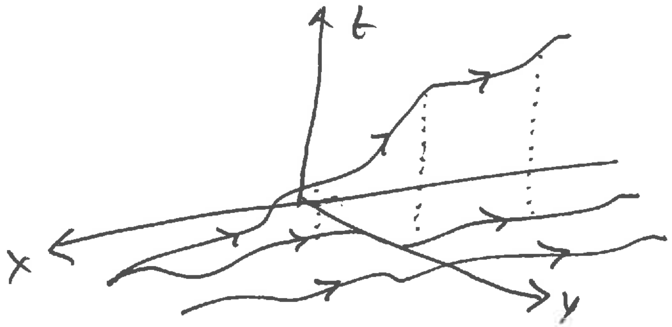
Integral curves of $\frac{D}{Dt} x^i = \partial_t x^i + v^j \partial_j x^i = v^i$ are the "fluidlines", ie trajectories in spacetime followed by fluid elements.



The spatial projections of fluid lines are called "streamlines"

Streamlines can be closed; fluid lines cannot.

Flow is said to be laminar if streamlines do not intersect.



- Moment equations require knowledge of equilibrium function and are therefore of limited use unless the distrib. function is known ($\langle\psi\rangle = \int \psi f d^3u, \dots$).

- Pressure tensor is nonzero even if fluid has zero bulk (average) velocity: $P_{ij} = \rho(\langle u^i u^j\rangle - v^i v^j) = \rho\langle u^i u^j\rangle$

  if $v^i = 0$

  Note that P_{ij} is sensitive to correlations and that $\langle u^i u^j\rangle \neq v^i v^j$ despite $\langle u^i\rangle = v^i$

  $$\langle u^i u^j\rangle = \frac{1}{n}\int f u^i u^j d^3u \neq \frac{1}{n}\int f u^i d^3u \cdot \frac{1}{n}\int f u^j d^3u.$$

- Similarly, the specific internal energy accounts for internal motions of the particles and is nonzero also for fluid at rest

  $$\epsilon = \langle|\vec{u} - \vec{v}|^2\rangle \overset{\vec{v}=0}{=} \langle|\vec{u}|^2\rangle = \langle\vec{u}^2\rangle \neq 0 \text{ in general.}$$

Let's concentrate on the 1st-moment equation and assume for simplicity there is no external force $F_j = 0$; (2) $\Leftrightarrow$

$$\partial_t \rho v_j + \partial_i (\rho v_i v_j) + \partial_i P_{ij} = 0 = \partial_t \rho v_j + \partial_i (\underbrace{\rho v_i v_j + P_{ij}}_{\text{single tensor}}) = 0$$

What does this equation express?

A conservation of the j-th component of linear momentum.

$\rho v_i v_j$ is: flux of j-th component of linear momentum in i-th direction; flux of momentum density tensor

In general $\partial_t \psi + \vec{\nabla}\cdot(\psi \vec{v}) = K$ expresses conservation of $\psi \cdot V$ (ψ is density)

$\hat{\psi}\vec{v} := \psi V \vec{v}$: flux of ψ $\Leftrightarrow$

$\psi := \frac{\hat{\psi}}{V}$

(change of $\hat{\psi}$ per unit time and area)

Proof

$$\int \partial_t \psi \, d^3x = \partial_t \psi V = -\int \underbrace{\psi \vec{v}}\cdot \vec{n} \, d^2s$$

$$\frac{\psi V}{\Delta t} = \psi v \Sigma \Leftrightarrow \frac{\psi (v \Delta t)\Sigma}{\Delta t} = \psi v \Sigma$$

Recap

- Moment equations. $\partial_t f + \vec{u} \cdot \vec{\nabla} f + \frac{\vec{F}}{m} \vec{\nabla}_{\vec{u}} f = \Gamma(f)$

  Assume that collisions do not vary some property of the system, ie that they leave ψ unchanged

  ψ is collisional invariant if $\int \psi \, \Gamma(f) \, d^3u = 0$

- Recalling $\langle \psi \rangle = \frac{1}{n} \int \psi f \, d^3u$, the Boltzmann eq. yields

$$\partial_t (n\langle \psi \rangle) + \partial_i (n \langle u_i \psi \rangle) - n \langle u^i \partial_i \psi \rangle - \frac{n}{m} \langle F^i \partial_{u_i} \psi \rangle - \frac{n}{m} \langle \partial_{u_i} F^i \psi \rangle = 0$$

moment equations: express conservation of ψ

$$n \langle u_i \psi \rangle =: \phi_i = \int u_i \psi f \, d^3u \quad : \text{ transport flux of } \psi$$

- ψ can be cast as a hierarchy of "moments" of the velocity

(i) $\psi = m$ first moment m: const

(ii) $\psi = m u_i$ second moment

(iii) $\psi = m u_i u_j$ third moment

$\vdots$

(i) $\Longrightarrow$ $\partial_t (n m) + \partial_i (n \langle m u^i \rangle) = \partial_t \rho + \partial_i (\rho v^i) = 0$

continuity eq. (conservation of mass)

(ii) $\Longrightarrow$ $\partial_t (\rho v^i) + \partial_j (\rho v^i v^j) - \frac{\rho}{m} F^i + \partial_j P^{ij} = 0$

conservation of momentum

$P_{ij} := \rho \langle (u_i - v_i)(u_j - v_j) \rangle = \rho (\langle u_i u_j \rangle - v_i v_j)$ pressure tensor

$P_{ij} \neq 0$ even if fluid is at rest $v_i = v_j = 0$

P_{ij} accounts for correlations

(iii) · Define $\psi := \frac{1}{2} m |\vec{u} - \vec{v}|^2$; $\epsilon := \frac{1}{2} \langle |\vec{u} - \vec{v}|^2 \rangle$: specif internal energy

$$\vec{q} := \frac{1}{2} \rho \langle (\vec{u} - \vec{v}) |\vec{u} - \vec{v}|^2 \rangle$$

third-moment equation is then

$$\partial_t(\rho\epsilon) + \partial_i(\rho\epsilon v^i) + \partial_i q^i + P_{ij} \Lambda^{ij} = 0$$

conservation of internal energy

$\Lambda_{ij} := \frac{1}{2} (\partial_i v_j + \partial_j v_i)$: strain tensor

- $D_t := \partial_t + \vec{v} \cdot \vec{\nabla}$: Lagrangian or convective derivative

$D_t x^i$: fluid lines, ie trajectories of fluid elements in spacetime

- In general the moment equations can be seen as conservation of fluxes:

$$\partial_t \psi + \vec{\nabla}(\psi \vec{v}) = S$$

$\psi : \rho, \rho v_i, \rho \epsilon,$; $\overline{\psi} := \psi \vec{v}$: flux of ψ density

$$\int dV \, \partial_t \psi \underset{\psi = \text{uniform}}{=} \frac{\Delta(\psi V)}{\Delta t} \underset{S=0}{=} - \int \vec{\nabla}(\psi \vec{v}) \, dV = - \int \psi \vec{v} . \vec{n} \, d\Sigma$$

$$= \psi v \Sigma \Rightarrow$$

$$\psi v = \frac{\Delta(\psi V)}{\Delta t \Sigma} \underset{\tilde{\psi} = \frac{\psi}{V}}{=} \frac{\Delta \tilde{\psi}}{\Delta t \Sigma}$$: change of ψ density ($\tilde{\psi}$) per unit time and area

Maxwell - Boltzmann distribution

we have seen that the H theorem expresses the eq. distrib. function as a conservation law, ie

$$\ln(f_0(u)) = \sum_j \chi_j(\vec{u}) \qquad \text{where } \chi_j: \text{ conserved quantities}$$

eg mass, energy, momentum

As a result, a possible ansatz for the equilibrium function is

$$(*) \quad \ln(f_0(\vec{u})) = -A(\vec{u}-\vec{u}_0)^2 + \ln C \qquad A, C \in \mathbb{R} \text{ const.}$$

$$\Longleftrightarrow$$

$$f_0(\vec{u}) = C\, e^{-A(\vec{u}-\vec{u}_0)^2} \qquad \vec{u}_0: \text{ velocity at equilibrium.}$$

Let's now consider a monoatomic[1] fluid of identical particles, with negligible external forces $\vec{F}=0$

[1] translational motion is only degree of freedom; molecules would also have rotation, vibrations

Using (*) and the fact that $n = \int f d^3u = \int c e^{-A(\vec{u}-\vec{u}_0)^2} d^3u$ one obtains

$$A = \frac{3}{4\epsilon} \qquad c = n\left(\frac{3}{4\pi\epsilon}\right)^{3/2} \qquad (\underline{\text{Exercise}})$$

after exploiting the identities

$$\int_{-\infty}^{\infty} e^{-A\vec{u}^2} d^3u = \frac{\pi^{3/2}}{A^{3/2}} ; \quad \int_{-\infty}^{\infty} \vec{u}\, e^{-A\vec{u}^2} d^3u = 0 ; \quad \int_{-\infty}^{\infty} \vec{u}^2 e^{-A\vec{u}^2} d^3u = \frac{3\pi^{3/2}}{2A^{5/2}}$$

Such a fluid has also experimental (phenomenological) expressions for the specific internal energy and pressure, ie

$$\epsilon = \frac{3}{2}\frac{K_B T}{m} ; \qquad p = \frac{2\epsilon}{3nm} = n K_B T$$

where T: temperature ; K_B: Boltzmann const $k_B = 1.38 \times 10^{-16}$ erg K^{-1}

from which we deduce the equilibrium distrib. function for a monoatomic fluid

$$\boxed{f_0(\vec{u}) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot e^{-\left(\frac{m(\vec{u}-\vec{v})^2}{2k_B T}\right)}} \qquad (**)$$

(**) : absolute Maxwellian or Maxwell-Boltzmann distribution

Now that we have an explicit expression, we can calculate a number of different quantities

$\vec{v}^2_{rms}$: root mean square velocity (2nd moment of distribution)

$$:= \langle \vec{u}^2 \rangle = \frac{1}{n}\int \vec{u}^2 f(u)\, d^3u \neq \vec{v}^2 := \langle \vec{u} \rangle^2 = \left(\frac{1}{n}\int u f\, d^3u\right)^2$$

Using the identities above, we can write

$2\epsilon = 3k_BT/m$ ①

$$\langle \vec{u}^2 \rangle = \frac{3k_BT}{m} + \langle \vec{u} \rangle^2 = \frac{3k_BT}{m} + \vec{v}^2$$

① $2\epsilon = \langle |\vec{u}-\vec{v}|^2 \rangle$

$= \langle u^2 \rangle - 2\langle u v \rangle + v^2$

$= \langle u^2 \rangle - 2v^2 + v^2 = \langle u^2 \rangle - v^2$

which we can invert to obtain

$$T = \frac{m}{3k_B}\left(\langle \vec{u}^2\rangle - \langle \vec{u}\rangle^2\right) = \frac{m}{3k_B}\left(\vec{v}_{rms}^{\,2} - \vec{v}^2\right) \quad (*)$$

Expression (*) clarifies that the temperature is a measure of the kinetic energy (in root mean square sense) of the system and this is nonzero even for a system with a zero average velocity ($\vec{v}=0$). Indeed, the measure of the temperature in comoving frame removes this contribution if present. This remarks the importance of working in a frame comoving with the fluid to measure its thermodynamic properties.

We can also calculate other quantities if we make use of the distribution function in terms of the velocity norm $u := |\vec{u}|$

Recall that for any distribution function

$$\int_{-\infty}^{\infty} f(\vec{u})\, d^3u = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} u^2 f(u)\, du = 4\pi \int_0^{\infty} u^2 f(u)\, du$$

we can calculate $f_0(u)$ for a system with zero translational velocity with $f_0(\vec{u})\, d^3u = f_0(u) u^2 du\, d\theta\, d\phi$ and where

$$4\pi u^2 f_0(u) = 4\pi n u^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\left(\frac{mu^2}{2k_BT}\right)} \qquad (\ast\ast)$$

Using $(\ast\ast)$ we can calculate the entropy of the system as

$N = nV$

$$S := -k_B V H_0 \underset{\text{def}}{=} -k_B \frac{N}{n} \int f_0 \ln(f_0)\, d^3u$$

$$\overset{\odot}{=} \frac{3}{2} N k_B \ln\left(p V^{5/3}\right) + \text{const}$$

exercise

similarly, we can compute the average speed (scalar field)

$$v := \langle u \rangle = \frac{4\pi}{n} \int_0^\infty u^3 f_0 \, du = \left(\frac{8 k_B T}{\pi m} \right)^{1/2}$$

or the most probable speed $u_p := \frac{\partial f_0}{\partial u} = 0$ maximum of distribution function

$$= \left(\frac{2 k_B T}{m} \right)^{1/2}$$

So we have introduced many different measures of the velocity of the system:

$\vec{v} := \langle \vec{u} \rangle$: macroscopic mean velocity (vector field)

$v := \langle u \rangle$: average speed (scalar field)

$\vec{v}_{rms} := \sqrt{\langle \vec{u}^2 \rangle} := \left[\frac{1}{n} \int u^2 f \, d^3u \right]^{1/2}$: root mean square velocity

u_p : most probable velocity

They are all different but, not surprisingly, they are all $\propto \sqrt{\frac{k_B T}{m}}$

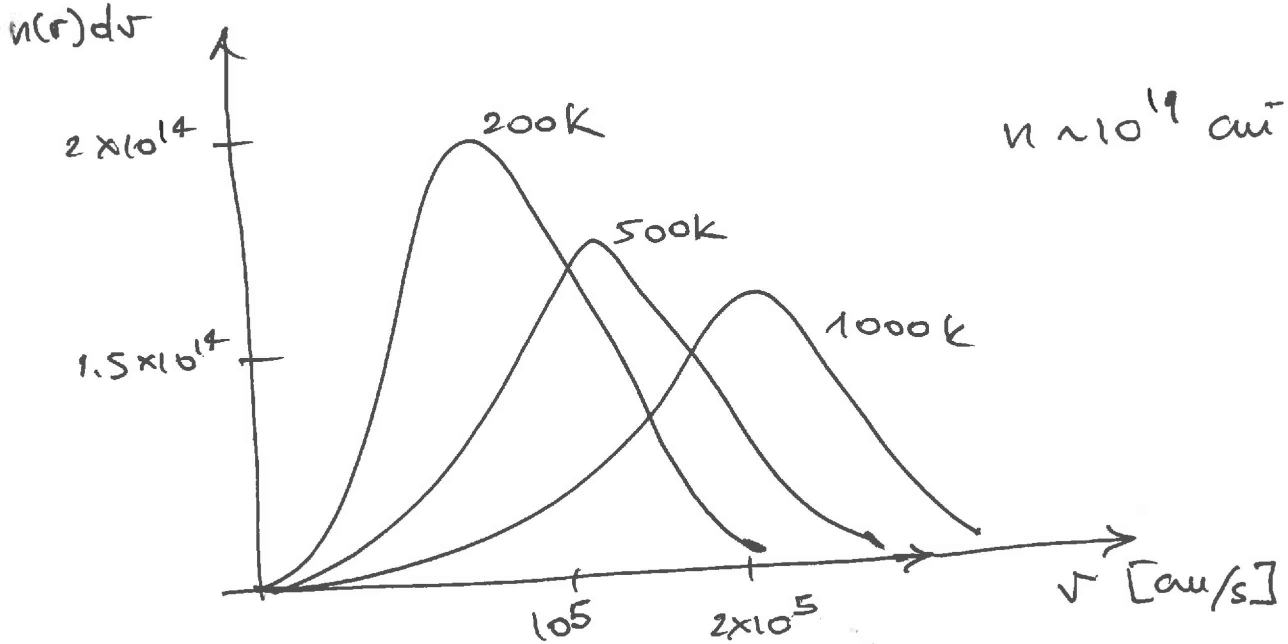
Example

- Fluid of O_2 at room temperature (273 K)

$$\sqrt{\frac{k_B T}{m}} \sim 10^5 \, \mathrm{cm\,s^{-1}} \sim 10^{-5} c$$

- What temperatures are necessary to reach relativistic speeds?

$$k_B T \sim mc^2 \Rightarrow T \sim 10^{13} \, K \quad \text{(realistic?)}$$



$n \sim 10^{19} \, \mathrm{cm}^{-3}$ $\quad f_0(u) = 4\pi n u^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{mu^2}{2k_B T}}$

zero-th order approximation: perfect fluids

We have seen that the moment eqs are useful only when a distribution function is selected.

A well-known case is obtained when the dist. fct. is the Maxwell-Boltzmann: $f(t,\vec{x},\vec{u}) = f_0(\vec{u})\Big|_{MB}$ and is referred to as the zero-th order approximation.

Two assumptions are required for its validity

- $Kn \ll 1$; ie $l_{mfp} \ll L$: scale of the system (collisional fluid)
- as result of frequent collisions the system will reach the Maxwell-Boltzmann equilibrium on a timescale

$$\tau = \frac{l_{mfp}}{v} \qquad v: \text{average particle speed}$$

$l_{mfp} \sim 100\,nm = 10^2 \cdot 10^{-9}\, 10^{-2}\,cm = 10^{-9}\,cm$ $\qquad \tau = 10^{-9}\,cm / 10^5\,cm\,s^{-1} \sim 10^{-14}\,s$!

These are the conditions for the establishment of the LTE: local thermodynamic equilibrium. Note that this equilibrium is local and the distribution function can depend on T, n which vary on scales of order L.

To calculate the moment eqs we need to compute the pressure tensor

$$P_{ij} := \rho\langle (u_i - v_i)(u_j - v_j)\rangle = \rho\left(\frac{m}{2\pi k_B T}\right)^{3/2}\int w_i\, w_j\, e^{-\left(\frac{m w^2}{2 k_B T}\right)}\, d^3 w$$

$$= \rho\langle w_i\, w_j\rangle$$

$\vec{w} = \vec{u} - \vec{v}$

$w = |\vec{w}|$

$$\vec{q} := \frac{1}{2}\rho\langle \vec{w}\,|\vec{w}|^2\rangle = \frac{\rho}{2}\left(\frac{m}{2\pi k_B T}\right)^{3/2}\int \vec{w}\, w^2\, e^{-\left(\frac{m w^2}{2 k_B T}\right)}\, d^3 w$$

Since $\int w^n e^{-\frac{w^2}{k}} d^3w = 0$ if n is odd $\Rightarrow$

* $\vec{q} = 0$ for a Maxwell-Boltzmann distribution

* $P_{ij} = p\delta_{ij}$ where $p := \frac{1}{3}\rho \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int w^2 e^{-\left(\frac{w^2}{2k_BT}\right)} d^3w$

(Exercise) $= n k_B T$: isotropic pressure

Both are important results worth remarking:
for a MB distribution the fluid has zero energy (heat) flux and isotropic pressure

$$P_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix};$$

the fact that non-diagonal terms of the pressure tensor are zero implies that there is no flux of momentum j in the direction i

Let's go back to the moment equations. We have seen that the Boltzmann equation in the presence of collisionally invariant quantities yields:

$$\begin{cases} \partial_t \rho + \partial_i (\rho v^i) = 0 & (1) \\ \partial_t (\rho v_j) + \partial_i (\rho v^i v_j) + \partial_i P^{ij} - \frac{\rho}{m} F_j = 0 & (2) \\ \partial_t (\rho \epsilon) + \partial_i (\rho \epsilon v^i) + \partial_i q^i + P_{ij} \Lambda^{ij} = 0 & (3) \end{cases}$$

: moment equations

which can also be written as a set of two equations only

$$\begin{cases} \partial_t v_j + v^i \partial_i v_j + \frac{1}{\rho} \partial_i P^{ij} - \frac{1}{m} F_j = 0 \\ \partial_t \epsilon + v^i \partial_i \epsilon + \frac{1}{\rho} \partial_i q^i + \frac{1}{\rho} P_{ij} \Lambda^{ij} = 0 \end{cases}$$

$P_{ij} := \rho (\langle u_i u_j \rangle - v_i v_j)$

$\epsilon := \frac{1}{2} \langle |\vec{u} - \vec{v}|^2 \rangle$

In the zeroth-order approximation : $\partial_i P^{ij} = \partial_i (p\delta^{ij}) = \partial_i p$

$$\Lambda_{ij} P^{ij} = \frac{1}{2} p \delta^{ij} (\partial_i v_j + \partial_j v_i)$$
$$= \frac{2}{2} p \partial_i v^i = p \partial_i v^i$$

eqs (1)-(3) reduce to

$$\begin{cases} \partial_t \rho + \partial_i (\rho v^i) = 0 & (4) \\ \partial_t v_j + v^i \partial_i v_j = -\frac{1}{\rho} \partial_j p + \frac{\rho}{m} F_j & (5) \\ \rho (\partial_t \epsilon + v^i \partial_i \epsilon) + p \partial_i v^i = 0 & (6) \end{cases}$$

Introduce now the total energy density

$$e_N := \rho\epsilon + \frac{1}{2} \rho \vec{v}^2$$

(7)

Eq. (6) becomes $\partial_t \left(\frac{1}{2}\rho v_i v^i + \rho\epsilon\right) + \partial_i \left[\left(\frac{1}{2}\rho v_i v^i + \rho\epsilon + p\right) v^i\right] = \frac{\rho}{m} F_i v^i$

conservation equation for the energy density so that if $\frac{1}{2}\rho v^i v^i + \rho\epsilon$ is the energy density,

$\left(\frac{1}{2}\rho v^i v^i + \rho\epsilon + p\right) v^i$ energy density flux : flux of energy per unit time and area.

Equations (4), (5), (7) are the classical (Newtonian) hydrodynamic eqs. and express conservation of mass, momentum and energy.

Mathematically, they are 5 equations in 6 unknowns

ρ, v^i, ϵ, p

A closure is needed to complete the system : equation of state $\boxed{p = p(\rho, \epsilon, \dots)}$

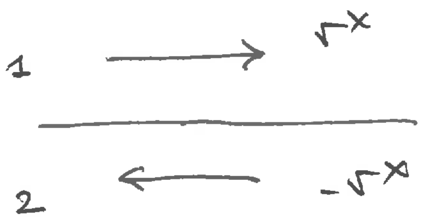
This information is not contained in the moment eqs. and is the result of a closure at a given order.

Fluids for which $P_{ij} = \delta_{ij}\, p$ and $q_i = 0$ are called <u>perfect fluids</u> and we will study more extensively later. <u>Remember</u> $\boxed{\text{ZEROTH-ORDER APPROX} \Longrightarrow \text{PERFECT FLUIDS}}$

What does P_{ij} = isotropic pressure tensor mean?

Consider simple 1-D flow in two dimensions ie

$\vec{v} = (v^x, 0)$ and v^x const in x : $\partial_x v^x = 0$



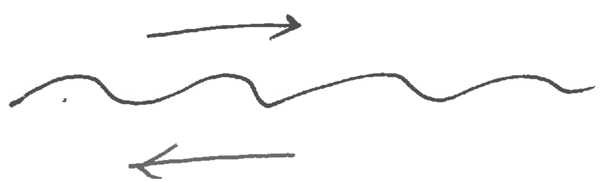
$$\partial_t \rho v_j \propto \partial_i P_{ij} \iff$$

$$\partial_t \rho v_x \propto \partial_i P_{xi} = \partial_y P_{xy} = 0 \quad \text{if } P_{ij} = p\delta_{ij}$$

so the flow remains laminar and stationary. However, if $P_{ij} \neq p\delta_{ij}$

eg $P_{xy} \neq 0$ then $\partial_t \rho v_x \propto \partial_y P_{xy} \neq 0$

momentum flux in direction orthogonal to flow is possible



Kelvin Helmholtz instability

Recap

- Moment equations need knowledge of a distribution function for definition of moments $\Psi := \frac{1}{n}\int \Psi f\, d^3u$

- Distribution function should be of equilibrium, $\Gamma(f_0)=0$

- H-theorem expresses conservation of quantities before - after collisions (eg mass, momentum, energy)

$$\Rightarrow f_0 = C\, e^{-A(\vec{u}-\vec{u}_0)^2} \qquad A, C \in \mathbb{R}$$

monoatomic gas
Absolute
$\boxed{\text{Maxwell-Boltzmann dist.}}$

$$\Rightarrow f_0(\vec{u}) = n \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\left(\frac{m(\vec{u}-\vec{v})^2}{2k_BT}\right)}$$

T: temperature
p: pressure

$$\epsilon = \frac{3}{2}\frac{k_BT}{m} \qquad p = \frac{2\epsilon}{3nm} = nk_BT \quad \Big\} \text{ phenomenological laws}$$

$$\langle \vec{u}^2 \rangle := \frac{1}{n}\int \vec{u}^2 f_0(\vec{u})\, d^3u = \frac{3k_BT}{m} + \vec{v}^2 \qquad \vec{v} := \langle \vec{u} \rangle$$

so that $$T = \frac{m}{3k_B}\left(\langle \vec{u}^2\rangle - \vec{v}^2\right)$$

- Temperature is a measure of mean square velocity and hence of the kinetic energy; microscopic interpretation of temperature.

- Only reasonable measure of T (and of other fluid properties) is locally comoving frame.

- $S = -k_B V H_0 = \frac{3}{2} N k_B \ln\left(p V^{5/3}\right) + \text{const}$

- $\langle u\rangle = v = \left(\frac{8k_BT}{\pi m}\right)^{1/2}$; u_p @ $\left.\frac{\partial f_0}{\partial u}\right|_{u_p} = 0 \Rightarrow u_p = \left(\frac{2k_BT}{m}\right)^{1/2}$

In other words, many different ways of looking at average velocities; all are proportional to $T^{1/2}$

• zeroth-order approximation

$$f(t,\vec{x},\vec{u}) = f_0(\vec{u})\Big|_{MB}$$

note lack of dependence on t and $\vec{x}$ (absolute MB)

Then the pressure tensor $P_{ij} := \rho(\langle u_i u_j\rangle - v_i v_j)$

$$= \rho\left(\frac{m}{2k_BT}\right)^{3/2}\int w_i w_j \, e^{-\left(\frac{mw^2}{2k_BT}\right)} d^3w = p\delta_{ij} \text{: diagonal}$$

energy flux

$$\vec{q} := \frac{1}{2}\rho\langle \vec{w}\,|\vec{w}|^2\rangle = \frac{\rho}{2}\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int \vec{w}\, w^2 e^{-\left(\frac{mw^2}{2k_BT}\right)} d^3w \overset{!}{=} 0$$

• A fluid within the zeroth-order approximation has zero energy flux and diagonal pressure tensor: p is isotropic pressure; such a fluid is said to be <u>perfect</u>

- In this case the moment equations have

$$\partial_i q^i = 0 \; ; \quad \partial_i P^{ij} = \partial_i p \; ; \quad \Lambda_{ij} P^{ij} = p \, \partial_i v^i$$

and reduce to

$$\begin{cases} \partial_t \rho + \partial_i (\rho v^i) = 0 \\ \partial_t v_j + v^i \partial_i v_j = -\frac{1}{\rho} \partial_j p + \frac{\rho}{m} F_j \\ \partial_t \left(\frac{1}{2} \rho v^i v_i + \rho \epsilon \right) + \partial_i \left[\left(\frac{1}{2} \rho v_i v^i + \rho \epsilon + p \right) v^i \right] = \frac{\rho}{m} F_i v^i \end{cases}$$

Hydrodynamical equations (classical)

5 eqs in 6 unknown : ρ, ϵ, p, v^i

$\Rightarrow$ EOS $p = p(\rho, \epsilon, \dots)$ is needed

- Perfect fluids are simplest approximation; anisotropic flux of momentum is very important: need more general description

To account for these effects we need to move away from the zeroth-order approximation and consider more general distribution functions

$$g(t,\vec{x},\vec{u}) = f(t,\vec{x},\vec{u}) - f_0(\vec{u})$$

approximated dist. funct to be used for averages

true distribution function

local Maxwell-Boltzmann

The approximated dist. function is obtained after expanding $f(t,\vec{x},\vec{u})$ in powers of l_{mfp}/L in the so called Chapman Enskog expansions. I will not go through the derivation but just report the final result: Navier-Stokes equations.

First we need to extend the pressure tensor to include non-diagonal terms:

$$P_{ij} := p\delta_{ij} - S_{ij}$$

symmetric tensor
(viscous) stress tensor

$$S_{ij} := \eta\left[\partial_i v^j + \partial_j v^i - \frac{2}{3}\partial_k v^k \delta_{ij}\right] + \zeta \underbrace{\partial_k v^k}_{\Theta} \delta_{ij}$$

$$= 2\eta \Lambda_{ij} + \left(\zeta - \frac{2}{3}\eta\right)\Theta\,\delta_{ij}$$

strain tensor

where $\Theta := \partial_k v^k$ is called fluid expansion (scalar)

η : coefficient of shear viscosity

ξ : " " bulk viscosity

$\xi - \frac{2}{3}\eta$: dilatational viscosity.

in comoving frame

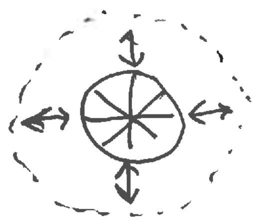
Let's have a closer look at these tensors

Θ measures the variations in density of the fluid; to see this, start from the continuity eq

$$\partial_t \rho + \vec{\nabla}\cdot(\rho \vec{v}) = 0 = \partial_t \rho + \rho\, \partial_i v^i + v^i \partial_i \rho \Rightarrow$$

$$\partial_i v^i = \Theta = -\frac{1}{\rho}\left(\partial_t \rho + v^i \partial_i \rho\right) = -\frac{1}{\rho} D_t \rho$$

Lagrangian derivative



Θ measures these changes of volume but not of shape

Hence $\Theta = 0 \iff D_t \rho = 0$ the fluid is <u>incompressible</u>

$\Theta \simeq 0$ for a liquid, eg water. When $\Theta = 0$, the hydrodynamic equations simplify considerably and can even be cast into an elliptic form (more later).

σ_{ij} : <u>stress tensor</u>

$$\sigma_{ij} = 2\eta \Lambda_{ij} + \left(\zeta - \tfrac{2}{3}\eta\right)\Theta\delta_{ij}$$

$$= 2\eta\left(\Lambda_{ij} - \tfrac{1}{3}\Theta\delta_{ij}\right) + \zeta\Theta\delta_{ij}$$

$$= 2\eta\,\Lambda_{ij}^{TF} + \zeta\Theta\delta_{ij}$$

Λ_{ij}^{TF} measures changes of shape but not of volume

$\Lambda_{ij}^{TF} := \Lambda_{ij} - \frac{1}{3}\Theta\delta_{ij}$: <u>shear tensor</u>, ie strain tensor without its trace (expansion) we can now write the Navier Stokes eqs.

$$\partial_t \rho + \partial_i (\rho v^i) = 0 \tag{8}$$

$$\partial_t v_j + v_i \partial_i v^j + \frac{1}{\rho}\Big[\partial_i p - \partial_i \big(\underbrace{2\eta \Lambda_{ij}^{TF}}_{\text{shear vis. term}} + \overbrace{\zeta \Theta \delta_{ij}}^{\text{bulk viscosity term}}\big)\Big] - \frac{F_j}{m} = 0 \tag{9}$$

$$\partial_t \epsilon + v^i \partial_i \epsilon + \frac{1}{\rho}\partial_i q^i + \frac{1}{\rho} P_{ij} \Lambda^{ij} =$$

$P_{ij}\Lambda^{ij} = (p\delta_{ij} - S_{ij})\Lambda^{ij} = p\Theta - S_{ij}\Lambda^{ij}$

$S_{ij}\Lambda^{ij} = (2\eta \Lambda_{ij} + (\zeta - \frac{2}{3}\eta)\Theta\delta_{ij})\Lambda^{ij}$

$= 2\eta \Lambda_{ij}\Lambda^{ij} - (\zeta - \frac{2}{3}\eta)\Theta^2$

$$\partial_t \epsilon + v^i \partial_i \epsilon + \frac{1}{\rho}\partial_i q^i + \frac{p}{\rho}\Theta \underbrace{- \frac{2\eta}{\rho}\Lambda_{ij}\Lambda^{ij} - \frac{1}{\rho}\left(\zeta - \frac{2}{3}\eta\right)\Theta^2}_{\text{viscous energy losses}} = 0 \tag{10}$$

(8), (9), (10) are the Navier Stokes eqs and again are 5 eqs for ρ, v^i, ϵ.

Notes

- Viscous corrections to momentum conservations

$$\partial_t v_j \propto \partial_i \eta \Lambda_{ij}^{TF} \propto \eta \partial_i \partial_j v^i$$: second derivatives of velocity

This is a general feature: viscous dissipative effects in lin. mom. are proportional to higher-order velocity spatial derivatives $\sim V^{-2}/L^2$

Similarly for energy conservation

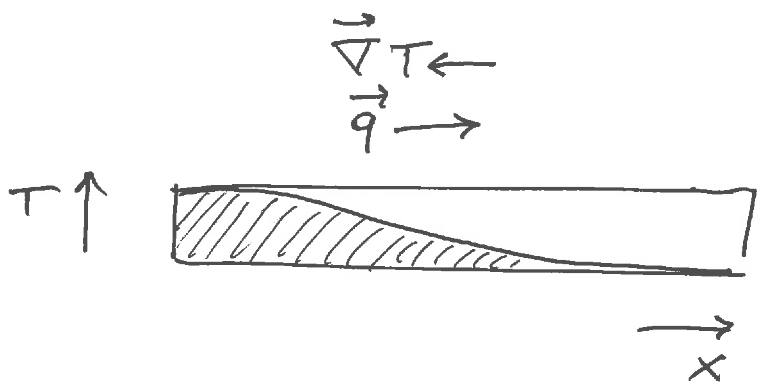
$$\partial_t \epsilon \propto 2\eta \Lambda_{ij} \Lambda^{ij} \propto \eta (\partial_i v^i)^2$$

$$\propto \xi \Theta^2 \qquad \propto \xi (\partial_i v^i)^2$$

viscous dissipative effects in internal energy are proportional to square of first derivatives $\sim \frac{V^{-2}}{L^2}$

Heat flux vector is seen (experimentally) to follow the Fourier law

$$\vec{q} = -K \vec{\nabla} T \qquad K\text{: thermal conductivity}$$



heat flux goes from large to small temperatures.

If fluid is static ($v^i = 0 = \Lambda_{ij}$) and incompressible ($\theta = 0$), the energy equation (10) can be written as $\left(\epsilon = \frac{3}{2}\frac{k_B T}{m}\right)$

$$\partial_t T = \frac{2m}{3k_B\rho}\vec{\nabla}\cdot\left(K\vec{\nabla}T\right) \overset{K=\text{const}}{=} + \frac{2mK}{3k_B\rho}\nabla^2 T \qquad (11)$$

This is the "diffusion" equation; it's a parabolic eq. we will discuss also later on; for the time being remember

$$\partial_t T \propto \nabla^2 T$$

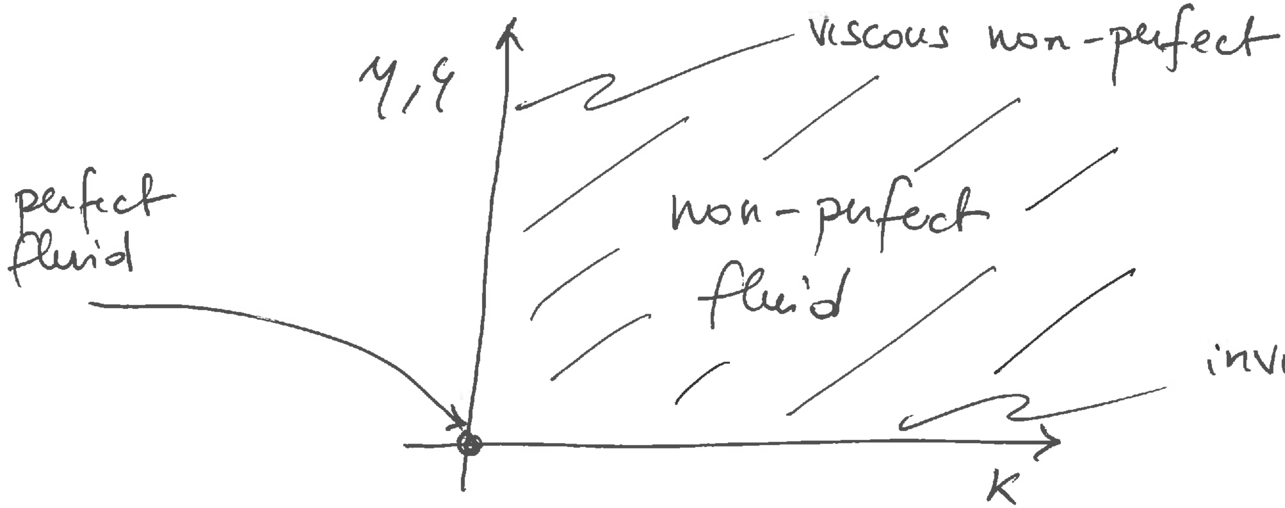
We can now classify fluids according to their properties wrt viscosity and heat conduction

$\kappa = 0 \wedge \eta = 0 \wedge \zeta = 0$: perfect fluid

$\kappa \neq 0 \vee \eta \neq 0 \vee \zeta \neq 0$: non-perfect fluid

$\kappa \neq 0 \quad \eta = 0 \wedge \zeta = 0$: inviscid non-perfect fluid

$\kappa = 0 \quad \eta \neq 0 \vee \zeta \neq 0$: viscous non-perfect fluid



$\boxed{\text{perfect fluids are a very special case of generic fluids}}$

Relativistic Boltzmann Equation

$\vec{x}, \vec{u} \longrightarrow x^\mu, \; p^\mu = m c u^\mu$: coordinates and 4-momentum in phase space

$= (p^0, p^i)$

u^μ: particle four velocity

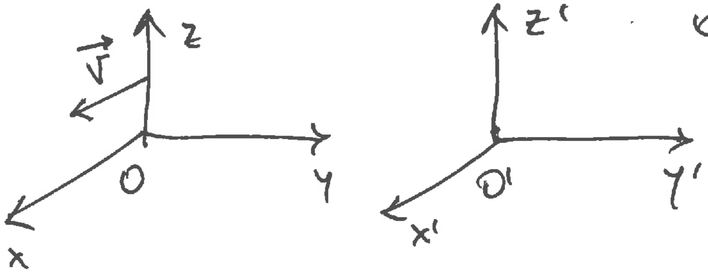
NOTE: this is special relativity (SR), ie $g_{\mu\nu} = \delta_{\mu\nu}$ but $v \sim c$

$u^\mu u_\mu = -1 \Rightarrow p^\mu p_\mu = -m^2 c^2$

As for the classical kinetic theory

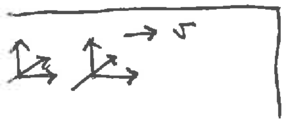
$f \, d^3x \, d^3p$: no of particles in volume $d^3x \, d^3p$ in phase space.

Complication! the volume is not Lorentz invariant because of Lorentz contraction; consider observer O′ comoving with particle and O moving at $\vec{v}$ along x wrt to O′



$$d^3x' = W \, d^3x$$

W: Lorentz factor $\quad W = (1-v^2)^{-1/2}$



However, the ratio $\frac{d^3p}{p_0}$ is Lorentz invariant, ie

$$\frac{d^3p}{p_0} = \frac{d^3p'}{p_{0'}} \qquad (1)$$

(Exercise)

Using the standard Lorentz transformation matrix $\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} W & -WV & 0 & 0 \\ -WV & W & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$p_\mu = \Lambda^{\nu'}{}_{\mu}\, p_{\nu'}$$

we obtain ($p_{x'} = 0$ since O' is comoving)

$$p_{0'} = p_0 / W \quad \Longrightarrow \quad d^3p' \overset{(1)}{=} p_{0'} \frac{d^3p}{p_0} = \frac{p_0}{W} \frac{d^3p}{p_0} = \frac{d^3p}{W}$$

so that

$$d^3x'\, d^3p' = W dx^3 \cdot \frac{d^3p}{W} = d^3x\, d^3p$$

ie the product $d^3x\, d^3p$ is Lorentz invariant despite neither d^3x nor d^3p is invariant.

of course the two observers must agree on the no of particles in the volume, ie ($\underline{A}$ is four-vector)

$$f'(\underline{x}', \underline{p}')\, d^3x'\, d^3p' = f(\underline{x}, \underline{p})\, d^3x\, dp$$

$$\Rightarrow \quad \boxed{f'(\underline{x}', \underline{p}') = f(\underline{x}, \underline{p})} \Leftrightarrow$$

the dist. function itself is Lorentz invariant.

The <u>relativistic</u> <u>Boltzmann eq.</u> can be obtained following the same logic and we conclude

$$\boxed{p^\mu \frac{\partial}{\partial x^\mu} f + m \frac{\partial}{\partial p^\mu}(F^\mu f) = \Pi(f)} \qquad (*)$$

$\underline{F}$: four-force acting on particle

$\Pi(f)$: collision integral $\quad \Pi(f) = \left(\frac{\partial f}{\partial t}\right)_{coll} := \int \frac{d^3 p_2}{(p_2)^0} \int d\Omega\, \sigma\, k\, (f_2' f_1' - f_2 f_1)$

where $K := \left((p_1)^\alpha (p_2)_\alpha - M^2 c^2 \right)^{1/2}$.

If $\Pi(f) = 0$, the relativistic collisionless Boltzmann eq. can be seen as the conservation of f along the particle trajectories in phase space

$$\frac{d}{d\lambda} f\left(x^\mu(\lambda), p^\mu(\lambda)\right) = 0$$

λ — affine parameter along particle trajectory

As a result, the density of points in phase space is constant and move like an incompressible fluid (Liouville theorem, obviously also classic).

Relativistic transport fluxes

Recall classical concepts first.

$$\langle\psi\rangle = \frac{1}{n}\int \psi f\, d^3u \; ; \quad \phi_i = n\langle u_i \psi\rangle = \int u_i \psi\, d^3u \quad : \text{transport flux of } \psi$$

In relativistic context, the transport flux of a generic rank-k tensor $\underline{G}$, $\phi^\mu(\underline{G})$ is defined as

(k + 1) indeces

rank (k+1) tensor

$$\phi^{\mu\alpha_1\ldots\alpha_k}(\underline{G}) := \int G^{\alpha_1\ldots\alpha_k}\, p^\mu \frac{d^3p}{p^0} . \qquad (**)$$

Question: What are the integration limits in the integral above?

In SR the velocity is upper limited but not the momentum

$$p^i = m v^i W \qquad W \in [1, \infty) \Rightarrow p^i \to \pm\infty \text{ as } v^i \to \pm c$$

$$\Rightarrow \int \frac{d^3p}{p_0} \longleftrightarrow \int_{-\infty}^{+\infty} \frac{d^3p}{p_0}$$

Let's consider the first transport fluxes.

(1) $K=0$; ie $G = c\,\mathbb{1}$ where $\mathbb{1}$ unit tensor. Then

$$\phi^\mu = N^\mu := c\int p^\mu f \frac{d^3p}{p^0}$$: number-density current four vector : first-moment

$$N^0 = c\int p^0 f \frac{d^3p}{p^0} = c\int f d^3p = cn \; ; \quad \boxed{N^0 = cn}$$

Note that the definition of n is different from the classical one because we are integrating in momentum space here ($n = \int f d^3u$) ; as a result, the dimensions of f in SR are different.

$$\int f_{SR} \frac{d^3p}{m^3} = \int f_N d^3u = n \Rightarrow \left[\frac{f_{SR}}{m^3}\right] = \left[f_N\right] = L^{-3}$$

$$\left[f_{SR}\right] = L^0$$

Similarly

$$N^i = c\int p^i f \frac{d^3p}{p^0} = \cancel{c}\int W\cancel{m}v^i f \frac{d^3p}{\cancel{Wmc}} = \int v^i f\, d^3p$$

$$p^0 = \frac{E}{c} = \frac{Wmc^2}{c} = Wmc$$

In other words, N^i is the flux of particle number per unit time and unit area along i-direction.

Once I have N^μ, I can construct the rest-mass density current

$$J^\mu := mN^\mu = mc\int p^\mu f \frac{d^3p}{p^0} \qquad \square$$

(2) $k=1$: <u>second moment</u>, rank-2 tensor

$\underline{G} = c\,\underline{p}$

$$\phi^{\mu\nu} = T^{\mu\nu} := c \int p^\mu p^\nu f \frac{d^3p}{p^0}$$

: flux of momentum μ in direction ν (cf momentum density tensor $\rho v_i v_j + P_{ij}$)

$T^{\mu\nu}$: energy-momentum tensor

(3) $k=2$: third moment

$\underline{G} = c\,\underline{p} \otimes \underline{p}$

$$F^{\mu\nu\sigma} := c \int p^\mu p^\nu p^\sigma f \frac{d^3p}{p^0}$$

one can continue in this hierarchy, although the physical meaning of moments higher than second becomes difficult

Relativistic moment eqs.

Recap classical treatment: Boltzmann eq ⇒ integral over d^3u of collisional invariant ψ ⇒ moment eqs.

$$\int \Gamma(u)\, \psi \, d^3u = 0$$

Do the same here but we have $\psi \longrightarrow G^{\alpha_1 \dots \alpha_k}$; $d^3u \longrightarrow d^3p/p_0$

$$\int \Pi(f)\, G^{\alpha_1 \dots \alpha_k} \frac{d^3p}{p_0} = 0 \qquad G^{\alpha_1 \dots \alpha_k} \text{ collisional invariant}$$

and Boltzmann eq ⟺ $\left[p^\mu \frac{\partial f}{\partial x^\mu} + m \frac{\partial (f F^\mu)}{\partial p^\mu} = \Pi(f) \right]$

$$\int G^{\alpha_1 \dots \alpha_k} \left(p^\mu \frac{\partial f}{\partial x^\mu} + m F^\mu \frac{\partial f}{\partial p^\mu} + m f \frac{\partial F^\mu}{\partial p^\mu} \right) \frac{d^3p}{p_0} = 0$$

Recap

- When using $f(t,\vec{x},\vec{u}) = f_0(\vec{u})$: Maxwell-Boltzmann distribution
  We obtain perfect fluids $\Longleftrightarrow$ fluids for which $\vec{q}=0$; $P_{ij} = p\delta_{ij}$

- Deviations away from Maxwell Boltzmann distribution are necessary to describe non-perfect fluids

strain tensor

$$P_{ij} = p\delta_{ij} - 2\eta\Lambda_{ij} - \left(\zeta - \frac{2}{3}\eta\right)\Theta\delta_{ij} = p\delta_{ij} - \underbrace{S_{ij}}_{\text{stress tensor}}$$

η : shear viscosity ; ζ : bulk viscosity ; $\Theta := \partial_i v^i$: expansion

$$= p\delta_{ij} - \left[2\Lambda^{TF}_{ij} + \zeta\Theta\delta_{ij}\right]$$

P_{ij} is no longer diagonal

shear tensor

- Conservation of mass, momentum and energy are then expressed by the Navier-Stokes eqs.

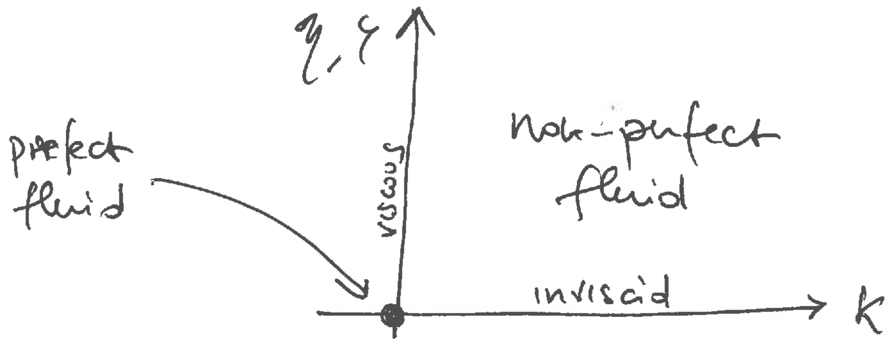
$$\partial_t \rho + \partial_i (\rho v^i) = 0$$

$$\partial_t v_j + v^i \partial_i v^j + \frac{1}{\rho}\left[\partial_i p - \partial^i \left(\eta \Lambda_{ij}^{TF} + \zeta \Theta \delta_{ij}\right)\right] - \frac{F_j}{m} = 0$$

$$\partial_t \epsilon + v^i \partial_i \epsilon + \frac{1}{\rho}\partial_i q^i + \frac{p}{\rho}\Theta - \frac{2\eta}{\rho}\Lambda_{ij}\Lambda^{ij} - \frac{1}{\rho}\left(\zeta - \frac{2}{3}\eta\right)\Theta^2 = 0$$

- viscous changes to velocity (acceleration) are $\propto \partial_i^2 v^i$

- viscous changes to internal energy are $\propto (\partial_i v^i)^2$



$$\partial_t T \propto -K \nabla^2 T$$

- Relativistic Boltzmann eq. : $\vec{x}, \vec{u} \longrightarrow \underline{x}, \underline{p}$: 4-vectors

d^3x, d^3p not Lorentz invariant

However $d^3x\, d^3p$ is Lorentz invariant; $\frac{d^3p}{p^0}$ is also

$f'(\underline{x}', p')\, d^3x'\, d^3p' = f(\underline{x}, p)\, d^3x\, d^3p$: no. particles seen by two different observers

$\Rightarrow$ $f'(\underline{x}', \underline{p}') = f(\underline{x}, p)$: distrib. function is LI.

Its conservation is expressed as

relativistic Boltzmann eq.

$$p^\mu \frac{\partial f}{\partial x^\mu} + m \frac{\partial}{\partial p^\mu}(F^\mu f) = \Pi(f)$$

collision integral

$$\Pi(f) = (\kappa f)_{coll} = \int \frac{d^3p_2}{(p_2)^0} \int d\Omega\, \sigma\, K\, (f_2' f_1' - f_2 f_1)$$

$$K := \left((p_1)^\alpha (p_2)_\alpha - m^2c^2\right)^{1/2}$$

- ## Relativistic transport fluxes

$\langle \psi \rangle := \frac{1}{n} \int \psi f \, d^3u \qquad \phi_i := n \langle u^i \psi \rangle = \int u^i \psi \, d^3u$ transport flux of ψ

$$\phi^\mu(\underline{G}) = \phi^{\mu \alpha_1 \dots \alpha_k} = \int G^{\alpha_1 \dots \alpha_k} p^\mu \frac{d^3p}{p^0}$$

tensor of rank k

$\underline{G} = c \, \underline{\mathbb{I}}$

- $k=0$ $\quad \phi^\mu = N^\mu := c \int p^\mu f \frac{d^3p}{p^0} \quad ; \quad N^\mu = (N^0, N^i) = c(n, N^i)$

$N^i = \int v^i f \, d^3p$: flux of particles no. along i-direction

$J^\mu := m N^\mu = mc \int p^\mu f \frac{d^3p}{p^0}$: rest-mass density current

- $k=1$

$\phi^{\mu\nu} = T^{\mu\nu} = c \int p^\mu p^\nu f \frac{d^3p}{p^0}$: flux of μ-mom. in ν-direction

$$F^{\mu\nu\sigma} := c \int p^\mu p^\nu p^\sigma f \frac{d^3p}{p^0}$$

: third-moment; no obvious physical interpretation

- the Boltzmann eq. becomes:

$$\int G^{\alpha_1 \dots \alpha_k} \left(p^\mu \frac{\partial f}{\partial x^\mu} + m F^\mu \frac{\partial f}{\partial p^\mu} + m f \frac{\partial F^\mu}{\partial p^\mu} \right) \frac{d^3p}{p^0} = 0$$

Assuming $\underline{F}$ does not depend on $\underline{p}$ (very reasonable!) the Boltzmann eq becomes

$$\int G^{\alpha_1 \dots \alpha_k} \left(p^\mu \frac{\partial f}{\partial x^\mu} + m F^\mu \frac{\partial f}{\partial p^\mu} \right) \frac{d^3 p}{p^0} = 0$$

The first term can be rewritten as

$$\int G^{\alpha_1 \dots \alpha_k} p^\mu \frac{\partial f}{\partial x^\mu} \frac{d^3 p}{p^0} = \frac{\partial}{\partial x^\mu} \int G^{\alpha_1 \dots \alpha_k} p^\mu f \frac{d^3 p}{p^0} - \int p^\mu f \frac{\partial G^{\alpha_1 \dots \alpha_k}}{\partial x^\mu} \frac{d^3 p}{p^0}$$

The second term is

$$m \int G^{\alpha_1 \dots \alpha_k} F^\mu \frac{\partial f}{\partial p^\mu} \frac{d^3 p}{p^0} = m \left(\underbrace{\int \frac{\partial (G^{\alpha_1 \dots \alpha_k} f F^\mu)}{\partial p^\mu} \frac{d^3 p}{p^0}}_{(A)} - \int F^\mu f \frac{\partial G^{\alpha_1 \dots \alpha_k}}{\partial p^\mu} \frac{d^3 p}{p^0} \right)$$

(Δ) is volume integral of a divergence $\Rightarrow$ Gauss theorem can transform it in a surface integral and we can choose the surface at $p^\mu \to \pm\infty$; f must vanish at such surface and so $(\Delta) = 0$.

As a result, the <u>relativistic transport equation</u> is

$$(\odot) \quad \boxed{\frac{\partial}{\partial x^\mu} \int G^{\alpha_1 \dots \alpha_k} p^\mu f \frac{d^3p}{p^0} - \int \left(p^\mu \frac{\partial G^{\alpha_1 \dots \alpha_k}}{\partial x^\mu} + m F^\mu \frac{\partial G^{\alpha_1 \dots \alpha_k}}{\partial p^\mu} \right) f \frac{d^3p}{p^0} = 0}$$

As in the classical theory, we can derive the moment eqs. from ($\odot$) by just considering

$$G^{\alpha_1 \dots \alpha_k} = \begin{cases} c & k=0 \\ p^\mu & k=1 \\ p^\mu p^\nu & k=2 \\ \vdots & \vdots \end{cases}$$

(1) $k=0$

$G^{\alpha_1 \dots \alpha_k} = c$; second term in ⊚) is zero

$$\frac{\partial J^\mu}{\partial x^\mu} := mc \frac{\partial}{\partial x^\mu} \int p^\mu f \frac{d^3p}{p^0} = 0$$

<u>1st moment eq.</u>

ind. variables

(2) $k=1$

$G^{\alpha_1 \dots \alpha_k} = p^\nu$

$$p^\mu \frac{\partial p^\nu}{\partial x^\mu} = 0 , \quad F^\mu \frac{\partial p^\nu}{\partial p^\mu} = F^\mu \delta_{\mu\nu} = F^\nu$$

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} := c \frac{\partial}{\partial x^\mu} \int p^\mu p^\nu f \frac{d^3p}{p^0} = cm \int F^\nu f \frac{d^3p}{p^0}$$

: <u>2nd moment eq.</u>

$$= 0 \text{ if } F^\nu = 0$$

⋮

As in the classical case, the moment eqs are useful as long as we have an expression for the distribution function.

(3) $k=2$

$$c \frac{\partial}{\partial x^\mu} \int p^\mu p^\nu p^\sigma f \frac{d^3p}{p^0} = \frac{\partial F^{\mu\nu\sigma}}{\partial x^\mu} = \int \left(p^\mu \frac{\partial (p^\nu p^\sigma)}{\partial x^\mu} + m F^\mu \frac{\partial (p^\nu p^\sigma)}{\partial p^\mu} \right) \frac{d^3p}{p^0}$$

⊛ %

Although the solution of these eqs. is not possible as long as an explicit expression for J^μ and $T^{\mu\nu}$ cannot be derived, they are useful as they can used to express conservation of $\underline{J}$ and $\underline{T}$

conservation of mass

$$0 = \int_\Sigma J^\mu \ell_\mu \, d^3x = \int_V \nabla_\mu J^\mu \, d^4x$$

Conser. of en. mom.

$$0 = \int_\Sigma T^{\mu\nu} \ell_\mu \, d^3x = \int_V \nabla_\mu T^{\mu\nu} \, d^4x$$

From these expressions we deduce the relativistic hydrodynamic equations as

$$\left.\begin{aligned} \nabla_\mu J^\mu &= 0 \\ \nabla_\mu T^{\mu\nu} &= 0 \end{aligned}\right\}$$

conservation of rest mass, energy and momentum

Relativistic 0-th order approximation

As in classical kinetic theory, the relativistic zeroth order approximation is obtained when a specific form is used for the distribution function. There are at least two different cases and attention needs to be paid on the quantum corrections:

A) non-degenerate fluid: quantum corrections not important; $p = p(T)$

B) degenerate fluid: quantum corrections cannot be ignored; pressure independent of temperature

A) non-degenerate fluid

$$f_0(\underline{p}) = \left(\frac{g_s}{h_p^3}\right) \frac{1}{\exp\left(-\alpha_F - c\, p^\mu U_\mu / k_B T\right)} \qquad (1)$$

equilibrium distribution function

: $\boxed{\text{Maxwell-Jüttner distribution function}}$

where

$$g_s = \begin{cases} 2s+1 & \text{for } m \neq 0 \quad \text{eg} \quad g_s = 2 \text{ for electron} \\ \underbrace{2s}_{\text{spin}} & \text{for } m = 0 \end{cases}$$

degeneracy factor

Note that $p^\mu = mc\,\underbrace{U^\mu}_{\text{four-vel. of particle}}$; u^μ : 4-velocity of fluid $\Rightarrow$ $p^\mu u_\mu \neq -mc$

α_F: fugacity

$$\alpha_F := \frac{m}{k_B T}\left(\frac{e+P}{\rho} - Ts\right) \quad : \text{dimensionless quantity}$$

$\rho := nm$; rest-mass density

$e := \underbrace{\rho(1+\epsilon)}_{\text{purely relativistic}}$ energy density $\qquad \rho\epsilon :=$ internal energy density

$\frac{e}{\rho} = 1+\epsilon$: specific energy

$$s := \frac{S}{Nm} \quad : \text{specific entropy}$$

The fugacity can also be expressed as ($\alpha_F \ll 1$ for non-degenerate fluid)

$$\alpha_F = \frac{\overbrace{G}^{\text{Gibbs free energy}}}{N k_B T} = \frac{\mu}{k_B T}$$ where $\mu = \frac{G}{N} = \left(\frac{\partial G}{\partial N}\right)_{p,T}$ is the chemical potential (energy change if N is changed)

With a little bit of algebra we can rewrite the expression of the M-J distribution. Recall that in comoving frame

$$-c\, p^\mu U_\mu = W m c^2$$ ($U_\mu = (-1,0,0,0)$; $p^0 = W m c$)

$W = (1 - v^i v_i / c^2)^{-1/2}$ is the Lorentz factor

and that $p^2 = p^i p_i = m^2 c^2 (W^2 - 1) \Longrightarrow W = \left(1 + \frac{p^2}{m^2c^2}\right)^{1/2}$

$$= m c^2 \left(1 + \frac{p^2}{m^2 c^2}\right)^{1/2}$$

As a result, (1) $\Longleftrightarrow$ $f_0(p) = \left(\frac{g_s}{h_p^3}\right) e^{\alpha_F} e^{-\left(1 + \frac{p^2}{m^2c^2}\right)^{1/2} \frac{mc^2}{2 k_B T}}$

$\xi_c := \frac{mc^2}{k_B T}$: coldness

$\xi_c \ll 1$ relativistic non-degenerate fluid

$\xi_c \gg 1$ non-relativistic " "

B) degenerate fluid: quantum corrections are important.

Two distinct distribution functions according to whether particles are bosons or fermions

$$f_0(p) = \left(\frac{g_s}{h_p^3}\right) \frac{1}{\exp\left(-\alpha_F - c p^\mu u_\mu / K_B T\right) \pm 1}$$

Bose-Einstein

Fermi-Dirac dist. function

+: fermions

−: bosons

$$= \left(\frac{g_s}{h_p^3}\right) \frac{1}{\exp\left[(E-\mu)/K_B T\right] \pm 1}$$

$$-\alpha_F - \frac{c p^\mu u_\mu}{K_B T} = -\frac{\mu}{K_B T} + \frac{c^2 m W}{K_B T}$$

$$= -\frac{\mu}{K_B T} + \frac{E}{K_B T} = \frac{E-\mu}{K_B T}$$

Notes

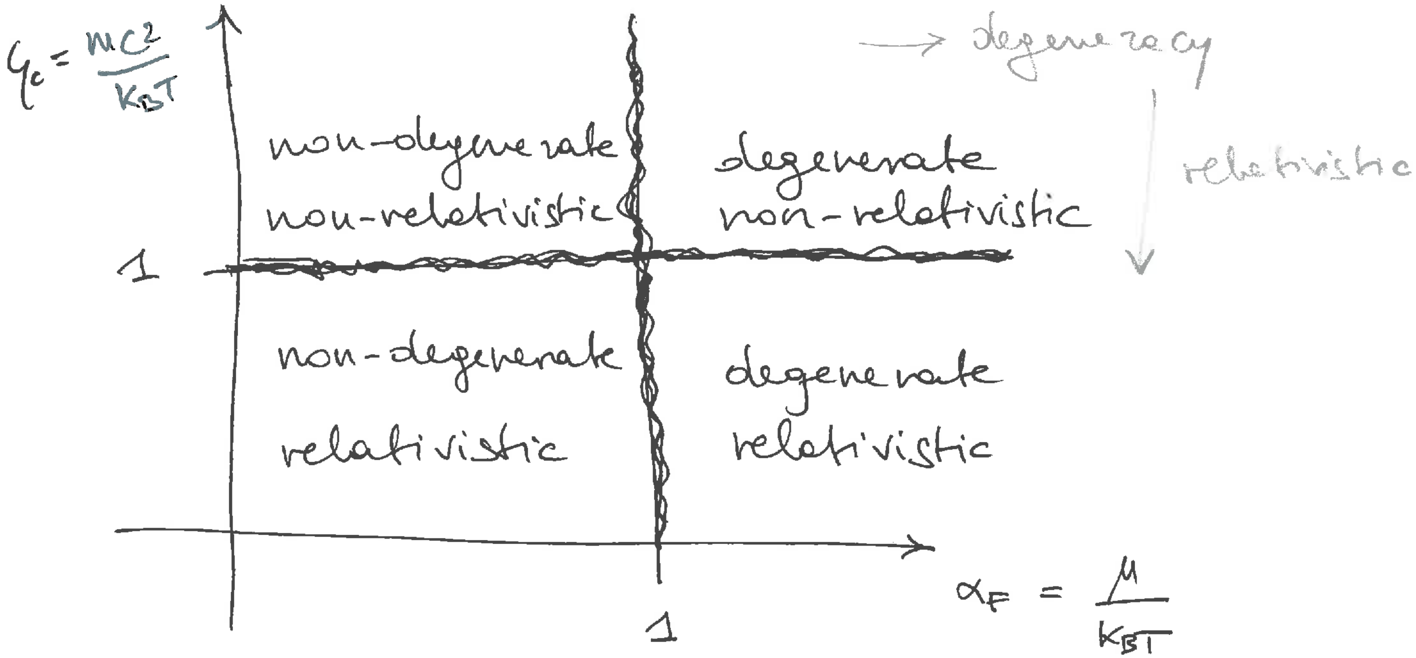
- As in the classical kinetic theory, also in the relativistic case the zeroth-order approximation in which the equilibrium distribution function is BE – FD leads to perfect fluids, ie fluids in which the energy fluxes as well as the viscous terms are zero.

- As in the classical kinetic theory, the moment eqs are hardly used and one resorts to hydrodynamic eqs. in which the thermodynamical quantities are expressed through the so-called "equation of state" ie a relation between the pressure and the other thermodyn. quantities

  $p = p(e, \epsilon, \ldots)$ (EOS)

  Different physical conditions will lead to different EOSs

In principle there should be an EOS for each physical condition. In practice, differences are not very large and EOSs are computed for fluids in specific states. These can be summarized using the fugacity and coldness



Expressions for these EOSs can be found in many books and I will concentrate only on a couple of them, but first a review of thermodynamics.

Laws of thermodynamics

Given a fluid element, a number of thermodynamic properties can be measured in the fluid's rest frame: $n, \rho, p, e, \mu, T \ldots$

For a single-component fluid there are only 2 independent thermodynamic potentials, eg ρ, ϵ

* $\boxed{\text{1st law}}$

$$\text{(0)} \qquad dU = \underbrace{T\,dS}_{\text{heat exchange}} - \underbrace{p\,dV}_{\text{work done}} + \underbrace{\mu\,dN}_{\text{creation/destruction of particles}}$$

$U = \rho \epsilon V = N m \epsilon$: internal energy

If $dN = 0$, (0) can be written as (exercise)

$$d\epsilon = T\,ds - p\,d\underbrace{\left(\frac{1}{\rho}\right)}_{\text{specific volume}}$$

$s = \dfrac{S}{Nm}$: specific entropy

Alternative and useful expressions are possible when using the specific enthalpy

$$h := \frac{e+p}{\rho} = \frac{1}{\rho}\left[\rho(1+\epsilon)+p\right] = 1+\epsilon+\frac{p}{\rho}$$

from which (exercise)

$$dp = \rho\, dh - \rho\, T ds$$

$$de = h\, d\rho + \rho\, T ds$$

(derive equiv. expressions for

$$H := N m h = V(e+p)$$

$$dp = \frac{n}{N}\left(dH - T d_{\rho} S\right)$$

$$de = \frac{1}{N}\left(H dn + n T ds\right)$$

)

Exercise

Note

- $h \neq h_N = \epsilon + \frac{p}{\rho}$: relativistic correction due to rest mass density $h \longrightarrow 1$ for $\epsilon, \rho \to 0$, $h_N \longrightarrow 0$

- $e \neq e_N = \rho\epsilon + \frac{1}{2}\vec{v}^2$: no rest-mass contribution!

* $\boxed{\text{2nd- law}}$ As in the classical theory, where $S = -k_B V H$

we can define a 4-vector S^μ : entropy current density

$$S^\mu = -k_B c H^\mu = -k_B c \int p^\mu f \left[\ln(f A_H) - \left(1 + \frac{1}{f B_H}\right) \ln(1 + f B_H) \right] \frac{d^3p}{p^0}$$

A_H, B_H : const.

The relativistic H-theorem states $\boxed{\nabla_\mu S^\mu \geq 0}$: entropy cannot decrease

Given the definitions of ρ, s we can rewrite

$$\boxed{S^\mu = s\rho\, u^\mu}.$$

Recap

- relativistic moment equation

$$\int G^{\alpha_1 \dots \alpha_k} \left(p^\mu \frac{\partial}{\partial x^\mu} f + m F^\mu \frac{\partial f}{\partial p^\mu} + m f \frac{\partial F^\mu}{\partial p^\mu} \right) \frac{d^3 p}{p^0} = 0 \quad \Longrightarrow$$

$$\frac{\partial}{\partial x^\mu} \int G^{\alpha_1 \dots \alpha_k} p^\mu f \frac{d^3 p}{p^0} - \int \left(p^\mu \frac{\partial G^{\alpha_1 \dots \alpha_k}}{\partial x^\mu} + m F^\mu \frac{\partial G^{\alpha_1 \dots \alpha_k}}{\partial p^\mu} \right) \frac{d^3 p}{p^0} = 0$$

relativistic moment equation

$$G^{\alpha_1 \dots \alpha_k} = \begin{cases} c & k = 0 \\ p^\mu & k = 1 \\ p^\mu p^\nu & k = 2 \end{cases}$$

$$\frac{\partial \bar{J}^\mu}{\partial x^\mu} = 0 \qquad : \text{1st-moment eq.}$$

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = cm \int F^\mu f \frac{d^3 p}{p^0} \qquad : \text{2nd-moment eq.}$$

- the moment eqs can be seen as conservation equations of the corresponding moments, ie

$$\int_\Sigma J^\mu \ell_\mu \, d^3x = \int_V \nabla_\mu J^\mu \, d^4x = 0 \iff \nabla_\mu J^\mu = 0$$

$$\int_\Sigma T^{\mu\nu} \ell_\mu \, d^3x = \int_V \nabla_\mu T^{\mu\nu} \, d^4x = 0 \iff \nabla_\mu T^{\mu\nu} = 0$$

will see these are cons. eq. for restmass energy and mom.

- relativistic 0-th -order approximation

A) non degenerate fluid : quantum corrections are small ; $p = p(T)$

B) degenerate fluid : " " not small

$p \neq p(T)$

A)

$$f_0(\underline{p}) = \left(\frac{g_r}{h_p^3}\right) \frac{1}{\exp\left(-\alpha_F - c p^\mu u_\mu / k_B T\right)}$$: Maxwell - Jüttner distribution

g_r: degeneracy factor

α_F: fugacity

$$\alpha_F = \frac{m}{k_B T}\left(\frac{e+p}{\rho} - Ts\right) = \frac{\mu}{k_B T}$$

$p^\mu = m U^\mu$; u_μ : fluid 4-vel.

$$-\frac{c p^\mu u_\mu}{k_B T} = \frac{m c^2}{k_B T}\left(1 + \frac{p^2}{m^2 c^2}\right)^{1/2} = \xi_c \left(1 + \frac{p^2}{m^2 c^2}\right)^{1/2}$$

ξ_c : coldness

$\xi_c \ll 1$ non-rel. fluid

$\xi_c \gg 1$ relativistic fluid

B)

$$f_0(\underline{p}) = \left(\frac{g_s}{h_p^3}\right) \frac{1}{\exp\left(-\alpha_F - c p^\mu u_\mu / k_B T\right) \pm 1}$$

+1 Bose-Einstein (bosons)

−1 Fermi-Dirac (fermions)

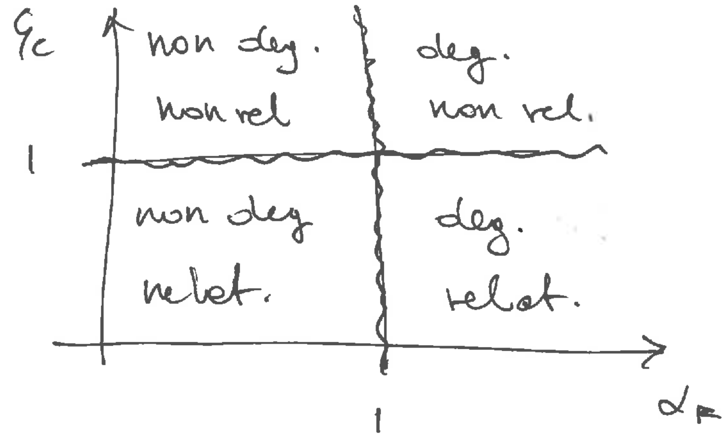
$$= \left(\frac{g_s}{h_p^3}\right) \frac{1}{\exp\left[(E-\mu)/k_B T\right] \pm 1}$$

$$-\alpha_F - \frac{c\, p^\mu u_\mu}{k_B T} = -\frac{\mu}{k_B T} + \frac{c^2 m W}{k_B T} = -\frac{\mu}{k_B T} + \frac{E}{k_B T} = \frac{E-\mu}{k_B T}$$

$\square$

- Equations of state

$$p = p(\rho, \epsilon, \dots)$$



(1st law)

$$dU = T\,dS - p\,dV + \mu\,dN \quad \overset{dN=0}{\Longleftrightarrow}$$

$$d\epsilon = T\,ds - p\,d\left(\frac{1}{\rho}\right) \qquad s: \text{ specific entropy}$$

$$dp = \rho\,dh - \rho T\,ds \qquad h: \text{ " enthalpy}$$

$$de = h\,d\rho + \rho T\,ds$$

$$h = \frac{e+p}{\rho} = \frac{1}{\rho}\left[\rho(1+\epsilon) + p\right] = 1 + \epsilon + \frac{p}{\rho} \longrightarrow 1 \quad \text{for } \rho, \epsilon \to 0$$

(2nd law)

$$\nabla_\mu S^\mu \geq 0 \qquad S^\mu = s\rho u^\mu$$

Note

when obtaining the relativistic moment eq. we have considered the term

$$\int \frac{\partial}{\partial p^\mu}\left(G^{\alpha_1\ldots\alpha_n} f F^\mu\right)\frac{d^3p}{p^0} \quad : \quad \text{4 div. in 3-space}$$

mom. space

this is fine because

$$\int \frac{\partial}{\partial p^\mu}\left(\quad\right)\frac{d^3p}{p^0} = \int \frac{\partial}{\partial p^0}\left(G^{\alpha_1\ldots\alpha_k} F^0 f\right)\frac{d^3p}{p^0} + \int \frac{\partial}{\partial p^i}\left(G^{\alpha_1\ldots\alpha_k} F^i p_0\right)\frac{d^3p}{p^0}$$

$F^\mu = m a^\mu$ and $\underline{F}$ is a purely spatial tensor

$F^\mu u_\mu = 0 \Rightarrow F^0 = 0$ and first term is identically zero

$F^\mu = (0, F^i)$

Representative EOSs

- First introduce <u>adiabatic</u> index $\gamma := \frac{c_p}{c_v} = \frac{\left(\begin{array}{c}\text{specific heat}\\ \text{constant pressure}\end{array}\right)}{\left(\begin{array}{c}\text{specific heat}\\ \text{constant volume}\end{array}\right)}$

c_{heat} : specific heat

$$c_{heat} = \frac{dq}{dT} = \frac{d(Q/m)}{dT}$$

$$dq = T\,ds = d\epsilon - \frac{p}{\rho^2}\,d\rho$$

$$= \frac{1}{dT}\left\{ \left(\frac{\partial \epsilon}{\partial T}\right)_\rho dT + \left[\left(\frac{\partial \epsilon}{\partial \rho}\right)_T - \frac{p}{\rho^2}\right] d\rho \right\}$$

so that

$$c_v = \left(\frac{\partial \epsilon}{\partial T}\right)_V$$: <u>change</u> in specific int. <u>energy</u> at const. volume

$$c_p = c_v + \left[\left(\frac{\partial \epsilon}{\partial \rho}\right)_T - \frac{p}{\rho^2}\right]\left(\frac{\partial \rho}{\partial T}\right)_p \overset{①}{=} \left(\frac{\partial h}{\partial T}\right)_p$$: change in specific <u>enthalpy</u> at constant pressure

Note that for a classical fluid: $c_p - c_V \overset{①}{=} -T \left(\frac{\partial p}{\partial T}\right)_V^2 \Big/ \left(\frac{\partial p}{\partial V}\right)_T > 0 \quad \Longrightarrow$

$\left(\frac{\partial p}{\partial V}\right)_T < 0$

$$\gamma = \frac{c_p}{c_V} > 1 : \begin{cases} \gamma = 5/3 \;\; \sim 1.66 & \text{monoatomic non-deg. non-rel. fluid} \\ \gamma = 4/3 \;\; \sim 1.33 & \text{ultrarelativistic degenerate fluid} \\ & (\varphi_c \to 0, \text{ eg massless particles}) \end{cases}$$

— Another important thermodynamic quantity is the "sound speed"

$de = h\,d\rho + \rho T ds = h\,d\rho$

$$c_s^2 := \left(\frac{\partial p}{\partial e}\right)_s = \frac{1}{h}\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{1}{h}\left(c_s^2\right)_N$$

We will appreciate the dynamical significance of this quantity later on, when discussing the nonlinear properties of the hydrodynamic eqs.

- Ideal fluid

monoatomic fluid: non deg. non-rel., or a non-deg. rel. "

$p = nk_BT = \rho k_BT/m$:

$p \propto T$: prototype EOS $\quad p = p(\rho, \epsilon)$; for a monoatomic fluid

$d\epsilon = c_V\, dT \implies \epsilon = c_V T$; $(\partial\epsilon/\partial\rho)_T = 0$; $(\partial\rho/\partial T)_p = -\rho/T$

$$c_p = c_V + \left[\left(\frac{\partial\epsilon}{\partial\rho}\right)_T - \frac{p}{\rho^2}\right]\left(\frac{\partial\rho}{\partial T}\right)_p = c_V + \frac{p}{\rho^2}\frac{\rho}{T} = c_V + \frac{p}{\rho T} = c_V + \frac{k_B}{m} \implies \gamma = \frac{c_p}{c_V} = 1 + \frac{k_B}{c_V m}$$

$$\implies c_V = \frac{k_B}{m(\gamma-1)}$$

Hence

$$p = nk_BT = n\,\overbrace{c_V\, m\,(\gamma-1)\,T}^{\epsilon} = nm\epsilon(\gamma-1) = \boxed{\rho\epsilon(\gamma-1) = p}$$

$$h = 1 + \epsilon + p/\rho = 1 + \gamma\epsilon \qquad p/\rho = \epsilon(\gamma-1)$$

$$e = \rho(1+\epsilon)$$

$$c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s = \frac{1}{h}\left[\left(\frac{\partial p}{\partial\rho}\right)_\epsilon + \frac{p}{\rho^2}\left(\frac{\partial p}{\partial\epsilon}\right)_\rho\right] = \frac{\gamma\epsilon(\gamma-1)}{1+\gamma\epsilon}$$

(Exercise)

Note that for an ultrarelativistic fluid $\gamma = 4/3$, $\gamma\epsilon \gg 1$

and so $c_s^2 \longrightarrow \gamma - 1 = 1/3$.

$$c_s^2 = \frac{\gamma\epsilon(\gamma-1)}{1+\gamma\epsilon} \sim \frac{\gamma\epsilon(\gamma-1)}{\gamma\epsilon} \sim \gamma - 1 = 1/3$$

Note that the adiabatic index of an ideal fluid is more than just the ratio of the specific heats. In particular, when the transformation are <u>isentropic</u>, the following relations are true

$$p V^{\gamma} = \text{const}; \qquad p^{(1-\gamma)/\gamma}\, T = \text{const}; \qquad V^{\gamma-1}\, T = \text{const}$$

From these relations it is possible to define three different indices

$$\Gamma_1 := \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s ; \qquad \frac{\Gamma_2 - 1}{\Gamma_2} := \left(\frac{\partial \ln T}{\partial \ln p}\right)_s ; \qquad \Gamma_3 - 1 := \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_s$$

and for an ideal fluid $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma$

- Polytropic fluid

Large class of fluids often used in modelling astrophysical fluids and plasmas. A polytropic transformation is one in which $\boxed{PV^\Gamma = \text{const.}}$ $\Longrightarrow$

$$P = p(e) = k e^\Gamma = k e^{1+1/N_p} \quad : \text{polytropic EOS}$$

Γ : polytropic exponent, $N_p = \frac{1}{\Gamma - 1}$: polytropic index ; k : polyt. const.

Relevant examples: (A) completely degenerate non-relativistic electron fluid : $\Gamma = 5/3$ ($N_p = 3/2$) and $k = \frac{1}{5}\left[\frac{3h_p^3}{8\pi m_e^4}\right]^{2/3}$

(B) completely degenerate ultrarelativistic electron fluid

$\Gamma = 4/3$ ($N_p = 3$) and $k = \frac{1}{4}\left[\frac{3c^3 h_p^3}{8\pi m_e^4}\right]^{1/3}$

Importance of a polytropic EOS pivots around flexibility in K, N_p ; note $[K] \overset{*}{=} L^{2/N_p}$ so that all quantities can be made dimensionless; let $M := \int d^3x\, \rho$

$\boxed{[\bar{M}] = L^{-1} L = L^0}$

$$\rho \longrightarrow \bar{\rho} = K^{N_p} \rho^{\circledcirc} \quad ; \quad M \longrightarrow \bar{M} = K^{-N_p/2} M$$

Similarly, the rescaling of quantities can be done trivially after rescaling the polytropic constant. Let L_1 be a lengthscale (eg a mass measurement) made with K_1 and N_1

$$L_2 = \left(\frac{K_2^{N_2}}{K_1^{N_1}}\right)^{1/2} L_1 = \left[\frac{K_2^{1/(\Gamma_2-1)}}{K_1^{1/(\Gamma_1-1)}}\right]^{1/2} L_1$$

Ex. $N_2 = N_1 = 1$; $L_2 = \left(\frac{K_2}{K_1}\right)^{1/2} L_1$

$[K] = L^{2/N_p}$

$\circledcirc$ $\boxed{\begin{aligned}[\bar{\rho}] &= [K]^{N_p}[\rho] \\ &= L^2 \cdot L^{-2} = L^0\end{aligned}}$

$*$ $[p] = [e] = \frac{L}{L^3} = L^{-2} = \left[K \rho^{1+\frac{1}{N_p}}\right] = [K]\left[L^{-2\left(1+\frac{1}{N_p}\right)}\right]$

$= [K]\, L^{-2-2/N_p} \Rightarrow$

<u>Note</u> : ideal-fluid and polytropic EOS coincide for isentropic transformations. Proof

$$= \epsilon(\gamma-1)d\rho + \rho(\gamma-1)d\epsilon = \epsilon(\gamma-1)d\rho + (\gamma-1)\left(\frac{p}{\rho}\right)d\rho \qquad d\epsilon = \frac{p}{\rho^2}d\rho$$

$$= \gamma\left(\frac{p}{\rho}\right) + \gamma\epsilon - \frac{p}{\rho} - \epsilon \qquad (\gamma-1)\epsilon$$

$$p = \rho\epsilon(\gamma-1) \quad ; \quad dp = (\gamma-1)\left(\frac{p}{\rho}+\epsilon\right)d\rho = \gamma\left(\frac{p}{\rho}\right)d\rho$$

where we have used $\rho\, d\epsilon = \frac{p}{\rho}d\rho - \rho T ds \overset{ds=0}{=} \frac{p}{\rho}d\rho$.

Direct integration then yields $\boxed{p = k\rho^{\gamma}}$

γ: adiabatic index $\qquad \gamma = \Gamma$ only for isentropic transformations

Γ ⌞ polytropic exponent

When this is the case, the specific internal energy can be computed as

$$\epsilon = \frac{p}{\rho(\Gamma-1)} = \frac{k\rho^{\Gamma-1}}{\Gamma-1}$$

In other words, ϵ is fully determined by ρ: compressions/rarefactions lead to changes in ϵ; in general ϵ is not determined in a polytropic EOS.

It's also easy to compute the sound speed

$$c_s^2 = \left(\frac{\partial p}{\partial e}\right) = \left(\frac{1}{\Gamma K \rho^{\Gamma-1}} + \frac{1}{\Gamma-1}\right)^{-1}$$

(EXERCISE)

ie fully determined by K and Γ.

Because in the polytropic EOS $p = p(\rho)$, it's easy to relate the ability of a fluid parcel to be compressed. In particular

$N_p \gtrsim 1$ ⓘ : soft EOS : high compressibility (compact relat. stars)

$N_p \lesssim 1$: stiff EOS : small " (extended relat. stars.

ⓘ

$p = K\rho^{1+\frac{1}{N_p}} = K\rho\,\rho^{\frac{1}{N_p}}$: the larger the N, the smaller the contribution of the second term, ie smaller pressure, hence radius

The polytropic EOS allows us to make also another important consideration on adiabaticity and isentropy. Let's start from the first law of thermodynamics: $T ds = d\epsilon - \left(\frac{p}{\rho^2}\right) d\rho$ to obtain

$$T = \frac{m}{k_B}(\Gamma - 1)\epsilon = \frac{m}{k_B}\frac{p}{\rho}$$

$$\frac{m}{k_B} ds = \frac{d\epsilon}{\epsilon(\Gamma-1)} - \frac{d\rho}{\rho} = d\left[\ln\left(\frac{\epsilon^{1/(\Gamma-1)}}{\rho}\right)\right]$$

(EXERCISE)

which can be integrated to yield

$$\underbrace{s = \frac{k_B}{m}\left[\ln\left(\frac{K}{\Gamma-1}\right)^{1/(\Gamma-1)} + \tilde{K}\right]}_{= \text{const.}}$$

integration constant

$$\hat{K} = \frac{\ln(\Gamma-1)}{\Gamma-1}$$

In other words, for each fluid element the entropy is constant if the fluid undergoes an isentropic transformation.

Of course K can differ from fluid element to fluid element, but it is normally assumed to be constant across the fluid.

In summary: perfect fluids are intrinsically adiabatic, ie the specific entropy is conserved along fluidlines and can be isentropic if the specific entropy is the same across the fluid.

* Radiation fluid

This is another important example: fluid composed of photons, thus following the Bose–Einstein statistics. This could be the case of a fluid of massive particles at ultrarelativistic energies and tightly coupled to radiation. If T is the temperature, the energy density per unit frequency ν is

$$e_R(\nu, T) = \left(\frac{8\pi h_P}{c^3}\right) \frac{\nu^3}{\exp(h_P\nu/k_B T) - 1}$$

Since for each photon $E = h_p\nu$, the no. of photons in the frequency interval $\nu, \nu + d\nu$ is given by

$$n(\nu)\,d\nu = \left(\frac{8\pi}{c^3}\right)\frac{\nu^2 d\nu}{\exp(h_p\nu/k_B T) - 1}$$

$e_R = \frac{N h\nu}{V} = n\, h_p\nu \Rightarrow n = \frac{e_R}{h\nu}$

The energy density is then

$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}$

$$e_R(T) = \int d\nu\; e_R(\nu, T) = a_R T^4$$

$$a_R := \frac{8\pi^5 k_B^4}{15 h_p^3 c^3} \quad : \text{ radiation constant}$$

Similarly, the isotropic pressure is $p_R = \frac{1}{3} a_R T^4 = \frac{1}{3} e_R$ which is typical of any ultrarelativistic fluid.

Notes

- Since the photons are massless, the concept of specific enthalpy does not apply. However, it is possible to consider the enthalpy

$$H_R = (e_R + p_R) V = \frac{4}{3} e_R V$$

- Similarly, the photon fluid will satisfy the first law of thermodynamics expressed not in terms of specific quantities

$$T dS = d(e_R V) + p_R dV \Rightarrow dS = d\left(\frac{4}{3} a_R T^3 V\right) \Rightarrow$$

$$S = \frac{4}{3} a_R T^3 V \propto e^{3/4}$$

$T_R \propto p_R^{1/4}$; $T_R^3 \propto p_R^{3/4}$

- For an isentropic transformation $(dS = 0)$ $T^3 V = \text{const} \Longleftrightarrow$

$p_R V^{4/3} = \text{const} \Longrightarrow$ photon fluid behaves as polytrope with $\Gamma = 4/3$

• When the photon fluid is in thermal equilibrium with an ideal fluid, the energy densities and pressures simply add

$$p = p_M + p_R = n k_B T + \frac{1}{3} a_R T^4$$

$$e = e_M + e_R = \rho \left[1 + \frac{n k_B T}{\rho (\gamma - 1)} \right] + a_R T^4$$

The dominance of one fluid over the other can be expressed via the ratio $\alpha = \frac{p_R}{p_M}$, so that $p = p_M (1 + \alpha)$

$\alpha \to 0$: fluid-pressure dominated

$\alpha \to \infty$: radiation " "

In this case $\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln e} \right)_s = \frac{5/2 + 20\alpha + 16\alpha^2}{(3/2 + 12\alpha)(1+\alpha)} = \begin{cases} \to 4/3 & \alpha \to \infty \\ \to 5/3 & \alpha \to 0 \end{cases}$

Recap

Adiabatic index $\gamma = \dfrac{c_p}{c_V}$

$$c_V = \left(\frac{\partial \epsilon}{\partial T}\right)_V \quad ; \quad c_p = c_V + \left[\left(\frac{\partial \epsilon}{\partial \rho}\right)_T - \frac{p}{\rho^2}\right]\left(\frac{\partial \rho}{\partial T}\right)_p$$

$$\gamma = \begin{cases} 5/3 & \text{non-deg. non-rel. fluid} \\ 4/3 & \text{ultrarelativistic fluid (eg photon fluid)} \end{cases}$$

$$c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s = \frac{1}{h}(c_s^2)_N$$

<u>Ideal fluid</u>

$$p = \rho\epsilon(\gamma - 1) \quad ; \quad c_s^2 = \frac{\gamma\epsilon(\gamma-1)}{1+\gamma\epsilon} \longrightarrow 1/3 \quad \text{for } \gamma\epsilon \gg 1$$

(radiation fluid)

Polytropic fluid

polytropic transformation $pV^{\Gamma} = \text{const}$

$p = p(e) = K e^{\Gamma} = K e^{1+\frac{1}{N_p}}$

N_p: polytropic index; K: polyt. constant; Γ: polyt. exponent

Because p does not depend on temperature, this EOS is representative of degenerate fluids, eg

$\Gamma = 5/3$: non-relativistic electron fluid (eg for white dwarfs)

$\Gamma = 4/3$: ultrarelativistic " "

$[K] = L^{2/N_p}$ $\Rightarrow$ • it is possible to work in dimensionless quantities $(M \to \bar{M} = K^{-N_p/2} M)$

• it is possible to scale from one polytropic EOS to the other: $(K_1, \Gamma_1) \longrightarrow (K_2, \Gamma_2)$ via algebraic relations

- Ideal fluid and polytropic EOS coincide for isentropic transformations $(ds=0)$ $\Longrightarrow$ $\epsilon = \dfrac{K\rho^{\Gamma-1}}{\Gamma-1} = \epsilon(\rho)$

$$c_s^2 = \left(\frac{1}{\Gamma K \rho^{\Gamma-1}} + \frac{1}{\Gamma-1} \right)^{-1} ;$$

$N_p \gtrsim 1$: soft EOS (high compressibility)

$N_p \lesssim 1$: stiff EOS (low ") ☆

$s = s(K)$: entropy is function of polytropic constant and conserved along fluid lines. Fluid is <u>adiabatic</u>. If s the same across fluid ($K = \text{const}$ for all elements), the fluid is <u>isentropic</u> ☆

Radiation fluid (BB photon gas, fluid of massless particles)

$e_R = a_R T^4$; $p = \frac{1}{3} e_R = \frac{1}{3} a_R T^4 \Longrightarrow c_s^2 = \frac{1}{3}$

Relativistic perfect fluids

- Kinematics

First fundamental quantity is fluid's <u>4-velocity</u>

$u^\mu := \dfrac{dx^\mu}{d\tau}$ $\quad\tau$: fluid proper time, ie time in fluid's reference frame

$a^\mu := u^\nu \nabla_\nu u^\mu$: fluid's 4-acceleration $\left(\begin{array}{l}\text{variation of } u \text{ along} \\ \text{fluid's worldline}\end{array}\right)$

$\underline{u}\cdot\underline{u} = -1$: timelike unit vector

proof $\quad u^\mu u_\mu = g_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = \dfrac{ds^2}{d\tau^2} = -1$

$\underline{a}$ and $\underline{u}$ are orthogonal, ie $\underline{a}\cdot\underline{u} = 0$

proof : first note that

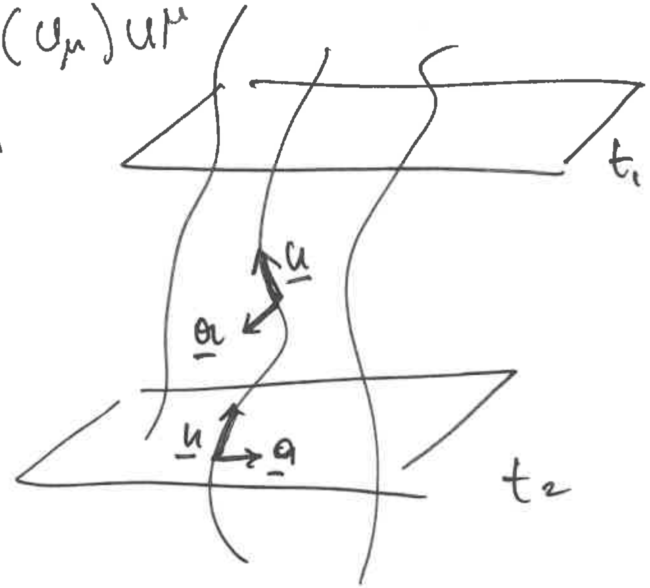
$$\nabla_\mu (u^\nu u_\nu) = \nabla_\mu(-1) = 0 = (\nabla_\mu u^\nu) u_\nu + (\nabla_\mu u_\nu) u^\nu = 2 (\nabla_\mu u^\nu) u_\nu \Rightarrow$$

$$\boxed{u^\nu \nabla_\mu u_\nu = 0};$$

Then

$$a^\mu u_\mu = (u^\nu \nabla_\nu u^\mu) u_\mu = 0 \quad \text{qed.}$$

$$= u^\nu \nabla_\nu (u_\mu u^\mu) - u^\nu \nabla_\nu (u_\mu) u^\mu$$
$$= 0 - u^\nu \nabla_\nu (u^\mu) u_\mu$$

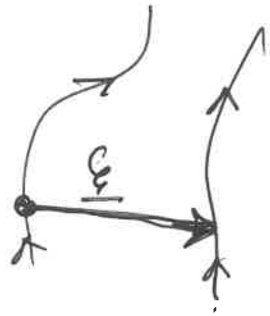


Without loss of generality we can write

$$u^\alpha = u^0 \left(1, \frac{dx^i}{dt}\right) = u^0 (1, v^i)$$

All kinematic properties of the fluid can be expressed in terms of $\underline{u}$ and $\underline{a}$

Let $\underline{\xi}$ be the 4-vector between two neighbouring worldlines



displacement vector

Before proceeding further it is useful to review the properties of a differential operator that is useful when considering the changes of a vector field relative to another vector field, ie the <u>Lie derivative</u>

I recall that the Lie derivative expresses the covariant derivative of a vector (tensor) field relative to another vector (tensor) field:

partial derivs replace cov. derivs.

$$\mathcal{L}_{\underline{V}}\,\underline{U} = \nabla_{\underline{V}}\,\underline{U} - \nabla_{\underline{U}}\,\underline{V} = \partial_{\underline{V}}\,\underline{U} - \partial_{\underline{U}}\,V = -[\underline{U}, V] = [\underline{V}, \underline{U}]$$

In component form

partial and covariant derivates can be exchanged

$$(\mathcal{L}_{\underline{V}}\underline{U})^{\mu} = V^{\nu}\partial_{\nu}U^{\mu} - U^{\nu}\partial_{\nu}V^{\mu} = V^{\nu}\nabla_{\nu}U^{\mu} - U^{\nu}\nabla_{\nu}V^{\mu}$$

$$(\mathcal{L}_{\underline{V}}\underline{U})_{\mu} = V^{\nu}\partial_{\nu}U_{\mu} + U^{\nu}\partial_{\nu}V_{\mu} = V^{\nu}\nabla_{\nu}U_{\mu} + U^{\nu}\nabla_{\nu}V_{\mu}$$

Properties

- $\mathcal{L}_{\phi\underline{V}}\underline{T} = \phi\,\mathcal{L}_{\underline{V}}\underline{T} - \underline{V}\,\mathcal{L}_{\underline{T}}\phi$
- $\mathcal{L}_{\underline{V}}\phi = V^{\nu}\partial_{\nu}\phi$
- $\mathcal{L}_{\underline{V}}\left(a\,Y^{\alpha\gamma} + b\,Z^{\beta\gamma}\right) = a\,\mathcal{L}_{\underline{V}}Y^{\alpha\gamma} + b\,\mathcal{L}_{\underline{V}}Z^{\beta\gamma}$

- $\mathcal{L}_{\underline{V}} (Z^{\mu\nu} Y_{\alpha\beta}) = \mathcal{L}_{\underline{V}} (Z^{\mu\nu}) Y_{\alpha\beta} + Z^{\mu\nu} \mathcal{L}_{\underline{V}} Y_{\alpha\beta}$

- $\mathcal{L}_{\underline{V}} T^{\alpha}{}_{\beta} = V^{\mu} \partial_{\mu} T^{\alpha}{}_{\beta} - T_{\beta}{}^{\mu} \partial_{\mu} V^{\alpha} + T^{\alpha}{}_{\mu} \partial_{\beta} V^{\mu}$

The displacement vector $\underline{\xi}$ just follows the two wordlines, that is, $\underline{\xi}$ is Lie dragged along $\underline{u}$ $\iff$

$$\mathcal{L}_{\underline{u}} \underline{\xi} = 0 \iff \mathcal{L}_{\underline{u}} \underline{\xi} = [\underline{u}, \underline{\xi}] = 0 \iff$$ (Exercise)

$$(\mathcal{L}_{\underline{u}} \underline{\xi})^\nu = u^\mu \nabla_\mu \xi^\nu - \xi^\mu \nabla_\mu u^\nu = 0 \Rightarrow$$

$$\dot{\xi}^\mu := \overbrace{u^\nu \nabla_\nu \xi^\mu}^{\text{convective derivative}} = \xi^\nu \nabla_\nu u^\mu \Longrightarrow$$

convective derivative of $\underline{u}$ along $\underline{\xi}$ = convective deriv. of $\underline{\xi}$ along $\underline{u}$

In full generality we can express $u^\nu \nabla_\nu \xi_\mu$ as:

$$\boxed{u^\nu \nabla_\nu \xi^\mu = \xi^\nu \nabla_\nu u_\mu = \left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} \Theta h_{\mu\nu} - a_\mu u_\nu\right) \xi^\nu}$$

after exploiting the irreducible decomposition of a generic rank-2 tensor $[\nabla_\nu u_\mu]$

In other words, we have shown that the convective derivative of the displacement vector can be decomposed in terms of three fundamental tensors $\underline{\omega}$, $\underline{\sigma}$, Θ.

where

$$\omega_{\mu\nu} := h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \nabla_{[\beta} u_{\alpha]} = h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \frac{1}{2}\left(\nabla_{\beta} u_{\alpha} - \nabla_{\alpha} u_{\beta} \right)$$

where $\underline{h}$ is the projector orthogonal to $\underline{u}$

$$h_{\mu\nu} := g_{\mu\nu} + u_{\mu} u_{\nu} \quad ; \quad \underline{h} \cdot \underline{u} = 0 = h_{\mu\nu} u^{\mu} = g_{\mu\nu} u^{\mu} + u_{\mu} u_{\nu} u^{\mu}$$
$$= u_{\nu} - u_{\nu} = 0.$$

$\boxed{\underline{\omega} : \text{kinematic vorticity tensor}}$

$$\sigma_{\mu\nu} := \nabla_{\langle\mu} u_{\nu\rangle} = \nabla_{(\mu} u_{\nu)} + a_{(\mu} u_{\nu)} - \frac{1}{3} \Theta h_{\mu\nu}$$
$$= \frac{1}{2}\left(\nabla_{\mu} u_{\nu} + \nabla_{\nu} u_{\mu} + a_{\mu} u_{\nu} + a_{\nu} u_{\mu} - \frac{2}{3} \Theta h_{\mu\nu} \right)$$

$\boxed{\underline{\sigma} : \text{shear tensor}}$

$$\Theta := h^{\mu\nu} \nabla_{\mu} u_{\nu} = \nabla_{\mu} u^{\mu}$$

$\boxed{\Theta : \text{expansion scalar}}$

$\underline{\omega}$, $\underline{\sigma}$ are anti-symmetric and symmetric tensors respectively, and satisfy a number of identities (EXERCISE)

If a fluid element is thought of as an ellipsoid, then the vorticity tensor represents rigid rotations of the principal axis wrt the inertial frame

In the frame determined by $\underline{u}$, the dual of $\underline{\omega}$ is defined as

$$ {}^*\omega^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta} $$

$\epsilon^{\mu\nu\alpha\beta}$: Levi-Civita tensor ⊙

from which

$\omega^{\mu} := {}^*\omega^{\mu\nu} u_{\nu}$ kin. vorticity 4-vector

$$ = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta} u_{\nu} $$

vorticity 4-vector ; equivalent of classical vorticity 3-vector

⊙

$$ \epsilon_{\alpha\beta\mu\nu} = -\sqrt{-g}\, \eta_{\alpha\beta\mu\nu} $$

$$ \epsilon^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{-g}} \eta^{\alpha\beta\mu\nu} $$

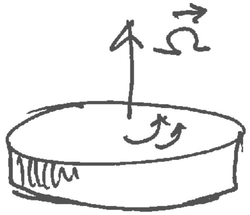
$$ \eta_{\alpha\beta\mu\nu} = \begin{cases} 1 & \text{even permutation of } 0123 \\ -1 & \text{odd} \quad \text{"} \quad \text{"} \ 0123 \\ 0 & \text{if } \alpha\beta\mu\nu \text{ not all diff.} \end{cases} $$

$\vec{\omega} = \vec{\nabla} \times \vec{v}$

$\vec{\omega} = (0, 0, \omega^z) = (0, 0, 2\Omega)$

cylind. coord sys

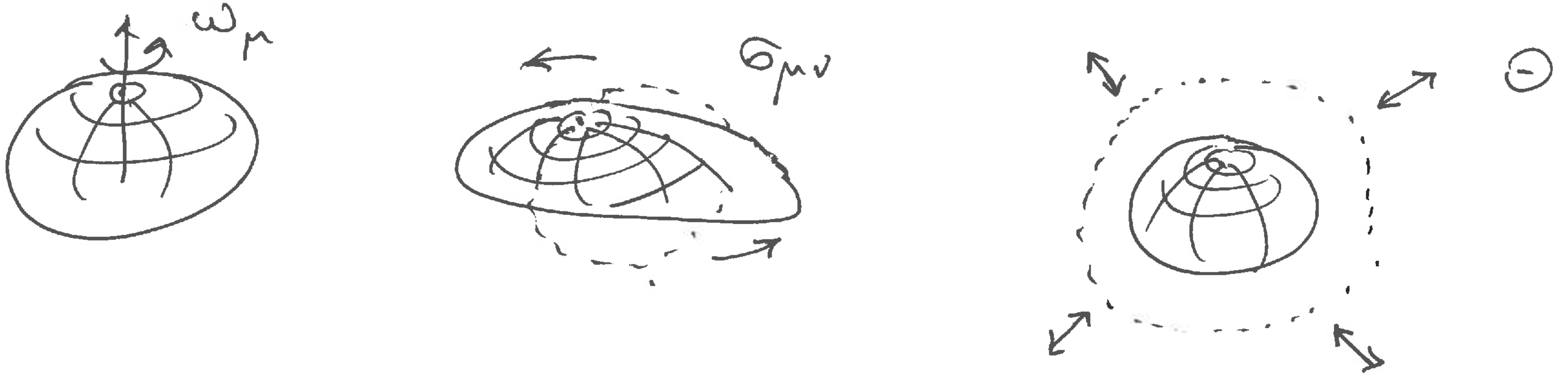
Eg



The $\underline{\text{shear tensor}}$ measures the changes in the $\underline{\text{ellipsoid's axes}}$ while $\underline{\text{preserving the volume}}$ (change in shape but not in volume).

$$\boxed{\text{cf } \Lambda^{TF}_{ij} = \Lambda_{ij} - \frac{1}{3}\Theta\delta_{ij}}$$

Conversely, the expansion tensor measures $\underline{\text{changes in volume without changing shape}}$



Energy momentum tensor

we have seen that if the distribution function is known, we can define the 1st, and 2nd moment of the distribution as

$$J^\mu = m N^\mu = mc \int p^\mu f \frac{d^3p}{p^0}$$: rest-mass density current (1st moment)

$$T^{\mu\nu} = c \int p^\mu p^\nu f \frac{d^3p}{p^0}$$: energy-momentum tensor (2nd moment)

However we can also obtain a different definition of these tensors without having to consider a distribution function. This is possible if we remember that in a rest-frame comoving with the fluid (indicated with hatted indices)

$J^{\hat{\mu}}$: flux of rest-mass density current in $\hat{\mu}$-direction

$T^{\hat{\mu}\hat{\nu}}$: flux of $\hat{\mu}$ momentum density in the $\hat{\nu}$-direction

In this way we have $J^{\hat{\mu}} = (\rho, 0, 0, 0)$

$T^{\hat{0}\hat{0}}$: (total) energy density

$T^{\hat{0}\hat{i}}$: flux of energy density in $\hat{i}$-th direction → energy flux

$T^{\hat{i}\hat{0}}$: flux of $\hat{i}$-th mom. density in $\hat{0}$-th direction } $T^{\hat{0}\hat{i}} = T^{\hat{i}\hat{0}}$

mom. density

$T^{\hat{i}\hat{j}}$: flux of $\hat{j}$-th " density in $\hat{i}$-th direction

More specifically if $E = \langle p^{\hat{0}} \rangle$: energy of particles in fluid element, then

$$T^{\hat{0}\hat{0}} = n\langle p^{\hat{0}}\rangle = e$$

$$T^{\hat{0}\hat{i}} = 0 \qquad \text{(perfect fluid)}$$

$$T^{\hat{i}\hat{j}} = 0 \; (i \neq j) \qquad \text{(symmetric tensor)}$$

$$T^{\hat{i}\hat{j}} = p \; (i = j) \qquad \text{isotropic pressure}$$

$$\Rightarrow \quad T^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

More generally

$$T^{\mu\nu} = \begin{pmatrix} \boxed{T^{00}} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & \cdots \\ T^{20} & \cdots & T^{22} & \cdots \\ T^{30} & \cdots & \cdots & T^{33} \end{pmatrix}$$

energy flux

momentum flux

isotropic pressure

momentum density

Without loss of generality we can express the above definitions using the fluid 4-velocity $\underline{u}$; in the comoving frame $u^{\hat{\mu}} = (1,0,0,0)$

$$(\cdot)\quad \begin{cases} J^{\hat{\mu}} = \rho u^{\hat{\mu}} \\ T^{\hat{\mu}\hat{\nu}} = e u^{\hat{\mu}} u^{\hat{\nu}} + p(\eta^{\hat{\mu}\hat{\nu}} + u^{\hat{\mu}} u^{\hat{\nu}}) \end{cases}$$

in the fluid rest-frame metric is locally flat

Eg $T^{\hat{i}\hat{j}} = e u^{\hat{i}} u^{\hat{j}} + p(\eta^{\hat{i}\hat{j}} + u^{\hat{i}} u^{\hat{j}}) = p\eta^{\hat{i}\hat{j}} = \begin{cases} p & \hat{i}=\hat{j} \\ 0 & \hat{i}\neq\hat{j} \end{cases}$

Because of covariance, we can now generalize $(\cdot)$ to any frame
ie $\eta^{\mu\nu} \to g^{\mu\nu}$; $u^{\hat{\mu}} \to u^{\mu}$

$$(\cdot\cdot)\quad \begin{cases} J^{\mu} = \rho u^{\mu} \\ T^{\mu\nu} = e u^{\mu} u^{\nu} + p(g^{\mu\nu} + u^{\mu} u^{\nu}) \end{cases}$$

$$= (e+p) u^{\mu} u^{\nu} + p g^{\mu\nu}$$

The definitions (..) can also be interpreted in the light of the considerations made above for the energy-mom. tensor. To this scope we need to decompose $\underline{T}$ in the direction defined by $\underline{u}$ and in the one orthogonal to it. To do this we can exploit the projection tensor $\underline{h}$

$$h_{\mu\nu} := u_\mu u_\nu + g_{\mu\nu} \implies h^\mu{}_\nu = u^\mu u_\nu + \delta^\mu{}_\nu$$

where $\underline{h} \cdot \underline{u} = 0 = (u_\mu u_\nu + g_{\mu\nu})u^\mu = -u_\nu + u_\nu$

As a result we can introduce

$$\underline{h} \cdot \underline{T} \cdot \underline{h}$$

$t_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta}$: fully spatial stress tensor $(t_{\mu\nu} u^\nu = 0)$

$t_\mu := -h^\alpha{}_\mu u^\beta T_{\alpha\beta}$: spatial momentum density

$t := t^\mu{}_\mu$: trace of $t_{\mu\nu}$

$e := u^\alpha u^\beta T_{\alpha\beta}$: energy density (projection of $\underline{T}$ in $\underline{u}$ direction)

With these definitions it is possible to construct a <u>general</u> <u>(geometric)</u> expression for the energy-momentum tensor

$$T_{\mu\nu} = e\, u_\mu u_\nu + 2\, u_{(\mu} L_{\nu)} + L_{\mu\nu}$$

Specializing to a perfect fluid: $L_\nu = 0$ and $L_{\mu\nu} = p h_{\mu\nu}$ $\square$

Having now an explicit expression for the energy density and pressure in terms of the energy-momentum density, we can go back and obtain these quantities directly from the distribution function

$$\varrho = -u_\mu J^\mu = -m\, u_\mu \int p^\mu f \frac{d^3p}{p^0}$$

$$e = u_\mu u_\nu T^{\mu\nu} = u_\mu u_\nu \int p^\mu p^\nu f \frac{d^3p}{p^0}$$

$$p = \frac{1}{3}\left(\eta_{\mu\nu} + u_\mu u_\nu\right) T^{\mu\nu} = \frac{1}{3}\left(\eta_{\mu\nu} + u_\mu u_\nu\right) \int p^\mu p^\nu f \frac{d^3p}{p^0}$$

these expressions relate quantities like ϱ, e, p directly to the distribution functions.

Energy conditions

If $e = u_\mu u_\nu T^{\mu\nu}$ is the energy density, it is reasonable to ask what conditions the projection of the energy-mom. tensor should satisfy.

$\boxed{\text{Weak energy condition}}$ requires $T_{\mu\nu}\xi^\mu \xi^\nu \geq 0 \overset{\xi = u}{\Longleftrightarrow} \boxed{e \geq 0}$

where $\underline{\xi}$ is future directed time-like vector $(\underline{\xi}\cdot\underline{\xi} < 0)$

A more stringent condition requires instead that

$$T^{\mu\nu}\xi_\mu \xi_\nu \geq -\frac{1}{2} T^\mu{}_\mu \quad \overset{\xi = u}{\Longleftrightarrow} \quad \boxed{e + 3p \geq 0}$$

$$e \geq -\frac{1}{2} T^\mu{}_\mu = -\frac{1}{2}(3p - e)$$

$\boxed{\text{strong energy condition}}$

$$e - \frac{e}{2} \geq -\frac{3}{2}p$$

$$e \geq -3p$$

$p = we$ $\quad w \geq -1/3$ weak condition

$e + 3p \geq 0 \Leftrightarrow e + 3we \geq 0 \quad e(1+3w) \geq 0 \quad w \geq -1/3 \,;\, e \geq 0$

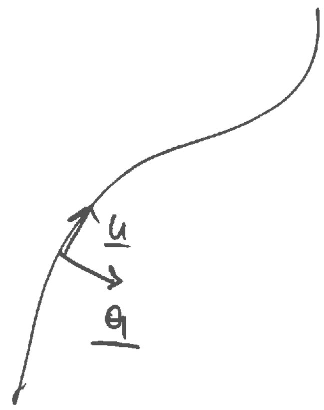
strong condition

Recap

kinematic properties of fluid

$u^\mu = dx^\mu / d\tau$ $\qquad$ $\underline{u} \cdot \underline{u} = -1$

$a^\mu = u^\nu \nabla_\nu u^\mu$ $\qquad$ $\underline{a} \cdot \underline{u} = 0$



Most general decomposition of $\nabla_\nu u_\mu$

$$\nabla_\nu u_\mu = \left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} \Theta h_{\mu\nu} - a_\mu u_\nu \right)$$

$\underline{h}$: projector orthog. to $\underline{u}$

$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$

$$\omega_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu \nabla_{[\alpha} u_{\beta]}$$

: kinematic vorticity tensor

$$\sigma_{\mu\nu} := \nabla_{(\mu} u_{\nu)} + a_{(\mu} u_{\nu)} - \frac{1}{3} \Theta h_{\mu\nu}$$

: shear tensor

$\Theta := h^{\mu\nu} \nabla_\mu u_\nu = \nabla_\mu u^\mu$: expansion

Consider fluid element as ellipsoid

$\underline{\omega}$: rotation around principal axes

$\underline{\sigma}$: changes in shape without change of volume

Θ: change in volume without change in shape $\square$

In comoving frame $\underline{J}$, $\underline{T}$ can be easily defined

$J^{\hat{\mu}} = (e, 0, 0, 0)$

$T^{\hat{0}\hat{0}}$: (total) energy density

$T^{\hat{0}\hat{i}}$: flux of en. den. in ith direction

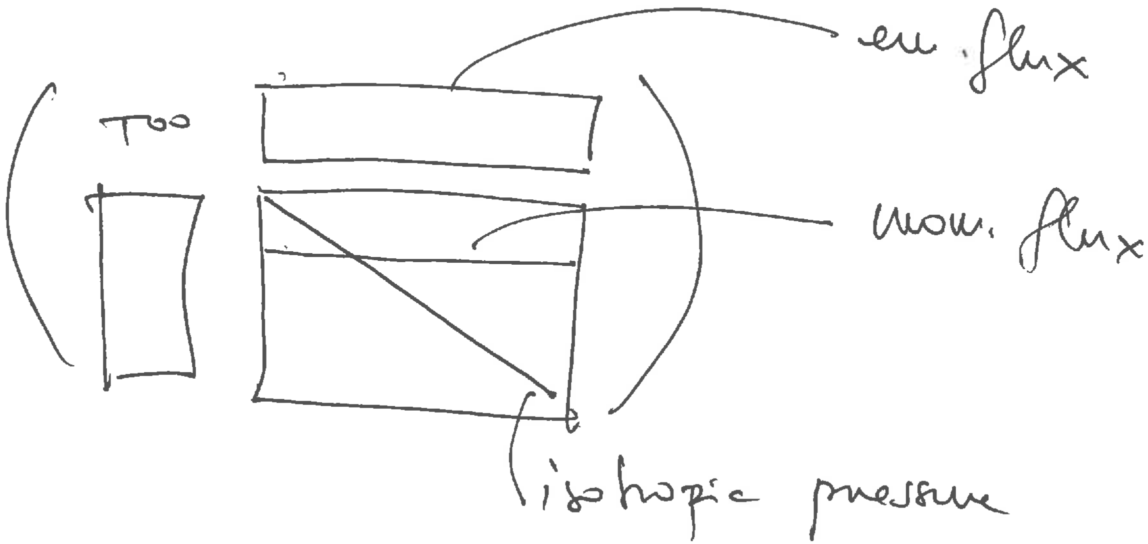
$T^{\hat{i}\hat{0}} = T^{\hat{0}\hat{i}}$

$T^{\hat{i}\hat{j}}$: flux of $\hat{j}$-th mom. den. in i-th direction

$$T^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e & & & \\ & p & & 0 \\ & & p & \\ 0 & & & p \end{pmatrix}$$

In general

$T^{\mu\nu} =$



$$\begin{cases} J^\mu = \rho u^\mu \\ T^{\mu\nu} = (e+p) u^\mu u^\nu + p g^{\mu\nu} \end{cases}$$

Most general expression

$$T_{\mu\nu} = e\, u_\mu u_\nu + 2 u_{(\mu} L_{\nu)} + L_{\mu\nu}$$

$$L_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta}$$

$$L_\mu = - h^\alpha{}_\mu u^\beta T_{\alpha\beta}$$: $L_\mu = 0$ for perfect fluid (no energy flux)

$$e = u^\alpha u^\beta T_{\alpha\beta}$$

$\square$

Relativistic hydrodynamic equations

$$\left.\begin{aligned} \nabla_\mu T^{\mu\nu} &= 0 \\ \nabla_\mu J^\mu &= 0 \end{aligned}\right\}$$ 5 eqs in 6 unknowns $\Rightarrow$ EOS needed to close system

Relativistic hydrodynamics equations

$u^\mu \nabla_\mu \rho = -\rho\,\Theta$

$$\nabla_\mu J^\mu = 0 = \nabla_\mu (\rho u^\mu) \iff u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu = 0 \quad : 1 \text{ eq.}$$

$$\nabla_\mu T^{\mu\nu} = 0 = \nabla^\mu \left[(e+p) u_\mu u_\nu + p g_{\mu\nu} \right] = 0$$

$$= \nabla^\mu (\rho h u_\mu u_\nu + p g_{\mu\nu}) = \underline{\nabla} \cdot \underline{T} = 0$$

4 eqs

5 eqs in 6 unknowns

$$\underbrace{(u^\mu,\ \rho,\ p,\ \epsilon)}_{3\ \ 1\ \ 1\ \ 1}$$

one EoS needed to close the system.

Conservation of energy and momentum are obtained after projecting $\underline{\nabla} \cdot \underline{T}$ in the direction along and orthogonal to $\underline{u}$

$$\underline{h} \cdot \underline{\nabla} \cdot \underline{T} = 0 \iff$$

$$h^\nu{}_\lambda \nabla_\mu T^{\mu\lambda} = h^\nu{}_\lambda \left[u^\lambda u^\mu \nabla_\mu (\rho h) + \rho h u^\mu \nabla_\mu u^\lambda + \rho h \Theta u^\lambda + g^{\lambda\mu} \nabla_\mu p \right]$$

$$= \rho h u^\mu \nabla_\mu u^\nu + h^\nu{}_\lambda g^{\lambda\mu} \nabla_\mu p = 0$$

$\underline{h} \cdot \underline{u} = 0$

$\Rightarrow$

(Exercise: derive Newtonian limit of eqs (Δ) and ($*$))

$$(\Delta) \quad \boxed{u^\mu \nabla_\mu u_\nu + \frac{h^\mu{}_\nu}{\rho h} \nabla_\mu p = 0} \iff$$

$$\boxed{\rho h\, a^\mu = -(g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu p}$$

relativistic Euler equation $\odot$

$a^\mu = 0$ if $p = 0$ or $\nabla_\mu p = 0$
i.e. geodesic motion for uniform pressure or dust.

Similarly

$$\underline{u} \cdot \underline{\nabla} \cdot \underline{T} = 0 \iff$$

$$u^\mu \nabla_\mu e + \rho h \Theta = 0$$

Recalling that the continuity equation implies $\Theta = -\frac{1}{\rho} u^\mu \nabla_\mu \rho$, so that

$$(*) \quad \boxed{u^\mu \nabla_\mu e - h u^\mu \nabla_\mu \rho = 0}$$

relativistic energy conservation eq. $\odot\odot$

$\odot$

$$\partial_t v^i + v^j \partial_j v^i = -\frac{1}{\rho} \partial_i p$$

$\odot\odot$

$$\partial_t \left(\tfrac{1}{2}\rho v^2 + \rho\epsilon\right) + \vec{\nabla}\cdot\left[\left(\tfrac{1}{2}\rho v^2 + \rho\epsilon + p\right)\vec{v}\right] = 0$$

Let's recall the 1st law of thermodynamics

$$de = h\,d\rho + \rho T ds$$

Then it's clear that (*) implies $\boxed{u^\mu \nabla_\mu s = 0}$ specific entropic is conserved along fluid lines

Stated differently: perfect fluids are adiabatic

($\nabla_\mu s = 0 \iff$ isentropic fluid).

Stationary flows

[rigorous definition will be made in future lecture]

The hydrodynamic eqs are non linear and their analytic solution is not possible in general. There are, however, regimes in which the equations simplify sufficiently that solutions can be found analytically.

Stationary flows (ie when $\frac{\partial}{\partial t} = 0$) are a good example.

Let's first work in Newtonian physics. The Euler equation reduces to

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\phi - \frac{\vec{\nabla}p}{\rho} \quad \Longleftrightarrow$$

$$\boxed{\begin{array}{ll} (ii) & \frac{1}{2}\vec{\nabla}(\vec{A}\cdot\vec{A}) = (\vec{A}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{A}) \\ (i) & \vec{A}\times(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}(\vec{A}\cdot\vec{B}) - (\vec{A}\cdot\vec{\nabla})\vec{B} \end{array}}$$

$$\vec{\nabla}\left(\frac{1}{2}\vec{v}^2\right) - \vec{v}\times(\vec{\nabla}\times\vec{v}) = -\vec{\nabla}\phi - \frac{\vec{\nabla}p}{\rho} \qquad (**)$$

1st law ; 2nd law for adiabatic flow

$$(i) \quad dh_N = d\left(\epsilon + \frac{p}{\rho}\right) \stackrel{ds=0}{=} \frac{1}{\rho}dp \qquad \frac{D}{Dt}s = \vec{v}\cdot\vec{\nabla}s = 0$$

$\partial_t s = 0$

As a result

$$\frac{D}{Dt}h_N = \frac{1}{\rho}\frac{D}{Dt}p \quad \Longleftrightarrow \quad (\vec{v}\cdot\vec{\nabla})h_N = \frac{1}{\rho}\vec{v}\cdot\vec{\nabla}p$$

Evaluating (**) along $\vec{v}$ direction yields $\quad \vec{v}\cdot(\vec{v}\times(\vec{\nabla}\times\vec{v})) = 0$

$$\vec{v}\cdot\vec{\nabla}\left(\frac{1}{2}\vec{v}^2 + \phi + h_N\right) = \vec{v}\cdot\vec{\nabla}\left(\frac{1}{2}\vec{v}^2 + \phi + \epsilon + \frac{p}{\rho}\right) = 0 \quad \Rightarrow$$

More in general (that is, also for non-stationary flows)

$$\frac{D}{Dt}\left(\frac{1}{2}\vec{v}^2+\phi+h_N\right)=\frac{D B_N}{Dt}=0 \iff \boxed{B_N=\frac{1}{2}\vec{v}^2+\phi+h_N=\text{const.}}$$

B_N: Bernoulli const., Bernoulli equation

Example

Cold fluid in gravitational potential, ie our atmosphere

$$\boxed{\frac{1}{2}\rho\vec{v}^2+\rho|\vec{g}|z+p=\text{const}}$$

$\vec{g}$: gravitational acceleration

(with this equation it is possible to model most stationary flows on Earth, eg flow past aeroplane wing or a car body. □

Stationary flows can be derived also for the relativistic hydrodynamics equations as the result of a timelike Killing vector. Because this is a generic result, we need to take a step back and recall the Killing equation; before that, let's look at Newtonian irrot. flows

Newtonian irrotational flows

Define

$\vec{\omega}_N := \vec{\nabla} \times \vec{v}$: vorticity vector

Start from Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} p + \frac{\vec{F}}{m}$$

Take the curl of both sides to obtain for a conservative force $(\vec{\nabla} \times \vec{F} = 0)$ (Exercise)

$$\frac{\partial \vec{\omega}_N}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}_N) - \vec{\nabla} \times \underbrace{\left(\frac{\vec{\nabla} p}{\rho}\right)}$$

$\Downarrow$ under assumptions on the right

$$\boxed{\partial_t \vec{\omega}_N = \vec{\nabla} \times (\vec{v} \times \vec{\omega}_N)}$$

this term vanishes if:

- flow is isentropic $(\vec{\nabla} p/\rho = \vec{\nabla} h_N;\ \vec{\nabla} \times \vec{\nabla} h_N = 0)$
- fluid is barotropic $p = p(\rho) \Rightarrow \vec{\nabla} p \parallel \vec{\nabla} \rho$
- fluid is incompressible $\rho = \text{const};\ \vec{\nabla} \times \vec{\nabla} p = 0$

Any flow that is initially irrotational, will remain irrotational: $\partial_t \vec{\omega}_N = 0$ if $\vec{\omega}_N = 0$

Stated differently no matter how complex the flow, vortices cannot be created if initially absent.

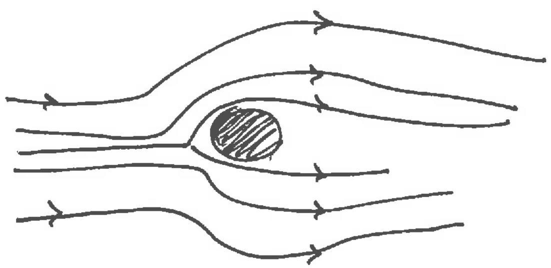
If flow is irrotational: $\vec{\nabla} \times \vec{v} = 0 \Rightarrow \vec{v} = \vec{\nabla} \varphi(t, r)$: grad. of potential $\Rightarrow$

$\boxed{\text{potential-flow equation}}$

If flow is also incompressible at all times, then $\vec{\nabla} \cdot \vec{v} = 0 \Leftrightarrow$

$$\boxed{\vec{\nabla} \cdot \vec{\nabla} \varphi = \Delta \varphi(r) = 0}$$

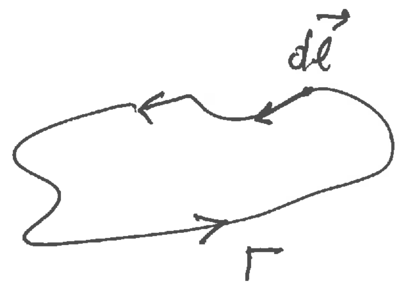
: elliptic equation with solution also in complex domains if boundary conditions are specified. Ex. flow past a solid body. ▫



Kelvin-Helmholtz theorem

Define the circulation $\varphi_N := \oint_\Gamma \vec{v} \cdot \vec{d\ell}$

Newtonian



KH theorem states

$$\frac{D}{Dt}\mathcal{C}_N = (\partial_t + \vec{v}\cdot\vec{\nabla})\oint_\Gamma \vec{v}\cdot d\vec{\ell} = 0$$

: law of conservation of circulation.

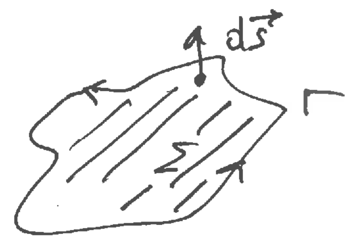
Proof.

comoving

$$\frac{D}{Dt}\oint \vec{v}\cdot d\vec{\ell} = \oint_\Gamma \frac{D}{Dt}\left(\vec{v}\cdot d\vec{\ell}\right) = \oint_\Gamma \frac{D\vec{v}}{Dt}\cdot d\vec{\ell} + \oint_\Gamma \vec{v}\cdot \overbrace{\left(\frac{D}{Dt}d\vec{\ell}\right)}^{d\vec{v}}$$

Stoke's theor.

$$= \iint_\Sigma \left[\vec{\nabla}\times\left(\frac{D}{Dt}\vec{v}\right)\right]\cdot d\vec{S} + \oint_\Gamma \vec{v}\cdot d\vec{v} \qquad (\Delta)$$



$$\oint_\Gamma \vec{v}\cdot d\vec{v} = \frac{1}{2}\oint d\vec{v}^{\,2} = 0$$

line integral of exact differential over closed curve

$$\frac{D\vec{v}}{Dt} = -\frac{\vec{\nabla}p}{\rho} \Rightarrow$$ the first term in (Δ) is

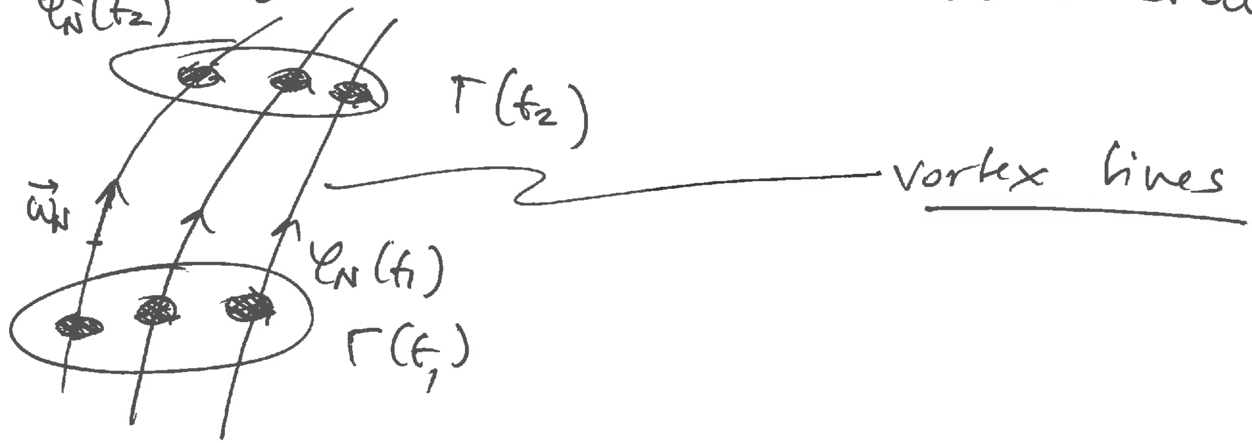
$$\frac{D}{Dt}\oint \vec{v}\cdot d\vec{\ell} = \int_\Sigma \vec{\nabla}\times\left(-\frac{\vec{\nabla}p}{\rho}\right)\cdot d\vec{S} = 0$$

for the cases considered above (isentropic, or barotropic or incompressible flow)

As a result, $\frac{D\mathcal{C}_N}{Dt} = 0$

$$= \frac{D}{Dt}\oint_\Gamma \vec{v}\cdot d\vec{\ell} = \frac{D}{Dt}\int_\Sigma (\vec{\nabla}\times\vec{v})\cdot d\vec{S} = \frac{D}{Dt}\int \vec{\omega}_N\cdot d\vec{S} = 0$$

In an irrotational flow, the circulation is conserved and the number of vortex lines (trajectories of vortices *) crossing any area comoving with fluid remains constant



* vortex: region of the flow with non zero vorticity

We next explore the relativistic equivalents of stationary and irrotational flows.

Before doing that, however, it is useful to derive a very general result for flows following the same symmetries of the spacetime.

Perfect fluids and symmetries.

Theorem

Let $\underline{T}$ satisfy a conservation equation $\underline{\nabla}\cdot\underline{T}=0$ and $\underline{\xi}$ be a killing vector field ①, then the quantity $\underline{Q}:=\underline{T}\cdot\underline{\xi}$ also satisfies a conservation law, ie $\underline{\nabla}\cdot\underline{Q}=0$

Proof.

$$\underline{\nabla}\cdot\underline{Q}=\nabla_\mu Q^\mu=\nabla_\mu(T^{\mu\nu}\xi_\nu)=(\nabla_\mu T^{\mu\nu})\xi_\nu+T^{\mu\nu}\nabla_\mu\xi_\nu$$

$$=0-T^{\mu\nu}\underbrace{\nabla_\mu\xi_\nu}_{\text{antisym.}}=0$$

Killing vector field ⊙

A vector field $\underline{\xi}$ is said to be a Killing field if

$$\mathcal{L}_{\underline{\xi}}\, \underline{g} = 0$$: the metric is Lie dragged along $\underline{\xi}$

$$\mathcal{L}_{\underline{\xi}}\, \underline{g} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\alpha\nu} \partial_\mu \xi^\alpha = 0$$

$\underline{\xi}$: generator of the associated symmetry group G

$$\mathcal{L}_{\underline{\xi}}\, \underline{g} = 0 \iff \nabla_{(\mu} \xi_{\nu)} = 0$$: Killing equations (define isometries of the spacetime).

Ex.

$\xi^\mu = (1,0,0,0)$: time like Killing vector. Then

$\partial_t g_{\mu\nu} = 0$: the metric is time independent

To derive the general result of perfect fluids in the presence of symmetries, let's go back to the momentum conservation equation

$$u^\mu \nabla_\mu u_\nu + \frac{1}{\rho h} h^\mu{}_\nu \nabla_\mu p = 0$$

which can be written as

$$\rho u^\mu \nabla_\mu (h u_\nu) - \rho u_\nu u^\mu \nabla_\mu h = -\nabla_\nu p - u^\mu u_\nu \nabla_\mu p$$

A perfect fluid is adiabatic, ie $u^\mu \nabla_\mu s = 0 \Rightarrow dp = \rho dh \Longleftrightarrow$

$u^\mu u_\nu \nabla_\mu p = u^\mu u_\nu \rho \nabla_\mu h$

$$\Rightarrow u^\mu \nabla_\mu (h u_\nu) = -\frac{\nabla_\nu p}{\rho} = -\nabla_\nu h \quad (*)$$

(2nd - 4th term cancel)

Recall now that

$$\mathcal{L}_{\underline{u}} u_\nu = u^\mu \nabla_\mu u_\nu + \underbrace{u_\mu \nabla_\nu u^\mu}_{=0\ \boxed{\cdot}} = u^\mu \nabla_\mu u_\nu = a_\nu$$

$\boxed{\cdot}$ $u_\mu \nabla_\nu u^\mu = \nabla_\nu (u_\mu u^\mu) - u^\mu \nabla_\nu u_\mu = -u^\mu \nabla_\nu u_\mu \Rightarrow 2 u^\mu \nabla_\nu u^\mu = 0$ qed

As a result (*) can be written as ⊙

$$\mathcal{L}_{\underline{u}}(hu_\mu) = -\frac{1}{e}\nabla_\mu p = -\nabla_\mu h \quad \mathcal{L}_{\underline{u}} h$$

Contract the LHS with a killing field

$$\xi^\mu \mathcal{L}_{\underline{u}}(hu_\mu) = \mathcal{L}_{\underline{u}}(\xi^\mu h u_\mu) - hu_\mu \mathcal{L}_{\underline{u}} \xi^\mu \qquad (\cdot\cdot)$$

$$= \mathcal{L}_{\underline{u}}(h\xi^\mu u_\mu) - h \underbrace{u^\mu u^\gamma}_{\text{sym.}} \underbrace{\nabla_\gamma \xi_\mu}_{\text{antisym.}}$$

$$= \mathcal{L}_{\underline{u}}(h\xi^\mu u_\mu)$$

As a result

$$\mathcal{L}_{\underline{u}}(h\xi^\mu u_\mu) = -\frac{\xi^\mu \nabla_\mu p}{e} = -\xi^\mu \nabla_\mu h = -\frac{1}{e}\mathcal{L}_{\underline{\xi}} p = -\mathcal{L}_{\underline{\xi}} h$$

If the fluid satisfies the same symmetries of the spacetime, then $\mathcal{L}_{\xi} \underline{B} = 0$ $\underline{B}$ generic fluid-related quantity

$\Rightarrow \mathcal{L}_{\underline{\xi}}\, p = 0 = \mathcal{L}_{\underline{\xi}}\, h \quad \Rightarrow$

$$\boxed{\mathcal{L}_{\underline{u}} \left(h\, \xi^{\mu} u_{\mu} \right) = 0 = \nabla_{\underline{u}} \left(h\, \xi^{\mu} u_{\mu} \right) = 0} \qquad (**)$$

In other words, if $\underline{\xi}$ is a symmetry generator of the spacetime and the fluid shares the same symmetry, then the scalar quantity $h\, \underline{u} \cdot \underline{\xi}$ <u>is conserved along the fluid lines</u>.

Expression $(**)$ is similar to the condition for geodesic curves with tangent $\underline{u}$; in that case infact $\left[\mathcal{L}_{\underline{u}} u_{\nu} = a_{\nu} \overset{\text{geod.}}{=} 0 \right]$

$\mathcal{L}_{\underline{u}} \left(u_{\mu} \xi^{\mu} \right) = 0 \qquad \underline{u} \cdot \underline{\xi}$ is conserved along the geodesic.

$(**)$ is the fluid extension of the conservation of some quantity

Ex. $\underline{\xi} = \frac{\partial}{\partial \phi}$, ie $\xi^{\mu} = (0, 0, 0, 1)$ in spherical coords.

Then $\quad h u_\mu \xi^\mu = h u_\phi$ is conserved

↳ specific angular momentum

$$\underset{\substack{\text{Newtonian}\\ \text{limit}}}{h u_\phi \longrightarrow \Omega r^2} = \text{conserved}, \quad \text{where } \Omega = d\phi/dt$$

All of this tensor algebra has not been in vain because the relativistic Bernoulli theorem is now trivial to derive.

For a stationary flow, there exist a timelike killing vector $\underline{\xi} = \partial_t$; $\xi^\mu = (1,0,0,0) \Rightarrow$

Bernoulli's theorem $\quad \boxed{\mathcal{L}_{\underline{u}} (h \underline{u} \cdot \underline{\xi}) = \mathcal{L}_{\underline{u}} (h u_t) = 0}$ $\quad \boxed{h u_t}$ is the Bernoulli constant and is conserved along fluid lines

The Newtonian limit of u_t is (Exercise)

$$-u_t \simeq 1 + \phi + \frac{1}{2} v^i v_i \quad \Rightarrow$$

$$-h u_t = \left(1 + \epsilon + \frac{p}{\rho}\right)\left(1 + \phi + \frac{1}{2} v^i v_i\right)$$

To take the Newtonian limit of $h u_t$ we need to neglect the terms related to the rest-mass density ie terms $O(1)$

$$\Rightarrow \quad -h u_t \longrightarrow \left(\frac{1}{2}\vec{v}^2 + \phi + \epsilon + \frac{p}{\rho}\right) + \text{higher-order terms} = \text{const}.$$

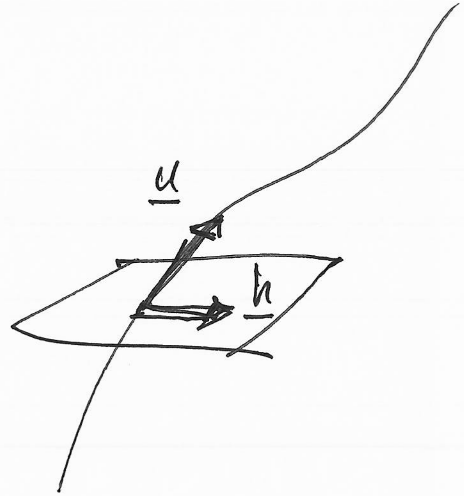
This is the Bernoulli eq. we have already encountered.

<u>B_N is replaced by $h u_t$.</u>

Recap

Relativistic hydrodynamic equations

$$\underline{\nabla}\cdot\underline{J}=0\ ; \quad \underline{\nabla}\cdot\underline{T}=0\ ; \quad p=p(\rho,\epsilon,\dots)$$



$\underline{h}\cdot\underline{\nabla}\cdot\underline{T}=0$: momentum conservation equation

$$u^\mu\nabla_\mu u_\nu = -\frac{h^\mu{}_\nu}{\rho h}\nabla_\mu p \quad \Longleftrightarrow \quad (\partial_t + \vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} p$$

$\underline{u}\cdot\underline{\nabla}\cdot\underline{T}=0$: energy conservation equation

$$u^\mu\nabla_\mu e = h\, u^\mu \nabla_\mu \rho \quad \Longleftrightarrow \quad \partial_t\left(\tfrac{1}{2}\rho\vec{v}^2+\rho\epsilon\right)+\vec{\nabla}\cdot\left[\left(\tfrac{1}{2}\rho\vec{v}^2+\rho\epsilon+p\right)\vec{v}\right]=0$$

Recalling that the 1st law states:

$$de = h\,d\rho + \rho T ds \quad \Longleftrightarrow \quad u^\mu\nabla_\mu e = h\,u^\mu\nabla_\mu\rho + \rho T u^\mu \nabla_\mu s$$

The energy conservation equation implies

$$\boxed{u^\mu \nabla_\mu s = 0}$$

perfect fluids are adiabatic, ie specific entropy is conserved along fluid lines □

* Stationary flows : $\partial_t = 0$

$$\frac{D}{Dt} B_N = (\partial_t + \vec{v}\cdot\vec{\nabla}) B_N = D_t \underbrace{\left(\frac{1}{2}\vec{v}^2 + \phi + \epsilon + \frac{p}{\rho}\right)}_{B_N \text{ : Bernoulli constant}} = 0$$

Useful for modelling aerodynamical flows

* Irrotational flows : $\vec{\omega}_N = \vec{\nabla}\times\vec{v} = 0$

$\partial_t \vec{\omega}_N = \vec{\nabla}\times(\vec{v}\times\vec{\omega}_N)$ ie $\partial_t \vec{\omega}_N = 0$ if $\vec{\omega}_N = 0$ initially

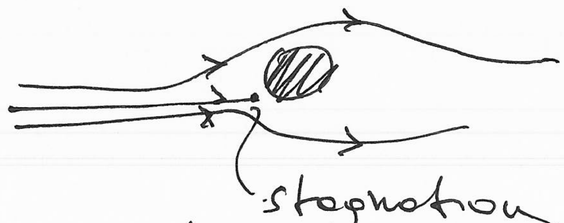
Vorticity cannot be produced if zero initially.

$\vec{\omega}_N = 0 \iff \vec{v} = \vec{\nabla}\varphi$ and if flow is also incompressible

$\vec{\nabla}\cdot\vec{v} = 0 \iff \nabla^2\varphi = 0$: equation for potential flow.

Elliptic problem with boundary conditions: $\varphi = \varphi(r)$

Potential flow is useful to study stationary motion past an object with boundary conditions on φ and its derivatives on the object



Kelvin-Helmholtz theorem

Given a circulation $\mathcal{C}_N := \oint_\Gamma \vec{v}\cdot d\vec{\ell}$

$\frac{D\mathcal{C}_N}{Dt} = 0$: circulation is conserved $\iff$ no. of vortex lines cannot change

Perfect fluids with symmetries

If $\underline{T}$ is a conserved rank 2 tensor ($\underline{\nabla} \cdot \underline{T} = 0$) and $\underline{\xi}$ is a killing field ($\mathcal{L}_{\underline{\xi}} \underline{g} = 0$) then also $\underline{Q} := \underline{T} \cdot \underline{\xi}$ is conserved: $\underline{\nabla} \cdot \underline{Q} = 0$

A bit of algebra shows

$$\mathcal{L}_{\underline{u}} (h \xi^\mu u_\mu) = -\frac{1}{\varrho} \mathcal{L}_{\underline{\xi}} p = -\mathcal{L}_{\underline{\xi}} h = 0$$

fluid obeys same sym. of spacetime

$\Rightarrow$ $h \underline{\xi} \cdot \underline{u}$ is conserved along fluid lines.

Ex: spacetime is stationary $\mathcal{L}_{\underline{\xi}} \underline{g} = 0 = \partial_t g_{\mu\nu} \Rightarrow$

$\xi^\mu = (1,0,0,0) \Rightarrow h u_t = \text{const.}$ Indeed $h u_t = B$: Bernoulli constant

$h u_t \to \frac{1}{2} \vec{v}^2 + \phi + \epsilon + \frac{p}{\varrho}$ in Newtonian limit

$\xi^\phi = (0,0,0,1)$: $\partial_\phi g_{\mu\nu} = 0$ $\qquad h u_\phi = \text{const}$

$h u_\phi$: specific angular momentum $\qquad h u_\phi \to \Omega r^2$ in Newtonian limit

Relativistic Irrotational flows.

First we introduce the concept of vorticity tensor (cf kinematic vorticity tensor)

$w_\mu := h u_\mu$: specific enthalpy current (canonical 1-form of fluid element)

$$\Omega_{\mu\nu} := 2 \nabla_{[\mu} w_{\nu]} = \nabla_\nu (h u_\mu) - \nabla_\mu (h u_\nu) = \partial_\nu (h u_\mu) - \partial_\mu (h u_\nu)$$

$\underline{\Omega}$ distinct from $\underline{\omega}$ (canonical vorticity two-form)

$\underline{\Omega}$ contains specific enthalpy and hence "inertial" role of fluid

It's possible to show that (Exercise)

$$\Omega_{\mu\nu} = 2h \left(\omega_{\mu\nu} - a_{[\nu} u_{\mu]} + u_{[\mu} \nabla_{\nu]} \ln h \right)$$

Only for a test cold fluid ($e = 0 = p$; $h = 1$) $\underline{\Omega}$ and $\underline{\omega}$ are proportional

$$\underline{\Omega} = 2\underline{\omega}.$$

Contract $\underline{\Omega}$ with $\underline{u}$ and use mom. conservation eq. to get

$$u^\nu \Omega_{\mu\nu} = \nabla_\mu h - \frac{1}{e}\nabla_\mu p$$ (Exercise)

and use 1st law of thermodynamics $e\nabla_\mu h - \nabla_\mu p = eT\nabla_\mu s$

so that

$$\boxed{\Omega_{\mu\nu} u^\nu = T\nabla_\mu s}$$ Carter – Lichnerowicz eq. of motion

Irrotational perfect fluids are isentropic ($\underline{\Omega} = 0$)

(cf perfect fluids are adiabatic $u^\mu \nabla_\mu s = 0$)

Note: $\underline{\Omega} \cdot \underline{u} \neq 0$ unless $\underline{\nabla} s = 0$

If $\underline{\xi}$ is Killing field, then (Exercise)

$$\nabla_\nu (h u_\mu \xi^\mu) = 0$$: in other words,

$h \underline{u} \cdot \underline{\xi}$ is conserved not only along fluid lines $(\nabla_{\underline{u}} (h \underline{u} \cdot \underline{\xi}) = 0)$ but is the same across the fluid. $\square$

In analogy with kinematic vorticity four-vector, we can define

$$\Omega^\mu := {}^*\Omega^{\mu\nu} u_\nu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \Omega_{\alpha\beta} u_\nu$$: vorticity four-vector

where $\Omega^\mu u_\mu = 0$ and $\boxed{\Omega^\mu = 2 h \omega^\mu}$ (Exercise)

and $\dot{\Omega}^\mu = u^\nu \nabla_\nu \Omega^\mu = 2 u^\nu \nabla_\nu (h \omega^\mu)$: in other words, vorticity will remain zero if initially zero.

If the vorticity tensor is zero, $\Omega_{\mu\nu} = 0$, then the specific enthalpy current can be taken as the gradient of a scalar function

$$w_\mu := h u_\mu = \nabla_\mu \psi$$

$$\boxed{\begin{array}{l} \text{cf. } \vec{\nabla} \times \vec{v} = 0 \iff \vec{v} = \vec{\nabla}\phi \\ \nabla_\mu w_\nu = \nabla_\nu w_\mu \end{array}}$$

$$\Rightarrow \quad u_\mu = \frac{1}{h} \nabla_\mu \psi \quad \Rightarrow \quad \rho u_\mu = \left(\frac{\rho}{h}\right) \nabla_\mu \psi .$$

Using the continuity equation $\nabla_\mu (\rho u^\mu) = 0$ we obtain

$$(\square) \qquad \boxed{\nabla_\mu \left[\left(\frac{\rho}{h}\right) \nabla^\mu \psi \right] = 0}$$

potential flow equation on a general curved background

Eq. $(\square)$ can be written alternatively as (Exercise)

$$(\square\square) \qquad \nabla^\mu \nabla_\mu \psi + \left(\frac{1}{c_s^2} - 1\right) (\nabla_\mu \ln h) \nabla^\mu \psi = 0$$

nonlinear eq. in the stream function ψ

Proof

$$(\square) \Longleftrightarrow \left(\frac{\rho}{h}\right) \nabla_\mu \nabla^\mu \varphi + \nabla_\mu \varphi \left[\frac{\nabla^\mu \rho}{h} - \frac{\rho}{h^2} \nabla^\mu h\right] = 0$$

$$\left(\frac{\rho}{h}\right) \nabla_\mu \nabla^\mu \varphi - \underbrace{(\nabla^\mu \varphi)(\nabla_\mu h)\left(\frac{\rho}{h^2}\right)\left(1 - \frac{h}{\rho}\frac{\nabla_\mu \rho}{\nabla_\mu h}\right)} = 0$$

$$(\nabla_\mu h)\left(\frac{\rho}{h^2}\right)\left(1 - \frac{\nabla_\mu \ln \rho}{\nabla_\mu \ln h}\right) =$$

$$c_s^2 = \left(\frac{\partial \ln h}{\partial \ln \rho}\right)_s$$

$$(\nabla_\mu h)\left(\frac{\rho}{h^2}\right)\left(1 - \frac{1}{c_s^2}\right) =$$

$$h = \frac{e+p}{\rho} = \frac{2e}{\rho} \propto \frac{\rho^2}{\rho} \sim \rho$$

$$\left(\frac{\rho}{h}\right) \nabla_\mu \ln h \left(1 - \frac{1}{c_s^2}\right) \qquad \text{qed}$$

Eq $(\square\square)$ can be simplified if $h \propto \rho$, eg. when $p = e$; $e \propto \rho^2$ (ultra-stiff EOS). $c_s^2 \longrightarrow 1$ and second term is zero.

$\Longrightarrow$ ($\square\square$) $\Longleftrightarrow$ $\boxed{\nabla_\mu \nabla^\mu \varphi = 0}$ linear eq. for a massless scalar field.

$\square$

Relativistic Kelvin-Helmholtz theorem

In analogy with what done in Newtonian hydrodynamics we can define a circulation of the four-velocity

$$\mathcal{C} := \oint_\Gamma h u_\alpha \, dx^\alpha$$

$\boxed{\text{cf. } \mathcal{C}_N = \oint_\Gamma \vec{v} \cdot d\vec{\ell} \qquad \vec{v} \leftrightarrow h\underline{u}}$

It is then possible to prove that

$$\boxed{u^\mu \nabla_\mu \mathcal{C} = u^\mu \nabla_\mu \oint_\Gamma h u_\alpha \, dx^\alpha = 0}$$

ie the circulation $\mathcal{C}$ is conserved along fluid lines.

Relativistic hydrodynamics

We have seen the corresponding eqs are

(*) $\nabla_\mu (J^\mu) = \nabla_\mu (\rho u^\mu) = 0$

(**) $\nabla_\mu T^{\mu\nu} = 0 \quad \begin{cases} h^\alpha{}_\mu \nabla_\nu T^{\mu\nu} = 0 & \text{mom. conservation} \\ u_\mu \nabla_\nu T^{\mu\nu} = 0 & \text{energy conservation} \end{cases}$

These equations are <u>nonlinear</u> and to better appreciate the implications of this statement, let me recall that (*), (**) can be written in a general form as

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = \underline{S}$$

A large class of equations in mathematical physics (eg. EFEs, eqs. of hydrodynamics and MHD)) can be written in a compact form as

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = \underline{S} \qquad (*)$$

$$\partial_t U_J + (A^i)_{JK} \nabla_i U_K = S_J$$

where $\underline{U} = \{U_1, U_2, \dots U_J\}$: state vector

$\underline{S} = \{S_1, S_2, \dots S_J\}$: source term

A : matrix of coefficients

The properties of the system (*) depend on the properties of A, and S.

(i) $\left\{ \begin{array}{l} a_{JK} : \text{elements of } A \\ a_{JK} = \text{const.} \; ; \quad S_J = \text{const} \end{array} \right\}$ (*) is a $\overset{\text{LINEAR}}{\vee}$ system of equations with constant coefficients

(ii) $\{ a_{JK} = a_{JK}(x,t) \; ; \quad S_J = S_J(x,t) \}$ (*) is a LINEAR system with variable coefficients

(iii) $A = A(\underline{u})$ (*) is a non LINEAR system $\overset{\text{or quasi-linear}}{}$

(often referred to as quasi-linear)

(129b)

More importantly, the system (*) is said to be (strongly) HYPERBOLIC if A is diagonalizable with a set of real eigenvalues $\lambda_1, \dots \lambda_N$ and a set of N linearly independent right eigenvectors, ie if

$$\Lambda := R^{-1} A R = \mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_N)$$

R: matrix of right eigenvectors $R^{(i)}$

$$A R^{(i)} = \lambda_i R^{(i)}$$

$\lambda_i \in \mathbb{R}$: real eigenvalues (can be the same)

(*) is said to be STRICTLY HYPERBOLIC if $\lambda_{(i)}$ are real and distinct

(*) is said to be SYMMETRIC HYPERBOLIC if A is symmetric, ie $A = A^T$

(*) is said to be WEAKLY HYPERBOLIC if A is not diagonalizable

Examples of hyperbolic equations are

- advection equation $\partial_t u + v\, \partial_x u = 0$
- wave equation $\partial_t^2 u - v^2 \partial_x^2 u = 0$
- hydrodynamic equations (inviscid)
- Einstein equations (only in harmonic coordinates $\Box x^\alpha = 0$)

The importance of hyperbolicity is strictly related with that of <u>WELL POSEDENESS</u> of the Cauchy initial - value problem.

$\underline{u}(x,0)$: initial data

$\underline{u}(x,t)$: solution of set (*) at time t

(*) is <u>well - posed</u> if

$$\| u(x,t) \| \leq k\, e^{at} \| u(x,0) \|$$

$k, a \in \mathbb{R}$: constants.

In other words the solution is always bounded by some exponential of the initial data ("it does not blow up..!")

An important theorem of hyperbolic systems states

$$\boxed{(*)\ \text{a hyperbolic set of equations} \Rightarrow (*) \text{ is well posed}}$$

Opposite implication not true.

It follows that a weakly hyperbolic system is not guaranteed to be well-posed and indeed the numerical solution leads to the growth of unstable modes ("codes crash"...)

Let's go back to Newtonian hydrodynamics

$$\begin{cases} \partial_t \rho + v^i \partial_i \rho + \rho\, \partial_i v^i = 0 \\ \partial_t v^i + v^j \partial_j v^i + \frac{1}{\rho} \partial^i p = 0 \\ \partial_t s + v^i \partial_i s = 0 \end{cases} \iff \begin{cases} \partial_t \rho + \vec{\nabla}\cdot(\rho \vec{v}) = 0 \\ \partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla} p = 0 \\ \partial_t s + (\vec{v}\cdot\vec{\nabla}) s = 0 \end{cases}$$

s: specif entropy $= \frac{S}{M}$

This set can be written in the general form (see previous lecture)

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = 0 \qquad (*)$$

where $\underline{u} = \{\rho\, v^i, s\}^T = \begin{pmatrix} \rho \\ v^1 \\ v^2 \\ v^3 \\ s \end{pmatrix}$: state vector

A : 3 5×5 matrices

$$A^1 = \begin{pmatrix} r' & \rho & 0 & 0 & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & r' & 0 & 0 & \frac{1}{\rho}\partial_s p \\ & & r' & & \\ & 0 & & r' & 0 \\ & & & & r' \end{pmatrix}$$

$A^2 = \ldots$

$A^3 = \ldots$

Recap

- Irrotational flows. Define vorticity tensor

$\Omega_{\mu\nu} := 2\nabla_{[\mu} w_{\nu]}$ where $w_\mu := h u_\mu$: specific enthalpy current

Note that $\underline{\Omega} \neq \underline{\omega}$: enthalpy introduce inertial features to kinematic vorticity $\underline{\omega}$

$\Omega_{\mu\nu} = \omega_{\mu\nu}$ only for $\epsilon = 0 = p$; $h = 1$ geodesic motion

- Eqs. of motion à la Carter-Lichnerowicz $\Omega_{\mu\nu} u^\nu = T \nabla_\mu s$

$\Rightarrow$ irrotational flows are isentropic

$\Omega^\mu := {}^*\Omega^{\mu\nu} u_\nu$: vorticity four-vector (not kinematical)

$\Omega^\mu u_\mu = 0$; $\Omega^\mu = 2 h \omega^\mu$

$u^\nu \nabla_\nu \Omega^\mu = 2 u^\nu \nabla_\nu (h \omega^\mu)$ vorticity remains zero if zero initially

If $\Omega_{\mu\nu}=0$, then $w_\mu = h u_\mu = \nabla_\mu \psi$ $\quad$ cf. $\vec{\nabla}\times\vec{v}=0 \Leftrightarrow \vec{v}=\vec{\nabla}\varphi$

Equations of hydrodynamics are expressed by single scalar elliptic equation

$$\nabla_\mu\left[\left(\frac{\rho}{h}\right)\nabla^\mu\psi\right]=0$$

potential flow equation (genuine curved background)

□

- Relativistic Kelvin-Helmholtz theorem

$$\mathcal{C}:=\oint h u_\alpha dx^\alpha \qquad \text{then} \qquad u^\mu\nabla_\mu \mathcal{C} = u^\mu \nabla_\mu \oint h u_\alpha dx^\alpha = 0$$

Circulation is conserved along fluid lines (no. of vortices does not change in perfect-fluid hydrodynamics).

- Relativistic hydrodynamic equations

$$\begin{cases} \nabla_\mu J^\mu = 0 \\ h^\alpha{}_\mu \nabla_\nu T^{\mu\nu} = 0 \\ u_\mu \nabla_\nu T^{\mu\nu} = 0 \end{cases}$$

can be cast as $\partial_t \underline{U} + \underline{A}\,\nabla\cdot\underline{U} = \underline{S}$ $\quad (*)$

$\underline{u}$: state vector; $\underline{S}$: source vector; A : matrix of coeffs.

(*) is linear if $a_{JK} \in A$ are constant or simple functions of space and time

(*) is non linear (quasi-linear) if $a_{JK} = a_{JK}(\underline{u})$

(*) is hyperbolic if A diagonalizable with N real eigenvalues and a set of lin. indep. right eigenvectors

$$\Lambda := R^{-1} A R \qquad \Lambda = \mathrm{diag}(\lambda_1, \dots \lambda_N) \qquad \lambda_i \in \mathbb{R}$$

$$A R^{(i)} = \lambda_i R^{(i)}$$

(*) is weakly hyperbolic if A non diagonalizable, ie the set of eigenvector is not complete

Theorem : a hyperbolic set of equation is well-posed.

Given an IVP $u = u(x,t)$ with $u_0 = u(x,0)$, the IVP is well-posed iff $\| u(x,t) \| \leq k e^{at} \| u_0 \|$

the solution grow at most exponentially.

We have already discussed that the system (*) is <u>non linear</u> if the coefficients of A are functions of the state vector $a_{jk} = a_{jk}(\underline{u})$ while the system is <u>linear</u> if the coefficients are constant.

There is no better way to appreciate the difference between linear and nonlinear hyperbolic equations than to consider some examples.

Let's start with a <u>linear</u> hyperbolic equation: advection equation in 1+1 spacetime

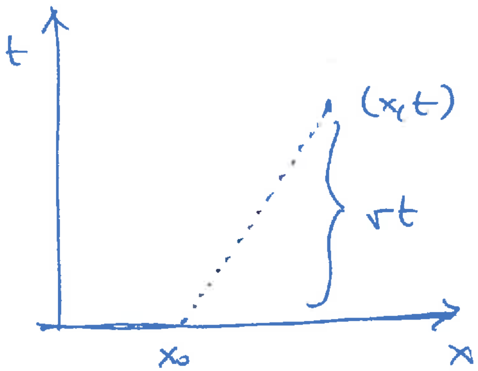
$$\boxed{\partial_t u + v \, \partial_x u = 0} \qquad v = \text{const.}$$

A has only one component $a_{11} = v = \text{const}$ (linear!)

This equation has the simple solution

$$u(x,t) = u(x_0, 0) = u_0(x_0) = u_0(x - vt)$$

$$x_0 = x - vt$$



$$v = \frac{dx}{dt}$$

In other words the solution at any new time and position $u(x,t)$ can be computed from the initial solution u_0 at the position x_0 suitably translated in space-time.

We can actually think that the initial solution is simply translated in spacetime along suitable directions.

What are these directions?

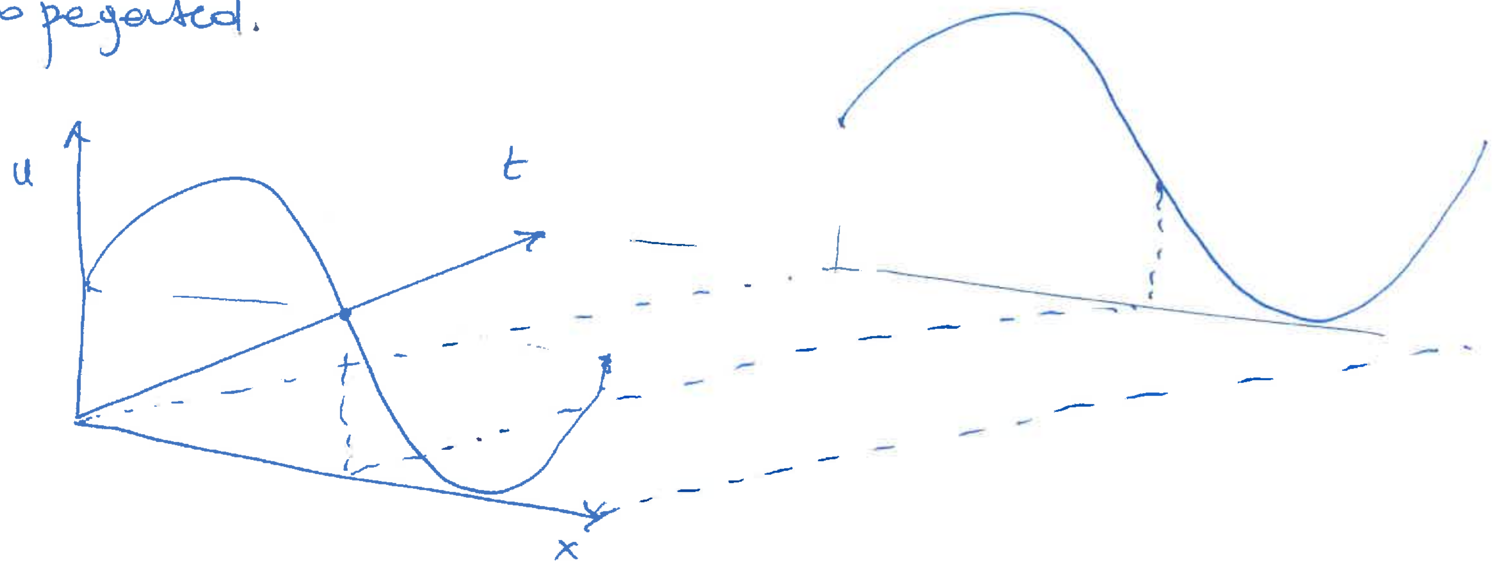
$$d_t u = \frac{du}{dt} = \partial_t u + \frac{dx}{dt}\,\partial_x u = 0 \quad \text{if} \quad \frac{dx}{dt} = \lambda = v = \text{const.}$$

the direction (straight line) $\frac{dx}{dt} = \lambda = v$ is called <u>characteristic</u> direction and it corresponds with the direction in spacetime along which the solution is propagated.

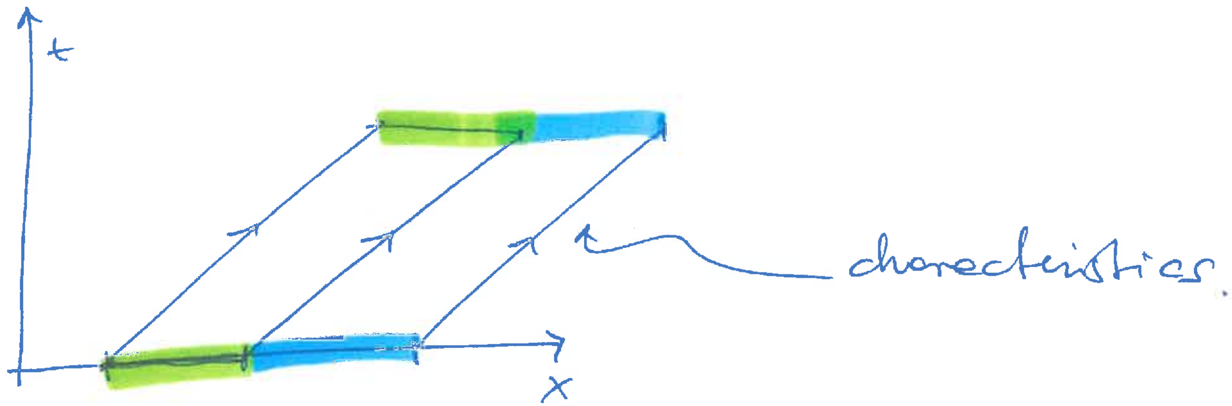
Eg

$u_0 = u_0(x) =$

$\sin(4\pi x)+1$



Because the solution is transported everywhere with the same velocity (ie along characteristic curves that are parallel) the solution is not distorted. This is all very clear and familiar.



We can generalize these results to a system of linear hyperbolic equations

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = 0$$

where A is a matrix of constant coefficients.
(ie the system is linear)

Under these conditions we can define the vector of characteristic variables

$$\underline{W} := R^{-1}\,\underline{U} \qquad : \text{characteristic vector}$$

where R is the matrix of right eigenvectors. Multiplying (*) by R^{-1} we obtain (assume $\underline{S}=0$)

$$R^{-1}\,\partial_t \underline{U} + R^{-1} A \cdot \nabla \underline{U} = 0$$

the right eigenvectors are constant

$$R^{-1}\,\partial_t \underline{U} = \partial_t \left(R^{-1}\underline{U}\right) = \partial_t \underline{W}$$

and $\quad R^{-1} A \nabla \underline{U} = R^{-1} A \cdot \nabla (R\underline{W}) = R^{-1} A R \cdot \nabla \underline{W} = \Lambda \cdot \nabla \underline{W}$

where $\quad \Lambda = R^{-1} A R \quad$: diagonal (1) matrix with constant coefficients.

(1) diagonal because hyperbolic system

Putting things together

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = 0 \iff \boxed{\partial_t \underline{W} + \Lambda \cdot \nabla \underline{W} = 0} \quad (**)$$

(**) are called <u>characteristic equations</u> and state that the characteristic vector is conserved along the directions given by the eigenvalues of A, ie

$$d\underline{W} = \partial_t \underline{W}\, dt + \partial_{\vec{x}} \underline{W}\, d\vec{x}$$

$$\frac{d}{dt}\underline{W} = \partial_t W + \Lambda \frac{\partial \underline{W}}{\partial \vec{x}} = 0$$

$$= \left(\partial_t W + \frac{d\vec{x}}{dt} \partial_{\vec{x}} \underline{W}\right) dt$$

along $\Lambda = \frac{\partial \vec{x}}{\partial t}$

Note: here $\vec{x}$ represents the spatial coordinates of an arbitrary coordinate system and is a matrix, like Λ is a matrix.

Since Λ is a diagonal matrix with coefficients $\lambda(i)$ the characteristic vector $\underline{W}$ is conserved along the directions

$$\lambda(i) = \frac{\partial \vec{x}(i)}{\partial t}$$: characteristic curves ("characteristics")

Note that (**) are N independent <u>advection</u> equations and hence the solution (or value of $\underline{W}$) at any given time can be computed from the initial solution, i.e. the solution at $t=0$.

$$\boxed{W^i(x^j,t) = W^i(x^j - \lambda_i t, 0)}$$

As a result, also the original state vector can be computed rather trivially as

(□) $$\boxed{\underline{U}(x^j,t) = \sum_{i=1}^{N} W^i (x^j,t)\,\underline{R}^{(i)} = \sum_{i=1}^{N} W^i (x^j - \lambda_i t, 0)\,\underline{R}^{(i)}}$$

In other words, once $\underline{W}(x^j,0)$ is known, $\underline{U}$ can be computed at any position in space and time.

This is a very powerful result, which is however restricted to linear problems as in this case the characteristics do not intersect and hence the expression (□) is not double valued.

What happens therefore in the case of nonlinear hyperbolic systems?

Once again it is simpler to understand this by starting from a simple example.

The simplest nonlinear hyperbolic equation is offered by the <u>inviscid</u>[⊙] <u>Burgers equation</u>

$$\partial_t u + u\, \partial_x u = 0$$

clearly, in this case the matrix A in (∗) has a non-constant coefficient $a_{11} = u(x,t)$: function of space and time!

As for the advection equation we can write

$$\frac{d}{dt}u = \partial_t u + \frac{dx}{dt}\partial_x u = 0$$

u is conserved along the direction $\lambda = \frac{dx}{dt} = u(x,t) = \text{const}$

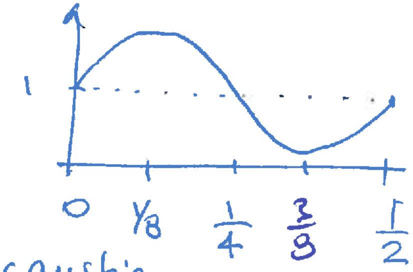
$$= u(x_0, 0) = u(x - \lambda t, 0)$$

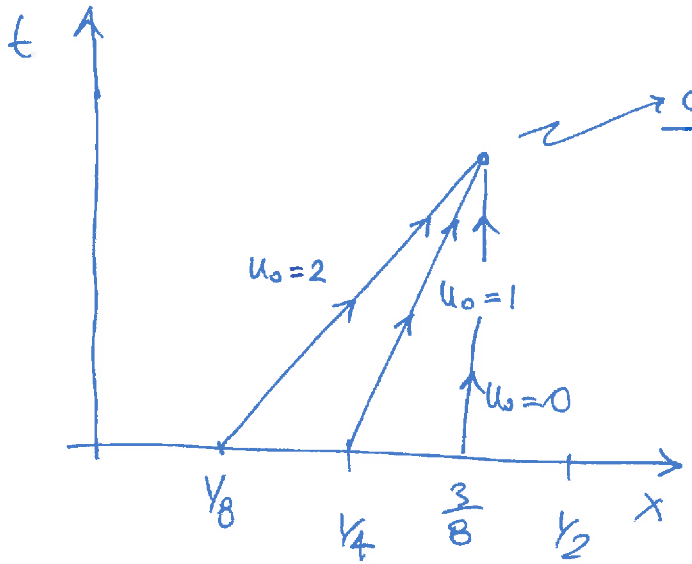
[⊙] The (viscous) Burgers equation is given by $\partial_t u + u\,\partial_x u = \nu\, \partial_x^2 u$

The characteristics are still straight lines, but they are no longer parallel!

Let's consider again the same initial data

$u_0 = \sin(4\pi x) + 1$





$$\frac{dx}{dt} = \lambda = u(x,t)$$

characteristics intersect!

When does this happen?

[one part of the solution moves faster than the other! Traffic.]

To calculate this we can write the implicit Burgers equation whose solution is

$$u(x,t) = u_0(x-\lambda t) = u_0(x - u(x,t)t)$$

Taking a time derivative yields ⊙

$$\partial_t u = [-u(x,t) - t\,\partial_t u]\,\partial_x u_0 \implies$$

$$\partial_t u(1 + t\,\partial_x u_0) = -u\,\partial_x u_0 \implies$$

$$\partial_t u = -\frac{u\,\partial_x u_0}{1 + t\,\partial_x u_0}$$; a caustic will be formed when the RHS will diverge,

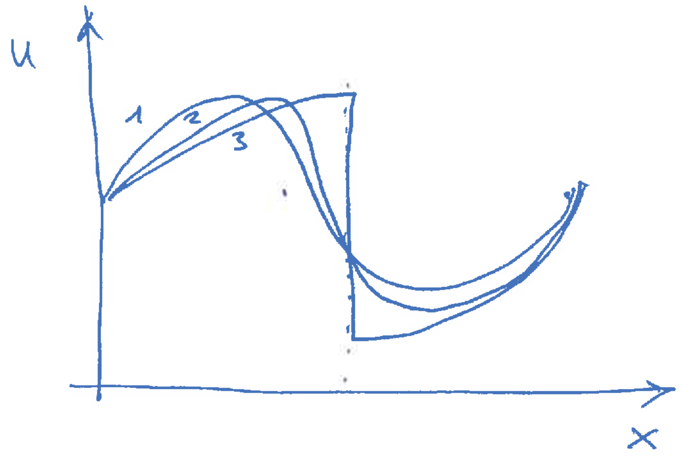
which will diverge for $t = -\dfrac{1}{\min(\partial_x u_0)} = \dfrac{1}{4\pi} \sim \dfrac{1}{0.08}$

$\min(4\pi\cos(4\pi x)) = -4\pi$

-1

$$\boxed{\begin{aligned} \odot\quad \partial_t u_0 &= \partial_x u_0\, \partial_t \dot{x} = \partial_x u_0 \left(\partial_t(-ut)\right) \\ &= \partial_x u_0 \left(-u - t\,\partial_t u\right) \end{aligned}}$$

How does the solution change?



what is this?

This is a discontinuity or shock! The solution is mathematically discontinuous (double valued.)

In other words, the development of a caustic is equivalent to the development of a discontinuity or shock!

This process is called "wave steepening" and is typical of nonlinear hyperbolic problems.

Note that the wave steepening is unavoidable and occurs also from data that is initially smooth.

Linear waves

Such a fluid is described by an energy-momentum tensor we have already encountered

$$T_{\mu\nu} = (e+p)\, u_\mu u_\nu + p\, g_{\mu\nu}$$

$$= he\, u_\mu u_\nu + p\, g_{\mu\nu}$$

Let's consider therefore the equations of conservation of energy and momentum

$$\nabla_\mu T^{\mu\nu} = 0$$

which are given by $(\mu = 0, 1)$

$$\begin{cases} \partial_t \left[(e + p v^2) W^2 \right] + \partial_x \left[(e+p) W^2 v \right] = 0 \\ \partial_t \left[(e+p) W^2 v \right] + \partial_x \left[(e v^2 + p) W^2 \right] = 0 \end{cases}$$

Exercise

where $u^\mu = W(1, v)$; $W = (1-v^2)^{-1/2}$

Let now e_0, p_0 and $v_0 = 0$ be the values of the energy density, pressure and velocity (fluid at rest) and introduce first-order perturbations of the type

$$e = e_0 + \delta e \,, \quad p = p_0 + \delta p \,, \quad v = v_0 + \delta v = \delta v \,.$$

The resulting set of perturbation equations will be

$$\begin{cases} \partial_t (\delta e) + (e_0 + p_0)\, \partial_x \delta v = 0 \\ \partial_t \delta v + \dfrac{1}{e_0 + p_0}\, \partial_x \delta p = 0 \end{cases}$$

where we have used

$$\left.\begin{array}{l} \partial_t e_0 = 0 = \partial_x e_0 \\ \partial_t p_0 = 0 = \partial_x p_0 \end{array}\right\} \text{stationary and uniform flow.}$$

Taking an additional time derivative and combining terms we obtain

$$(*) \qquad \partial_t^2 \delta e - \frac{\delta p}{\delta e} \partial_x^2 \delta e = 0 \iff \Box \delta e = 0$$

Similarly $\begin{cases} \Box \delta e = 0 \\ \Box \delta p = 0 \end{cases}$: wave equations with speed c_s.

$$c_s^2 = \delta p / \delta e$$

Eqs (*) show that perturbations propagate as waves with speed c_s. These are <u>acoustic waves</u> (or sound waves) and c_s is the <u>sound speed</u>. More generally

$$c_s^2 = (\partial p / \partial e)_s \qquad s: \text{specific entropy}$$

The linearization approach has clearly removed all the nonlinearities, but as mentioned before, the hydrodynamic equations are intrinsically nonlinear and generically lead to nonlinear waves. These can be distinguished as follows:

- simple waves

  are the nonlinear equivalent of sound waves but solutions of the full nonlinear eqs. They are always associated to a single eigenvalue, for which some quantities of the flow (called Riemann invariants) are conserved (see book for details)
  A theorem by Friedrichs states that: "any one-dimensional smooth solution neighbouring a constant state must be a simple wave". Examples of simple waves are rarefaction/compression waves

- <u>discontinuous waves</u>

are regions of the flow in which some of the fluid properties (e.g., velocity, rest-mass density, etc) are <u>taken</u> to be discontinuous. More on this later. They can be further distinguished into

  - <u>contact waves</u>: surfaces seperating two parts of the flow with different properties but without flow through the surface (contact discontinuities)
  - <u>shock waves</u>: same as above but with flow across the surface; fluid on either side have different properties but no chemical/physical change (density, energy, etc) takes place across surface.

- reaction fronts : same as shock waves but with chemical/physical changes taking place across surface (eg detonations, deflagrations). □

Having completed this classification we can now concentrate on the mathematical properties of simple waves and in particular of rarefaction waves.

They are simple waves that are characterized by the fact that the pressure and rest-mass density decrease in the region where the wave propagates. As simple waves they are adjacent to a constant state and are isentropic ($ds = 0$). The corresponding eigenvalues are the sonic ones

$$\lambda_\pm = \frac{v \pm c_s}{1 \pm v c_s} \simeq v \pm c_s$$ (cf velocity composition law in special relativity)

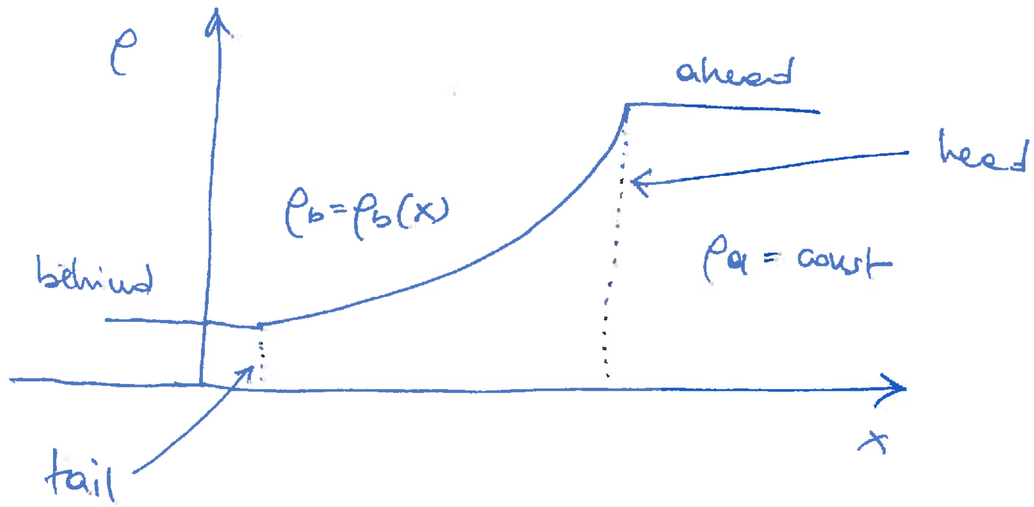
Newtonian limit

with Riemann invariants

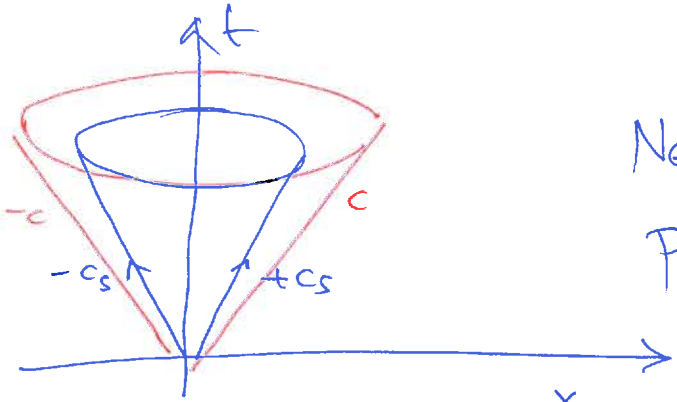
$$\mathcal{J}_\pm = \frac{1}{2} \ln \left(\frac{1+v}{1-v} \right) \pm \int \frac{c_s}{\rho}\, d\rho = \text{const.}$$

RWs have a head moving in the unperturbed medium and a tail, representing the portion of the flow moving with slowest velocity

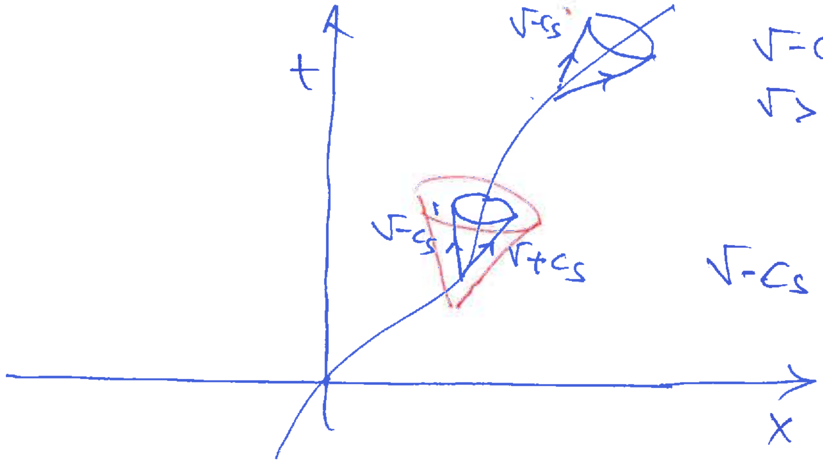
Exercise



Digression on sound cones and light cones



Newtonian physics



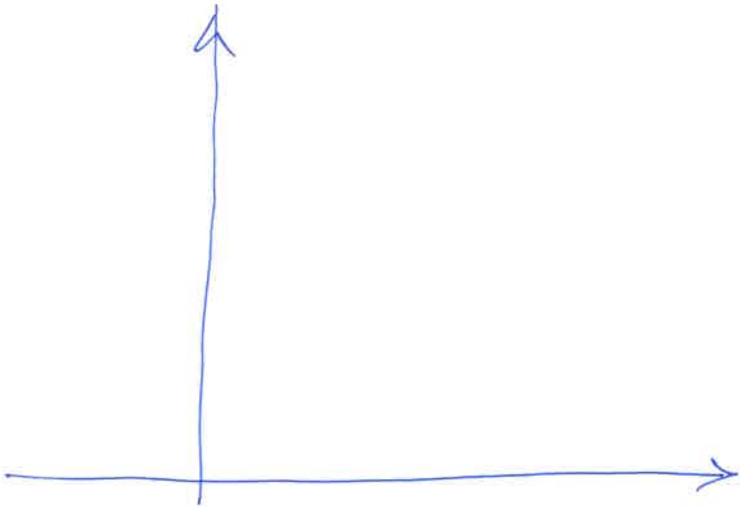
$v-c_s>0 \iff$
$v>c_s$: supersonic flow

$v-c_s<0$: subsonic flow

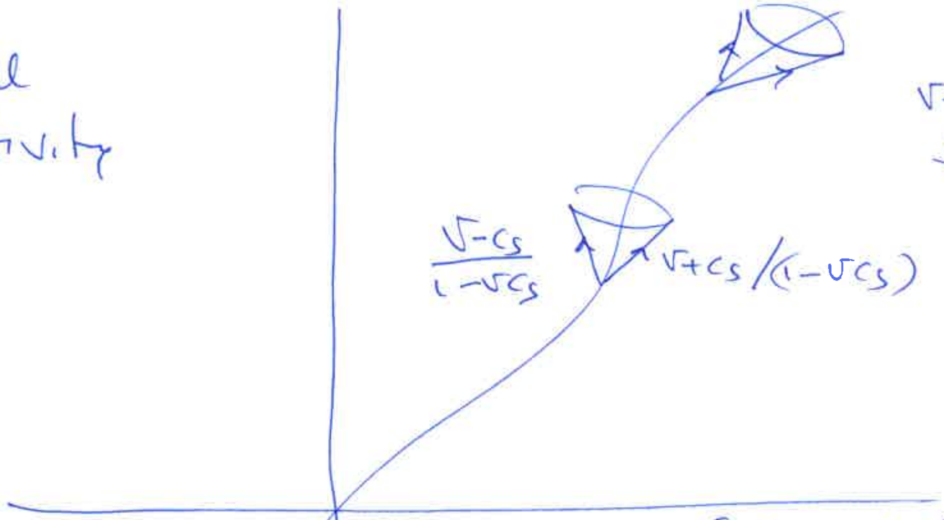
stationary flow:
sound cone symmetric

$\lambda = \pm c_s$

$\lambda = v \pm c_s$



special relativity



$v-c_s>0$
supersonic

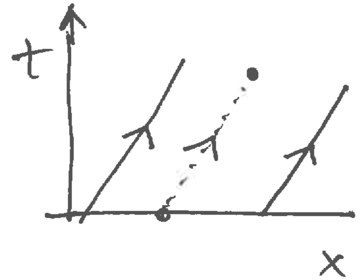
$\lambda = \pm c_s$

$\lambda = (v \pm c_s)/(1-vc_s)$

Recap

- Linear advection eq. : $\partial_t u + v\,\partial_x u = 0$ $\quad v = \text{const}$

$\iff \frac{d}{dt}u = 0$ along $\lambda : \frac{dx}{dt} = v$: straight lines all parallel __characteristics__



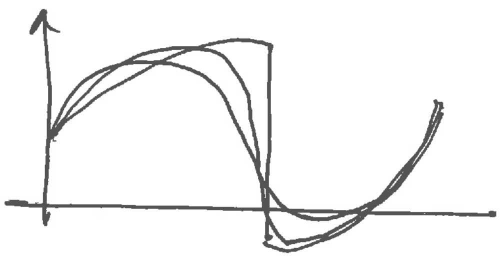
$u(x,t) = u(x - vt, 0)$

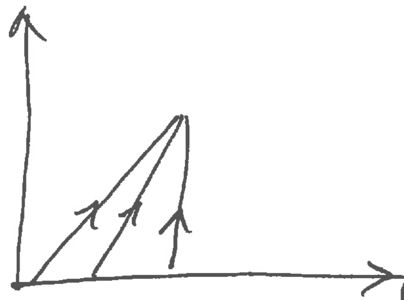
Solution is simply advected without distortions in spacetime

- Nonlinear equation: Burgers :: $\partial_t u + u\,\partial_x u = 0$

$\iff \frac{du}{dt} = 0$ along $\lambda : \frac{dx}{dt} = u(x,t)$ characteristics are still straight lines but they converge at one point: __caustic__

This leads to wave steepening and shock-formation even from smooth ID





• Sound waves : <u>linear</u> solutions of hydrodynamic equations

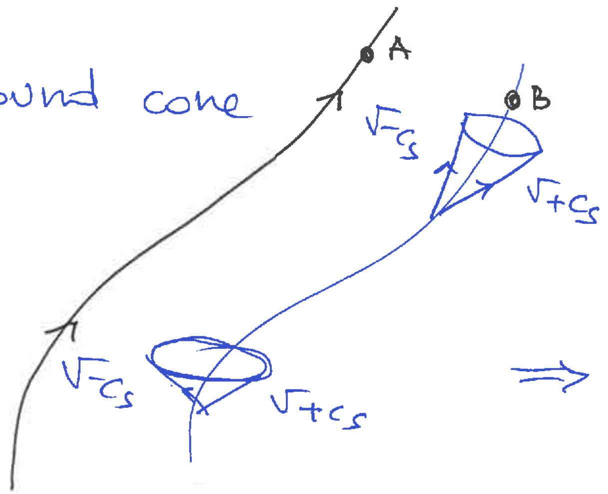
$$\Box\, \delta e = 0 = \left(\partial_t^2 - c_s^2\, \partial_x^2\right) \delta e .$$

perturbations propagate with sound speed $c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s$

$\lambda_\pm = \pm c_s$: characteristics for static fluid

$\lambda = \dfrac{v \pm c_s}{1 \pm v c_s}$: " " fluid moving at speed v

$\sim v \pm c_s$ sound cone



A, B causally disconnected

If $v - c_s < 0$: flow is subsonic

$v - c_s > 0$: " " supersonic

$\Rightarrow$ no signal can be sent upstream.

Nonlinear waves

- simple waves: continuous nonlinear equivalent of sound waves
  Eg compression or rarefaction waves

- discontinuous waves

  * contact waves (discontinuity): changes in rest-mass density but no mass flux

  * shock waves: mass flux but no chemical change

  * reaction waves: mass flux with chemical change
    - deflagration (subsonic front)
    - detonation (supersonic front)

- rarefaction waves : isentropic nonlinear waves preserving Riemann invariants

$$J^{\pm} = \frac{1}{2} \ln \left(\frac{1+v}{1-v} \right) \pm \int \frac{c_s}{\rho} d\rho = \text{const.}$$

An important property of simple waves and hence of RWs is that in the case of one-dimensional flows they can be written in a <u>self-similar form</u> ie in a form in which all quantities depend only on a <u>self similar variable</u>

$$\xi := \frac{x}{t}$$

ξ is dimensionally a velocity but can be seen as a position at any given time or as a time at any given position.

To derive the relevant expressions let's consider a one-dimensional flow in flat spacetime

$$u^\mu = W(1, v, 0, 0) \,, \qquad W = (1-v^2)^{-1/2}$$

Conservation of rest-mass (continuity eq.) and conservation of momentum

$$\begin{cases} \nabla_\mu J^\mu = \nabla_\mu (\rho u^\mu) = 0 \\ h^\nu{}_\lambda \nabla_\mu T^{\mu\lambda} = 0 \end{cases} \iff$$

Exercise

(RW1)
(RW2)
$$\begin{cases} \partial_t (\rho W) + \partial_x (\rho W v) = 0 \\ W \partial_t (W v) + W v\, \partial_x (W v) = -\dfrac{1}{\rho h} \left(\partial_x p + W^2 v \partial_t p + W^2 v^2 \partial_x p \right) \end{cases}$$

We now express the differential operators ∂_t and ∂_x as

$$\partial_t := -\frac{\xi}{t} \frac{d}{d\xi} \qquad \partial_x := \frac{1}{t} \frac{d}{d\xi}$$

so that the adiabaticity condition $u^\mu \nabla_\mu s = 0 = W \partial_t s + W v \partial_x s = 0$

can be written as

$$(v-\xi)\frac{ds}{d\xi} = 0 .$$

Exercise

Similarly, eqs (RW1) and (RW2) can be written as

Exercise

$$\begin{cases} (v-\xi)\dfrac{de}{d\xi} + e\left[W^2 v(v-\xi)+1\right]\dfrac{dv}{d\xi} = 0 \\ e h W^2 (v-\xi)\dfrac{dv}{d\xi} + (1-v\xi)\dfrac{dp}{d\xi} = 0 \end{cases}$$

these are two coupled ODEs providing the profiles of v and e (p) inside the RW.

The energy conservation eq. $u_\nu \nabla_\mu T^{\mu\nu} = 0$ is most easily deduced recalling the first law of thermodynamics for isentropic flows

$$de = h\,d\rho + \rho T\,ds = h\,d\rho$$ from which we obtain

(RW3) $$\frac{dp}{d\xi} = h c_s^2 \frac{d\rho}{d\xi}$$

A self-similar solution will exist if the determinant of the system (RW1) - (RW3) vanishes, ie if

$$c_s^2 = \left(\frac{v-\xi}{1-v\xi}\right)^2 \implies$$

$$c_s = \left|\frac{v-\xi}{1-v\xi}\right| = \frac{|v-\xi|}{1-v\xi} = \pm\frac{v-\xi}{1-v\xi}$$ where the $\pm$ signs have the following meaning

$+$: for $v > \xi$, ie $R_{\leftarrow}$ left-propagating RW wrt fluid

$-$: for $v < \xi$, ie $R_{\rightarrow}$ right-propagating RW wrt fluid

We can invert this relation to get

$$\xi = \frac{v \mp c_s}{1 \mp v c_s} \qquad \text{where now} \quad \begin{matrix} + : & R_{\rightarrow} \\ - & R_{\leftarrow} \end{matrix}$$

This expression allows us to compute the speeds of the head and tail of the RW.

Examples. Let $v_a = 0$ (fluid at rest)

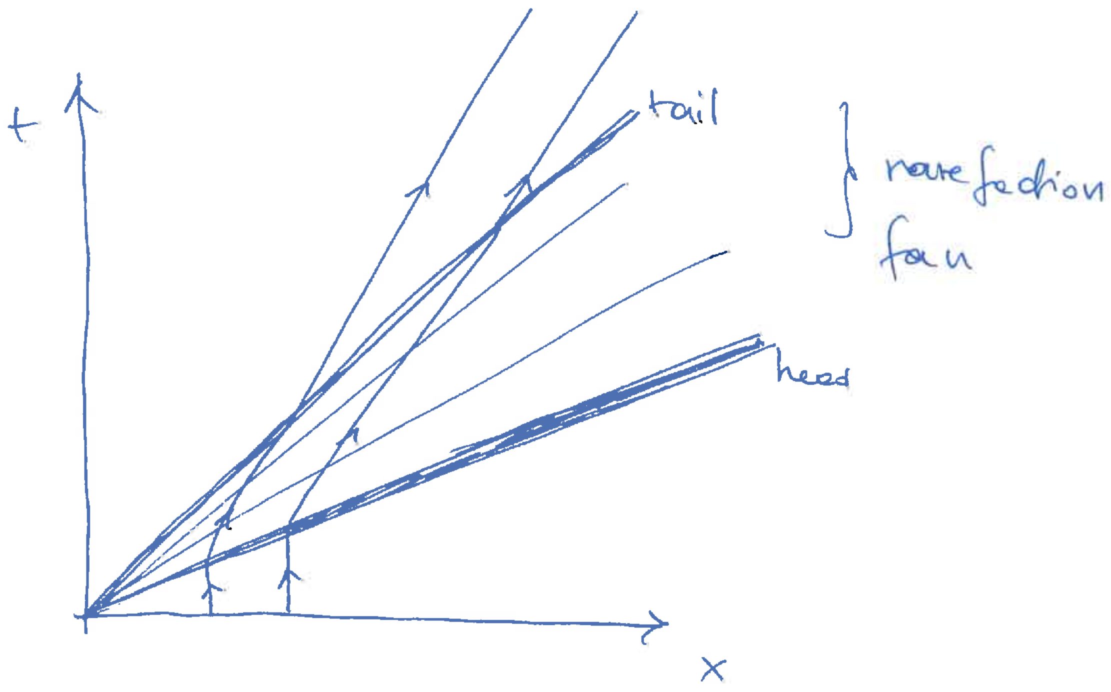
$$\xi_{head} = \mp c_s \qquad \begin{matrix} R_{\leftarrow} \\ R_{\rightarrow} \end{matrix}$$

in other words: the head of the RW moves at the local speed of sound

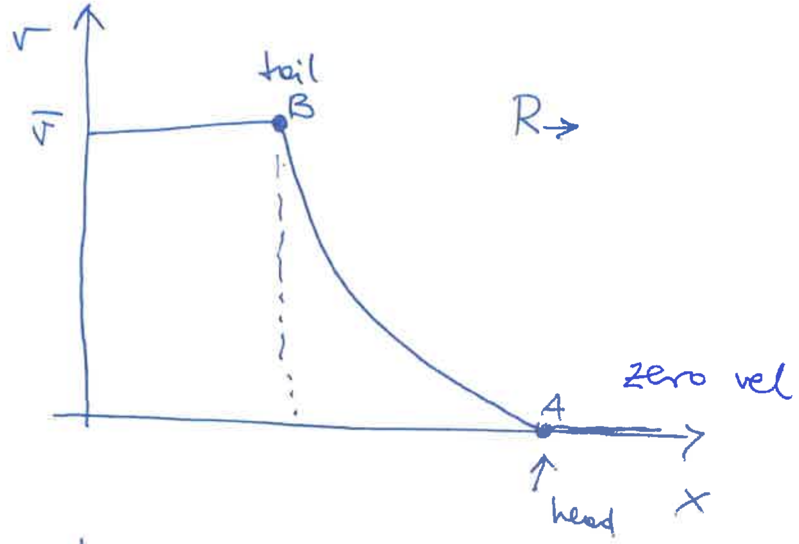
Similarly, let $v_b = \bar{v}$: const. velocity after the RW

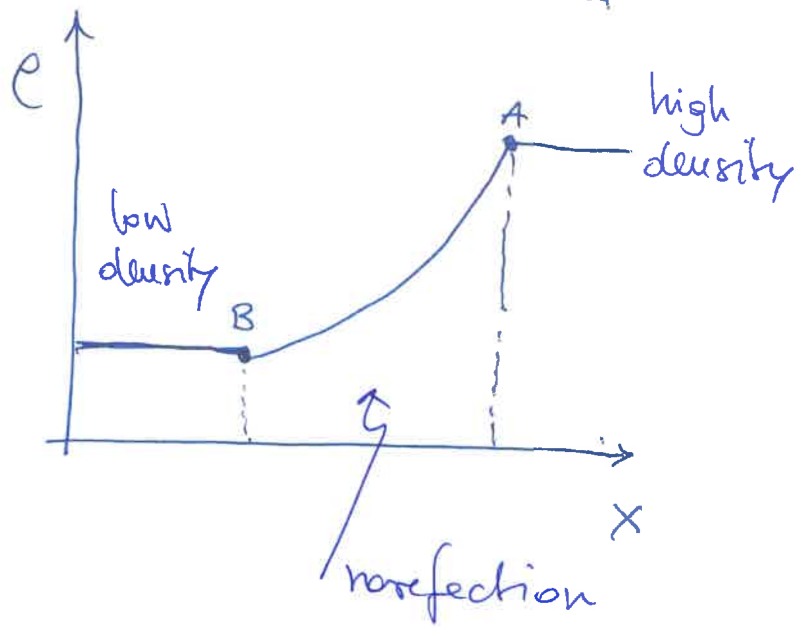
$$\xi_{tail} = (\bar{v} \mp c_s)/(1 \mp \bar{v} c_s)$$

Some diagrams will help fix ideas



nonzero const. vel.

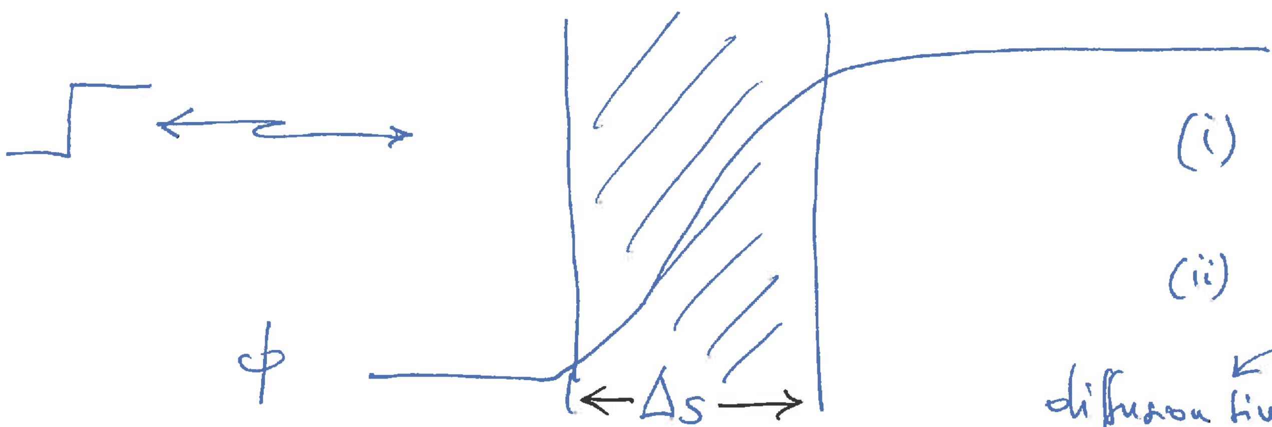




As discussed when considering the Burgers equation <u>discontinuous</u> waves can be produced from compressive motions having smooth initial data.

A discontinuous wave is a <u>mathematical artefact</u> to cope with flows in which the properties vary very rapidly on a very small lengthscale. No physics breaks down at a shock wave! Simply, complex and steep gradients are replaced by simple <u>junction conditions</u>.

The use of shock wave is reasonable when



(i) $\Delta s \approx l_{mfp} \ll \overbrace{\phi / \partial_x \phi}^{\text{lengthscale of variation}}$

(ii) $\underbrace{\tau}_{\text{diffusion timescale}} \ll \underbrace{\phi / \partial_x \phi}_{\text{timescale of front motion}}$

If these conditions are not met, then the discontinuous wave approximation is not valid and more sophisticated approaches are necessary, eg Boltzmann equation.

The junction conditions mentioned above are simple algebraic conditions that guarantee the conservation across the shock front of rest-mass, energy and momentum.

To derive these conditions in a covariant form we start from the conservation equations

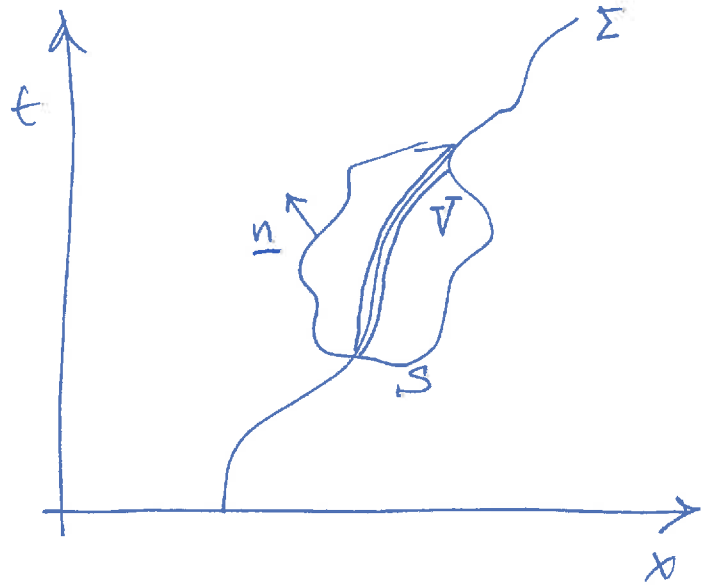
$$\begin{cases} \nabla_\mu (\rho u^\mu) = 0 \\ \nabla_\mu T^{\mu\nu} = 0 \end{cases}$$

which we rewrite as

$$\begin{cases} \nabla_\mu (\rho u^\mu f) = \rho u^\mu \nabla_\mu f \\ \nabla_\mu (T^{\mu\nu} \lambda_\nu) = T^{\mu\nu} \nabla_\mu \lambda_\nu \end{cases}$$

where f and λ_ν are an arbitrary scalar function and vector field, respectively

Let Σ be the history of a 2D spacelike surface representing the shock front. Let $\mathcal{V}$ be a 4D volume around Σ.



$$\int_{\mathcal{V}} \nabla_\mu (\rho u^\mu f)\, d^4x = \int_{\mathcal{V}} \rho u^\mu \nabla_\mu f \, d^4x$$

$$= \oint_S \rho u^\mu f n_\mu \, d^3x$$

Stoke's theorem

Similarly

$$\int_{\mathcal{V}} \nabla_\mu (T^{\mu\nu} \lambda_\nu)\, d^4x = \int_{\mathcal{V}} T^{\mu\nu} \nabla_\mu \lambda_\nu \, d^4x = \oint_S T^{\mu\nu} \lambda_\nu n_\mu \, d^3x$$

Consider now the limit in which $\mathcal{V} \to 0$. The first two integrals vanish, while the third ones reduce to the calculation of the integrand on both sides of Σ, ie

$$(*)\quad \begin{cases} \int_{\Sigma'} f\, [\![\rho u^\mu]\!]\, n_\mu\, d^3x = 0 \\ \int_{\Sigma'} \lambda_\mu\, [\![T^{\mu\nu}]\!]\, n_\nu\, d^3x = 0 \end{cases}$$

ahead ("1")

behind ("2")

where $[\![Q]\!] \equiv \underbrace{Q_a - Q_b}_{\text{jump of } Q \text{ across the shock front.}}$: "double bracket notation"

Given the arbitrariness in the choice of f and $\underline{\lambda}$, the conditions $(*)$ can be satisfied iff

$$(**)\quad \begin{cases} [\![\rho u^\mu]\!]\, n_\mu = 0 \\ [\![T^{\mu\nu}]\!]\, n_\nu = 0 \end{cases}$$

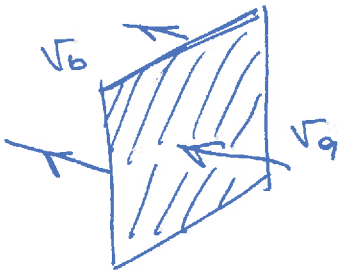
: (relativistic) Rankine Hugoniot conditions (junction / jump conditions)

Example: flat spacetime 1+1 flow, shock front moving in x direction. Hereafter use shock as rest frame to remove boost.

$u^\mu = W(1, v, 0, 0)$; $n_\mu = (0,1,0,0)$ and $(**) \iff$

$$\rho_a u_a^x = \rho_b u_b^x \; ; \; T_a^{xx} = T_b^{xx} \; ; \; T_a^{tx} = T_b^{tx}$$

$$\iff$$



$$J := \rho_a W_a v_a = \rho_b W_b v_b \qquad : \text{rest-mass} \quad (1)$$

$$\rho_a h_a W_a^2 v_a^2 + p_a = \rho_b h_b W_b^2 v_b^2 + p_b \qquad : \text{energy} \quad (2)$$

$$\rho_a h_a W_a^2 v_a = \rho_b h_b W_b^2 v_b \qquad : \text{momentum} \quad (3)$$

These equations can also be written as

$$[\![J]\!] = 0 \qquad \text{rest-mass jump}$$

$$J^2 = - \frac{[\![p]\!]}{[\![h/\rho]\!]} \qquad \text{energy jump}$$

$$T_{\alpha\beta} = \rho h u_\alpha u_\beta + p\eta_{\alpha\beta}$$

$$T^{\alpha\beta} = \rho h u^\alpha u^\beta + p\gamma^{\alpha\beta}$$

$$u^\alpha = W(1, v^i)$$

$$u_\alpha = W(-1, v^i) \qquad u_\alpha u^\alpha = -1$$

$$n_\alpha = (0,1,0,0)\,; \qquad n_\alpha n_\alpha = 1$$

$$[\![\rho u^\alpha]\!] n_\alpha = 0 \iff$$

$$\rho_a u_a^1 = \rho_b u_b^1 \iff \rho_a W_a v_a = \rho_b W_b v_b$$

$$[\![T^{\mu\nu}]\!] n_\mu = 0 \iff$$

$$[\![T^{\mu\nu}]\!] n_\mu = 0 \iff [\![T^{x0}]\!] = 0\,; \quad [\![T^{xx}]\!] = 0$$

$$T^{00} = \rho h (u^0)^2 + p\gamma^{00} = \rho h W^2 - p$$

$$T^{0x} = \rho h u^0 u^x + p\gamma^{0x} = \rho h W^2 v \implies [\![T^{0x}]\!] = 0 \iff \rho_a h_a W_a^2 v_a = \rho_b h_b W_b^2 v_b$$

$$T^{xx} = \rho h (u^x)^2 + p = \rho h W^2 v^2 \implies [\![T^{xx}]\!] = 0 \iff$$

$$\rho_a h_a W_a^2 v_a^2 + p_a = \rho_b h_b W_b^2 v_b^2 + p_b$$

$$[\![hW]\!] = 0 \quad \text{momentum jump}$$

Their combination can be used to derive a very important equation :

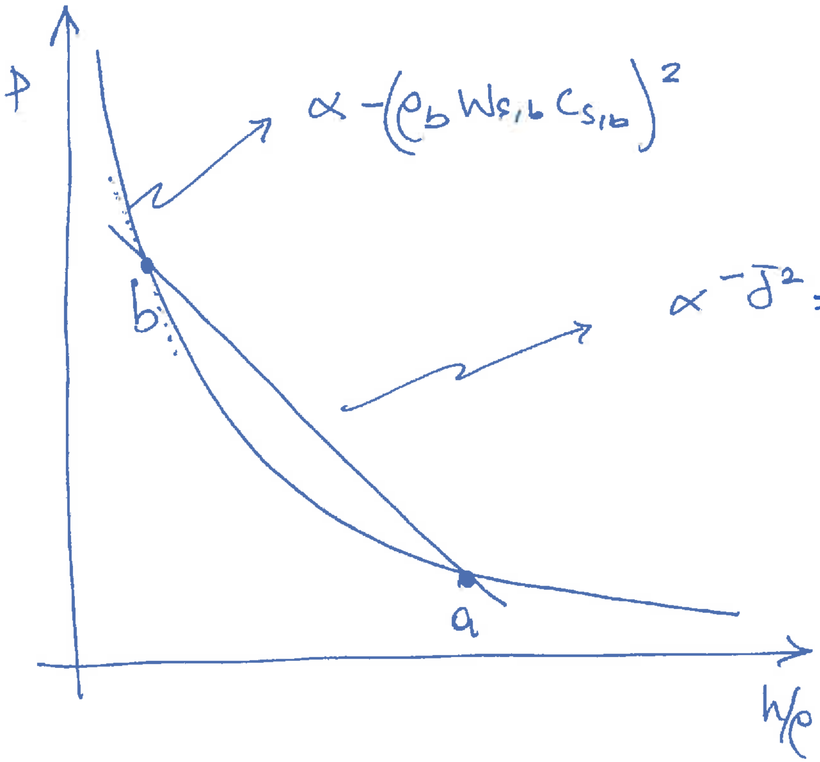
hyperbola in $(p, h/\rho)$ space

$$\boxed{[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b} \right) [\![p]\!]}$$

: Taub adiabat (relativistic equivalent of the classical Hugoniot adiabat)

$$[\![\epsilon + \frac{p}{\rho}]\!] = \frac{1}{2} \left(\frac{1}{\rho_a} + \frac{1}{\rho_b} \right) [\![p]\!]$$

The importance of the Taub adiabat is that provides a simple and graphical description of the fluid changes across a shock wave.



$\propto -(\rho_b W_{s,b}\, c_{s,b})^2$

$\propto -j^2 = -(\rho W v)^2_{a,b}$: mass flux

The slope of the chord joining two states is proportional to the mass flux. If state behind shock is at higher pressure, the mass flux will be larger

Slope of Taub adiabat at any point is negative and proportional to the local sound speed

$$\frac{dp}{d(h/\rho)} = -\rho^2 W_s^2 c_s^2 \qquad W_s = (1-c_s^2)^{-1/2} \qquad c_s^2 = \left(\frac{\partial p}{\partial e}\right)_s$$

$$= -\frac{\rho^2 c_s^2}{1-c_s^2} \propto -c_s^2$$

Exercise

$\boxed{\text{① Compute Taub adiabat for an ideal-fluid EOS}}$

How do we know if a discontinuity front is physically realistic? The answer comes from considering weak shocks, ie discontinuities in which the states ahead and behind are not very different (derivatives are discontinuous but flow is continuous). In this case, the jumps in the specific entropy are given by (for our ideal-fluid EOS)

$$[\![s]\!] = \left[\frac{1}{12\,hT}\left(\frac{\partial^2 (h/\rho)}{\partial p^2}\right)\right]_a [\![p]\!]^3 + O\left([\![p]\!]^4\right)$$

It is possible to show, using the Poisson adiabat, ie the equivalent of the Taub adiabat for states having the same entropy, that $\boxed{\partial^2 p / \partial (h/\rho)^2 > 0}$, ie, the second derivative of the Taub adiabat is always positive

The Second law of thermodynamics imposes that entropy must increase, ie

$$[\![s]\!] < 0 \quad \Rightarrow \quad \xrightarrow{\text{recall}} \quad s_a - s_b < 0$$

from which we deduce $[\![p]\!] < 0$

in other words, across a physically realistic shock

$$\boxed{s_b > s_a} \; ; \qquad \boxed{p_b > p_a}$$

so that the state behind the shock necessarily has higher pressure and entropy. A shock such that $[\![s]\!] > 0$ is unphysical. Since the new state must lie on the Taub adiabat

$$p_b > p_a \quad \Rightarrow \quad \frac{h_b}{\rho_b} < \frac{h_a}{\rho_a}$$

but $\quad [\![h^2]\!] \propto [\![p]\!] \Rightarrow$

$$h_b > h_a \quad \Longrightarrow$$

$\boxed{\rho_b > \rho_a}$: ie the fluid is <u>compressed</u> by the shock.

What about the <u>velocity</u>?

: momentum conservation implies

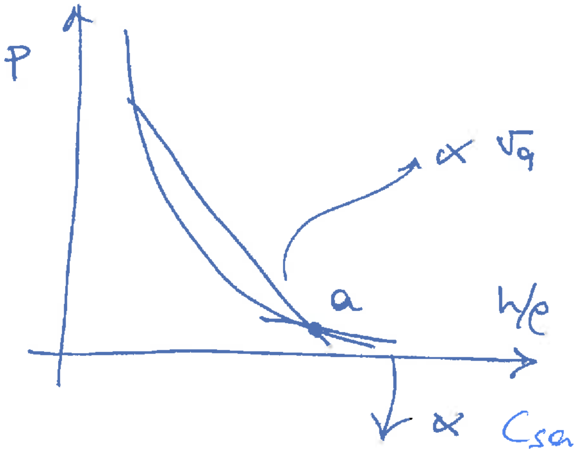
$$\rho_a h_a W_a^2 v_a^2 + p_a = \rho_b h_b W_b^2 v_b^2 + p_b$$

and since ρ, p, h increase across the shock, it must be that

$$W_a^2 v_a^2 < W_b^2 v_b^2 \quad \Longrightarrow$$

$\boxed{|v_a| > |v_b|}$ ie the fluid is <u>decelerated</u> across the shock

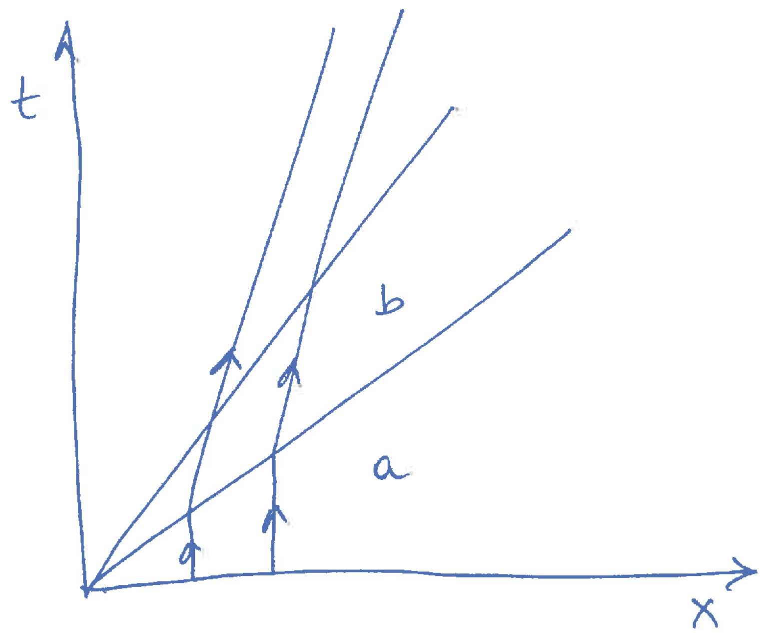
Furthermore it's easy to recognize that



$v_a \geq c_{s_a}$ $\quad$ ($\propto$ tg at a)

$v_b \leq c_{s_b}$ $\quad$ ($\propto$ tg at b)

in other words: flow is supersonic when entering the shock and subsonic when leaving the shock

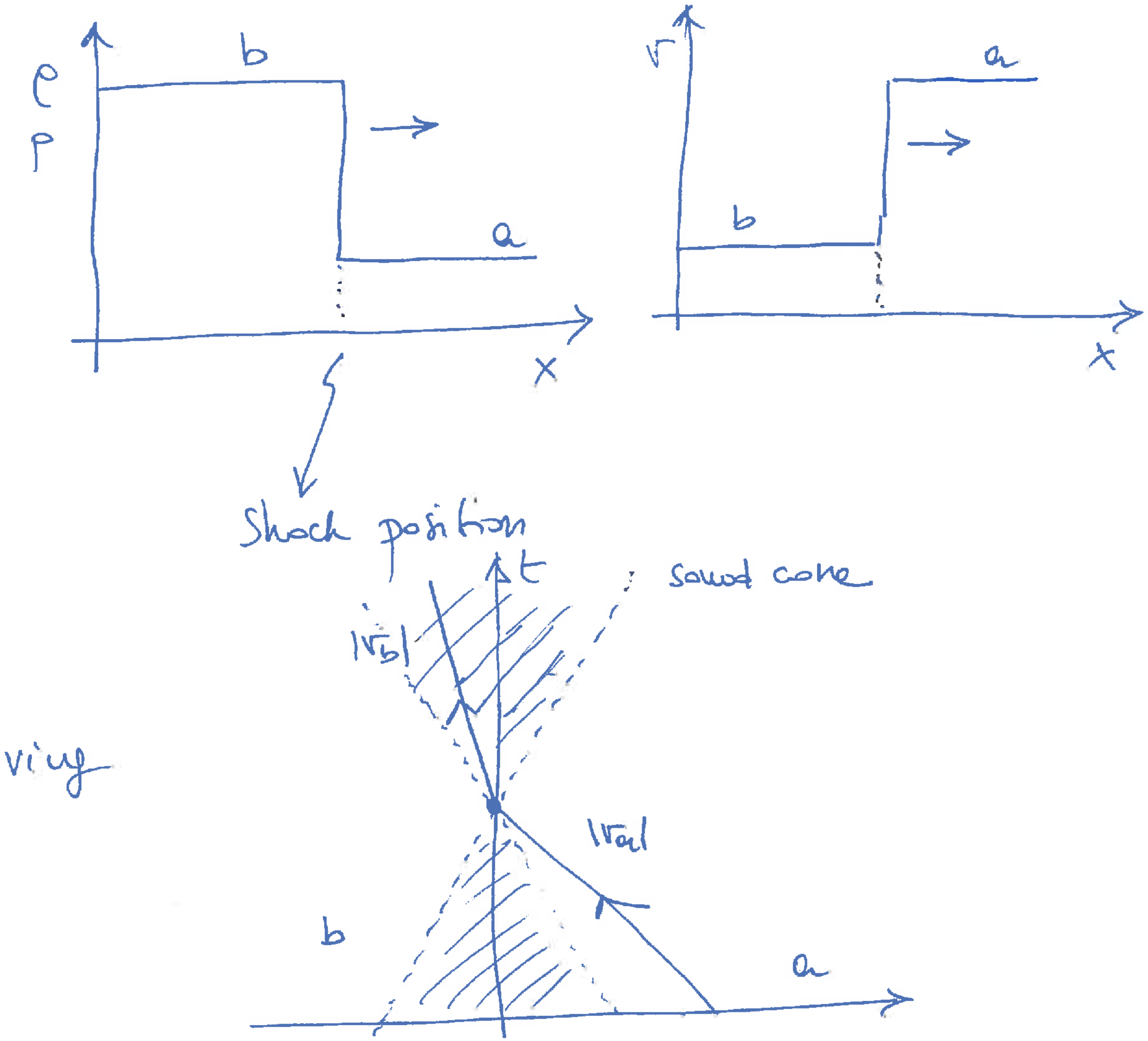


: spacetime diagram of shock moving to the positive x-direction

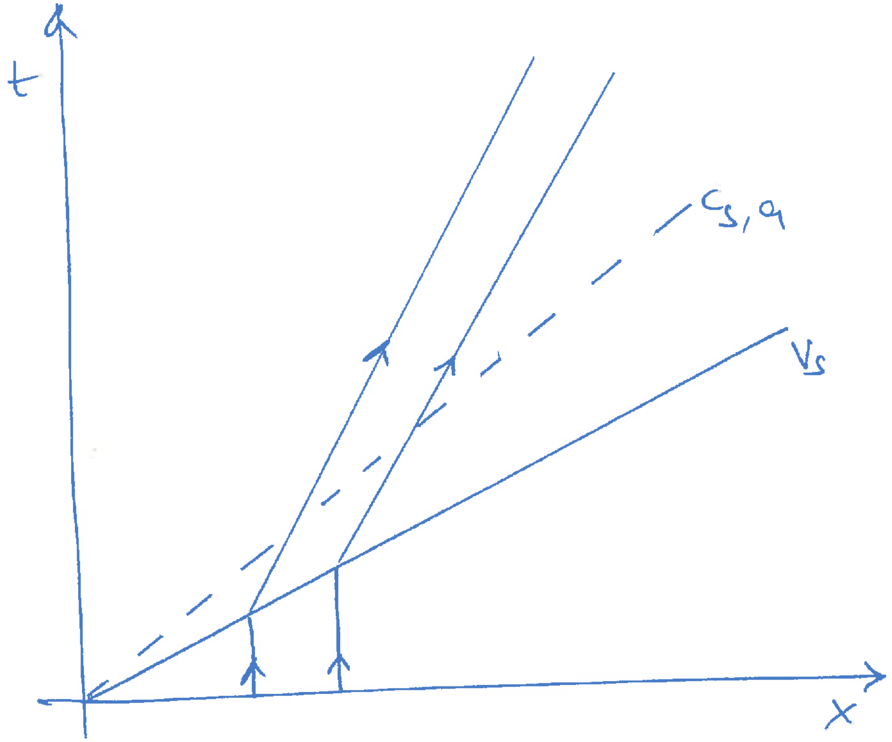
Recap: across a shock

$$\left\{ \begin{array}{l} s_b > s_a \\ p_b > p_a \\ h_b > h_a \\ \rho_b > \rho_a \\ \underbrace{|v_b| < |v_a|} \end{array} \right.$$

note that this is the velocity in a frame comoving with the shock!



The spacetime diagram showing representative fluidlines (ie worldlines of representative fluid elements) is shown below:



Note that in the Eulerian frame the shock is supersonic, ie moves with $V_s > c_s$ (cf the slope of the sound speed; this is the sound speed ahead of the shock even though it is marked in the shocked fluid.

Note also that the fluid is compressed, $\rho_b > \rho_a$ as deduced from the separation between fluidlines.

Recap

- covariant formulation of jumps across discontinuity

$$[\![\rho u^\mu]\!] n_\mu = 0$$

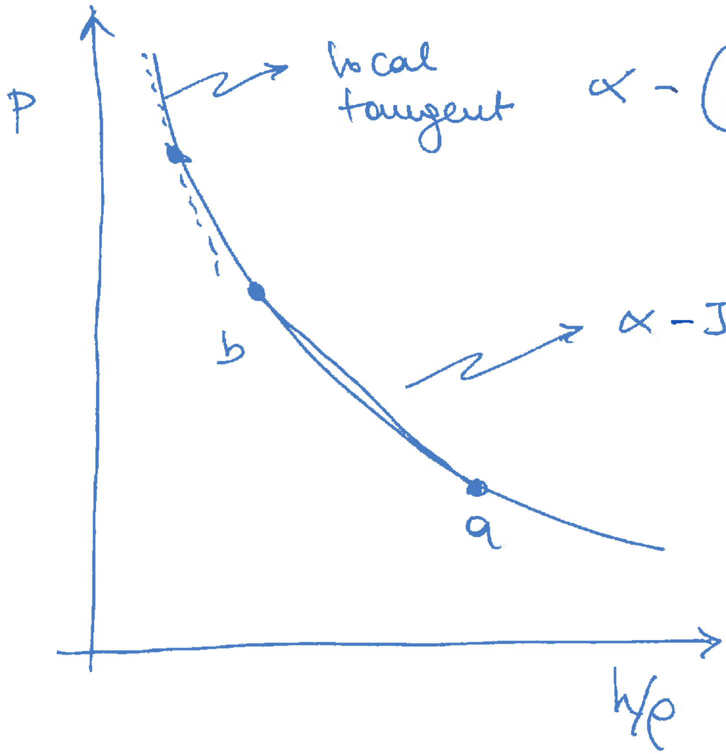
$$[\![T^{\mu\nu}]\!] n_\nu = 0$$

for 1-D flow in flat spacetime

$$\rho_a u_a^x = \rho_b u_b^x \; ; \quad \underbrace{T_a^{xx} = T_b^{xx}}_{\text{cons. of momentum (flux).}} \; ; \quad \underbrace{T_a^{0x} = T_b^{0x}}_{\text{cons. of energy (flux).}}$$

- using these equations one obtains Taub adiabat

$$[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right)[\![p]\!]$$



$\propto -(\rho W_s c_s)^2$ local tangent proportional to local sound speed.

$\propto -J^2 = (\rho W v)^2_{a,b}$: slope of chord is proportional to mass flux

Across a physical shock entropy must increase

$S_b > S_a \implies$

$p_b > p_a$

$\Rightarrow \rho_b > \rho_a$

$\Rightarrow |v_b| < |v_a|$: relative to shock front

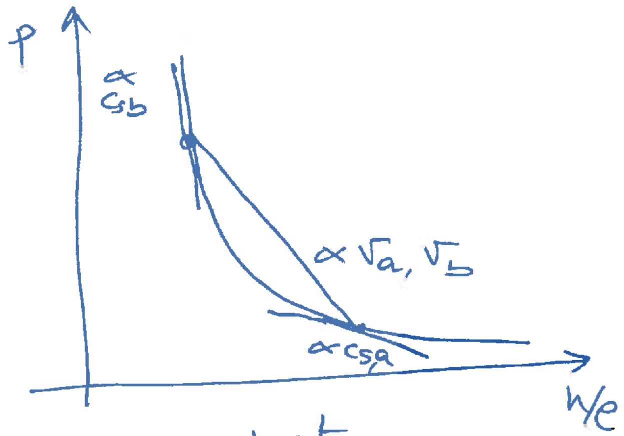
In other words, across a shock the fluid is compressed, heated up and decelerated.

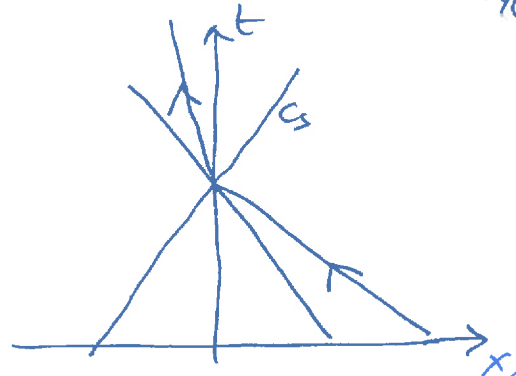
The Taub adiabat also reveals that

$|v_a| > c_{sa}$

$|v_b| < c_{sb}$

The flow entering the shock is supersonic and is subsonic when leaving the shock (relative to the shock)





The junction conditions can be manipulated to obtain expressions for the velocities on either side of the shock in terms of the quantities there, eg

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)}$$

$$v_b^2 = \frac{(p_a - p_b)(e_a + p_b)}{(e_a - e_b)(e_b + p_a)}$$

or the relative velocity

$$v_{ab} = \frac{v_a - v_b}{1 - v_a v_b} = \left(\frac{(p_a - p_b)(e_a - e_b)}{(e_a + p_b)(e_b + p_a)} \right)^{1/2}$$

contact discontinuity

Note that $v_a \to 0$ for $p_a \to p_b$

On the other hand $e_b > e_a$ for a physical shock or the ratio / product of velocities

$$\frac{v_a}{v_b} = \frac{e_b + p_a}{e_a + p_b} \quad ; \quad v_a v_b = \frac{p_a - p_b}{e_a - e_b}$$

These equations are particularly telling in the case of an ultrarelativistic fluid, ie $p = \frac{1}{3} e$ and $c_s = \frac{1}{\sqrt{3}}$, in which case

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b}\right) v_b \quad ; \quad v_a = \frac{1}{3 v_b}$$

while the corresponding Lorentz factors are

$$W^2_{a(b)} = \frac{3}{8}\left(\frac{3e_{a(b)} + e_{b(a)}}{e_{a(b)}}\right)$$

$$W^2_{ab} = \frac{(3e_a + e_b)(3e_b + e_a)}{16 e_a e_b} = \frac{4}{9} W_a^2 W_b^2$$

with the additional property that

$$W_a^2 - 2W_a{}^2{}_b W + W_b^2 = 1.$$

All of the relations described so far are measured in the shock rest frame, but in such a frame one cannot measure the shock velocity. To derive it, it is sufficient to bear in mind that the mass flux is invariant under Lorentz boosts in the x-direction, ie

$$J = \rho_a W_a v_a = \rho_a \tilde{W}_a \tilde{W}_s (V_s - \tilde{v}_a) = \rho_b \tilde{W}_b \tilde{W}_s (V_s - \tilde{v}_b)$$

when the tilde is meant to refer to velocities in the Eulerian frame.

This expression can be inverted to obtain:

$$V_s^{\pm} = \frac{\rho_a^2 \tilde{W}_a^2 \tilde{v}_a \pm |J| \sqrt{J^2 + \rho_a^2}}{\rho_a^2 \tilde{W}_a^2 + J^2}$$

so that the shock velocity is fully determined by the properties of the flow ahead.

$$J^2 = -\frac{[\![p]\!]}{[\![h/\rho]\!]}$$

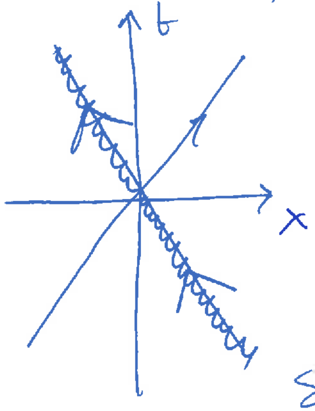
Eg.

$\tilde{v}_a = 0$; $\tilde{W}_a = 1$

$$V_s^{\pm} = \frac{\rho_b \tilde{W}_b \tilde{v}_b}{\rho_b \tilde{W}_b \mp \rho_a}$$

Let's consider some interesting limits (still for our ultrarelativistic fluid with $p = e/3$)

$\boxed{\text{weak shocks}}$: $e_b \sim e_a$

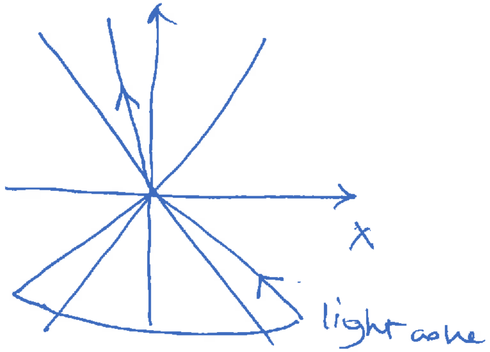


then

$$v_a = \left(\frac{3e_b + e_a}{3e_a + e_b}\right) v_b \sim v_b \quad ; \quad v_a \simeq \frac{1}{3 v_b} \Rightarrow$$

$$v_a \sim \frac{1}{\sqrt{3}} = c_{sa}$$

similarly $v_b \to c_{sb}$: a weak shock in ultrarelativistic fluid tends to a sound wave.



$\boxed{\text{strong shocks}}$: $e_b \to \infty$

$v_a \to 1$; $v_b \to \frac{1}{3}$; note that $v_b / c_{sb} = 1/\sqrt{3} \leq 1$

ie the shocked fluid is still subsonic despite $v_s \to 1$!

Contact discontinuities :

Discontinuous nonlinear waves with $[\![J]\!]=0 \Rightarrow$ $J^2 = -\dfrac{[\![p]\!]}{[\![h/\rho]\!]}$

$[\![p]\!]=0 \xRightarrow{\text{Taub adiabat}} [\![h]\!]=0$ but $[\![\rho]\!]\neq 0$

ie $\qquad [\![h^2]\!] = \left(\dfrac{h_a}{\rho_a}+\dfrac{h_b}{\rho_b}\right)[\![p]\!]$

$$[\![p]\!]=0=[\![h]\!]=[\![v]\!] \; ; \quad [\![\rho]\!]\neq 0$$

‖ pressure, specific enthalpy, (normal)⊙ velocities are continuous across CPs ‖ ; density is discontinuous across CDs

⊙ Tangent velocities can indeed be discontinuous across a CD. This is relevant for multi-dimensional Riemann problems.

Riemann problem

Determine the flow pattern (ie the number and type) of nonlinear waves that develops from constant and discontinuous initial data.

Riemann worked on this more than 150 years ago and the solution of this problem is the basis for many advanced numerical methods in (relativistic) hydrodynamics

Mathematically, this is defined as

$$\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x<0 \\ \underline{U}_R & \text{if } x>0 \end{cases}$$

$\underline{U}_L \neq \underline{U}_R$

$\underline{U}_L = \text{const};$

$\underline{U}_R = \text{const};$

Physically you can think of a tube containing a membrane and in which you can specify the properties of the fluid on either side of the membrane

The problem consists then in determining the evolution of the system if the membrane is removed istantaneously.

This solution is schematically represented as

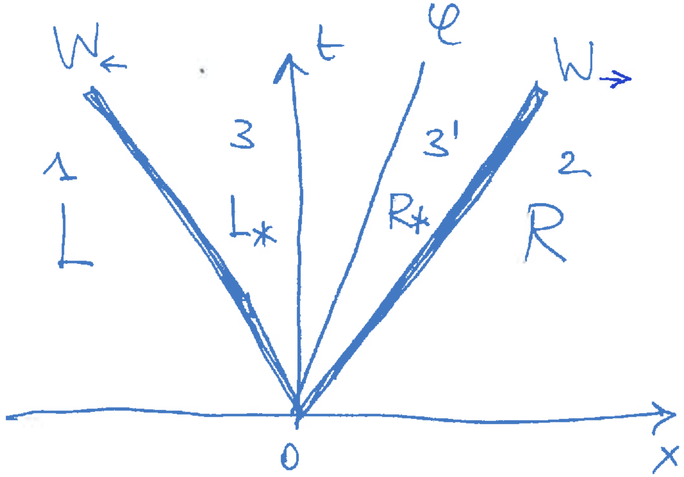
$L \; R$ at $t=0$

$L \; W_{\leftarrow} \; L_* \; C \; R_* \; W_{\rightarrow} \; R$ at $t>0$

left propagating nonlinear wave

contact discontinuity

right-propagating nonlin. wave



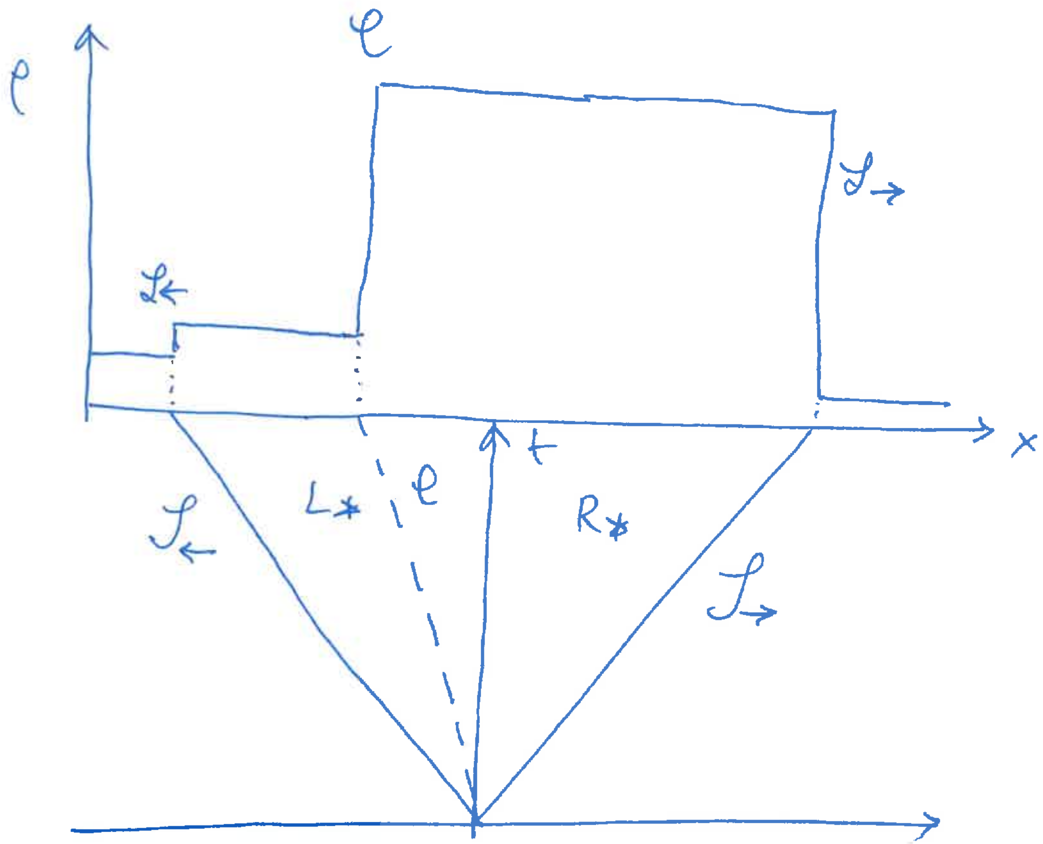
Note :

- the states L and R are the original ones as the waves have not yet reached them
- the regions L_* and R_* are separated by a contact discontinuity and hence $p_{L*} = p_{R*} = p_*$
- no assumption is made on the waves $W_\rightarrow$ and $W_\leftarrow$: these can be shocks or rarefaction waves.

In Newtonian hydrodynamics, Riemann concluded that the one-dimensional flow resulting from the initial data ($\triangledown$) will lead to <u>four different solutions</u>, or equivalently, <u>three wave patterns</u>.

(i) <u>two shock waves</u> moving to the right and to the left

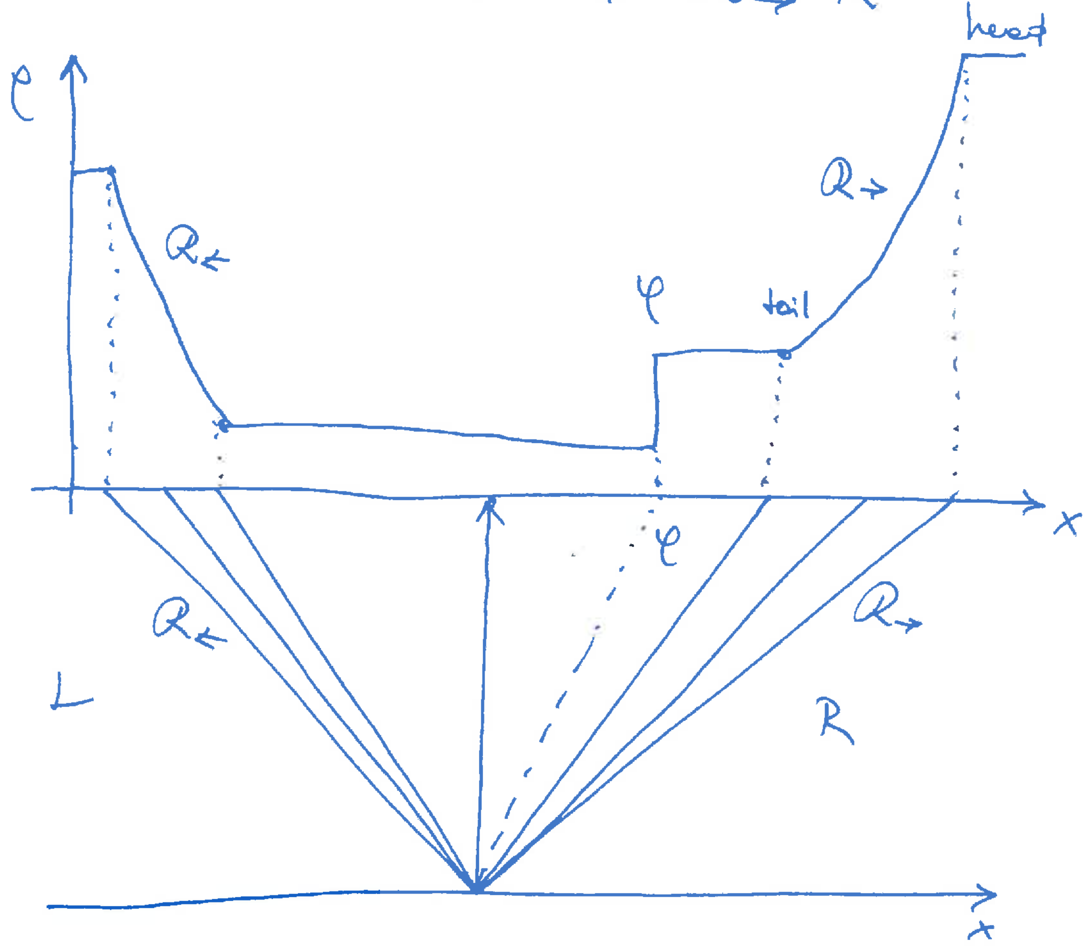
$$L \; S_{\leftarrow} \; L_* \; C \; R_* \; S_{\rightarrow} \; R$$



Note that the shocked fluid has larger density.

(ii) two rarefaction waves: moving to the right and to the left

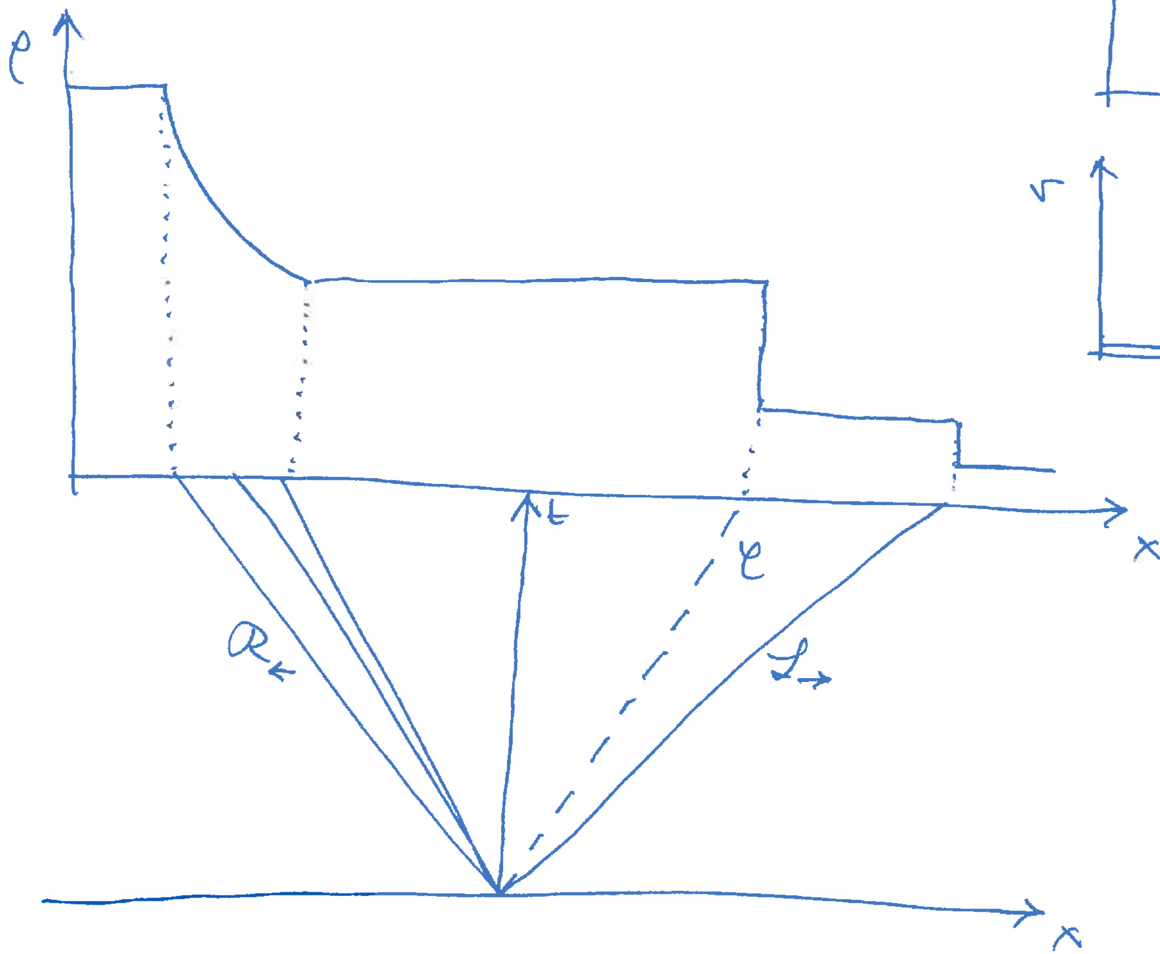
$$L \; \mathcal{R}_{\leftarrow} \; L_* \; \mathcal{C} \; R_* \; \mathcal{R}_{\rightarrow} \; R$$

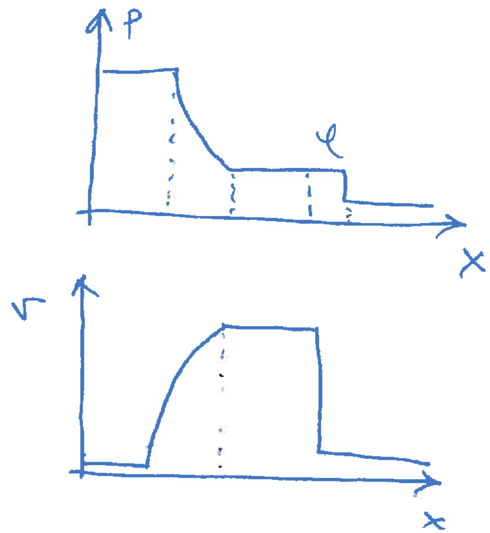


(iii) one shock and one rarefaction wave : moving to the right/left

$L \quad \mathcal{R}_{\leftarrow} \quad L_* \quad \mathcal{C} \quad R_* \quad \mathcal{S}_{\rightarrow} \quad R$

$L \quad \mathcal{S}_{\leftarrow} \quad L_* \quad \mathcal{C} \quad R_* \quad \mathcal{R}_{\rightarrow} \quad R$





Solution of the Newtonian Rieman problem (RP)

- no general solution of the RP is known in closed analytic form, not even in Newtonian hydrodynamics. Only exception is for an ultra relativistic fluid with EOS $p = w e$ $\quad 0 \leq w \leq 1$

- in practice numerical solution can be obtained to arbitrary accuracy and for this reason one talks of "exact" solution of the RP even if the solution is numerical (ie with a given truncation error).

- there are at least two different approaches to the solution of the RP and I will discuss the first proposed by Rezzolla and Zanotti (2001)

- importance of Riemann problem stems from the fact that it is the basis of many numerical algorithms

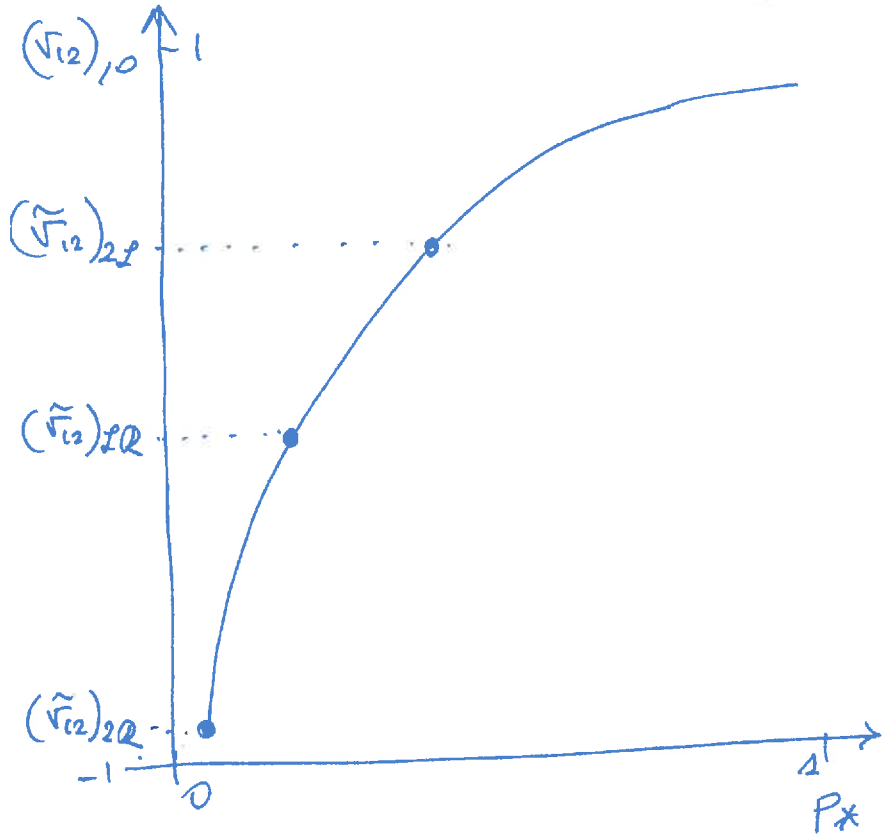
1) Using the initial states, ie $(v, p, \rho)_{1,2}$ ⊙ determine the wave pattern that will develop. This can be done after comparing the relative velocity of the initial states with the limiting values for the three different wave patterns.

$$(v_{12})_0 \equiv \frac{v_1 - v_2}{1 - v_1 v_2} \quad : \text{relativistic invariant}$$

tilde because these are reference values

Compare $(v_{12})_0$ with $(\tilde{v}_{12})_{2S}$, $(\tilde{v}_{12})_{SR}$, $(\tilde{v}_{12})_{2R}$, which is possible because $(v_{12})_0$ is a monotone function of p_* and the three branches smoothly join at the limiting values.

⊙ Note that I'm using the indices 1 and 2 to refer to the states on the L and R.



In other words: if

- $(V_{12})_0 > (\tilde{V}_{12})_{2L}$: $2L$ wave pattern
- $(\tilde{V}_{12})_{LR} < (V_{12})_0 \leq (\tilde{V}_{12})_{2L}$: LR wave pattern
- $(\tilde{V}_{12})_{2R} < (V_{12})_0 \leq (\tilde{V}_{12})_{LR}$: $2R$ wave pattern

The limiting values $(\tilde{V}_{12})_{2L}$, $(\tilde{V}_{12})_{LR}$ and $(\tilde{V}_{12})_{2R}$ are complex functions of the states in the RP, but they are analytic! For explicit expression see book.

$\square$

2) Compute the pressure in the * (3-3') region as the root of the nonlinear equation

$$\underbrace{v_{12}(p*)} - (v_{12})_0 = 0$$

different functional form for the three different wave patterns.

$v_{12}(p*)$ is an invariant and can be calculated in any frame, but there is a frame, the one comoving with the contact discontinuity, which is particularly useful, because in this frame there is no jump in the velocities! $v_{3,e} = 0 = v_{3',e}$

As a result, $(v_{13})_{,e} = \dfrac{v_{1,e} - v_{3,e}}{1 - v_{1,e} v_{3,e}} = v_{1,e} = v_{13} = \dfrac{v_1 - v_3}{1 - v_1 v_3}$

invariant

Similarly

$$(v_{23'})_{,e} = v_{2,e} = v_{23'} = \frac{v_2 - v_{3'}}{1 - v_2 v_{3'}}$$

so that

$$v_{12}(p_*) = (v_{12})_{,e} = \frac{v_{1,e} - v_{2,e}}{1 - v_{1,e} v_{2,e}}$$

where $v_{1,e}$ and $v_{2,e}$ involve only quantities in states 1,2 and then in $p_3 = p_{3'} = p_*$

$\square$

3) Once $p_3 = p_{3*} = p_*$ is computed, one can derive all quantities in the state. For example, if the wave is a shock the Taub adiabat allows to compute h_b ⊙, from which one can compute the density ⊙⊙

⊙ $[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right)[\![p]\!]$

⊙⊙ $\rho = \frac{\gamma p}{(\gamma-1)(h-1)}$ for ideal fluid EoS

$\boxed{\text{Recap}}$

- Riemann problem deals with evolution of a discontinuous initial problem

$$\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x < 0 \\ \underline{U}_R & \text{if } x > 0 \end{cases}$$

$\underline{U}_L = \text{const.}$

$\underline{U}_R = \text{const.}$

$\underline{U}_L \neq \underline{U}_R$

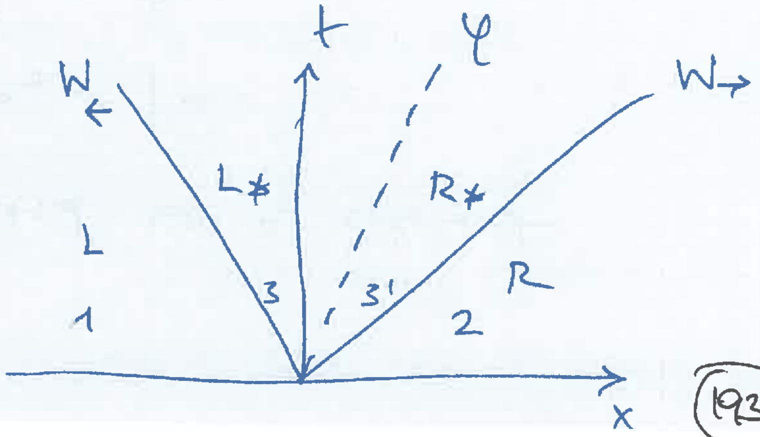
ie

$L \; R \qquad t = 0$

$L \; W_{\leftarrow} \; L_* \; C \; R_* \; W_{\rightarrow} \; R \qquad t > 0$

left prop. wave

contact discontinuity



- regions $L*$ and $R*$ are seperated by a contact discontinuity, hence with

$$p_{L*} = p_{R*} = p_*$$
$$v_{L*} = v_{R*} = v_*$$
$$\rho_{L*} \neq \rho_{R*}$$

- $W_{\leftarrow}, W_{\rightarrow}$ can be any nonlinear wave

- Riemann solution to the problem: four different solutions corresponding to three different wave patterns.

$L \ \mathcal{S}_{\leftarrow} \ L_* \ \mathcal{C} \ R_* \ \mathcal{S}_{\rightarrow} \ R$: 2 shocks

$L \ \mathcal{R}_{\leftarrow} \ L_* \ \mathcal{C} \ R_* \ \mathcal{R}_{\rightarrow} \ R$: 2 rarefaction waves

$\left.\begin{array}{l} L \ \mathcal{R}_{\leftarrow} \ L_* \ \mathcal{C} \ R_* \ \mathcal{S}_{\rightarrow} \ R \\ L \ \mathcal{S}_{\leftarrow} \ L_* \ \mathcal{C} \ R_* \ \mathcal{R}_{\rightarrow} \ R \end{array}\right\}$: 1 shock - 1 rarefaction wave

- no general analytic solution of the RP is known, but solution can be obtained numerically to arbitrary precision, hence the name of "exact" RP.

- simplest solution of RP in relativistic hydrodynamics consists in comparing the initial relative velocity with limiting values for the different wave patterns

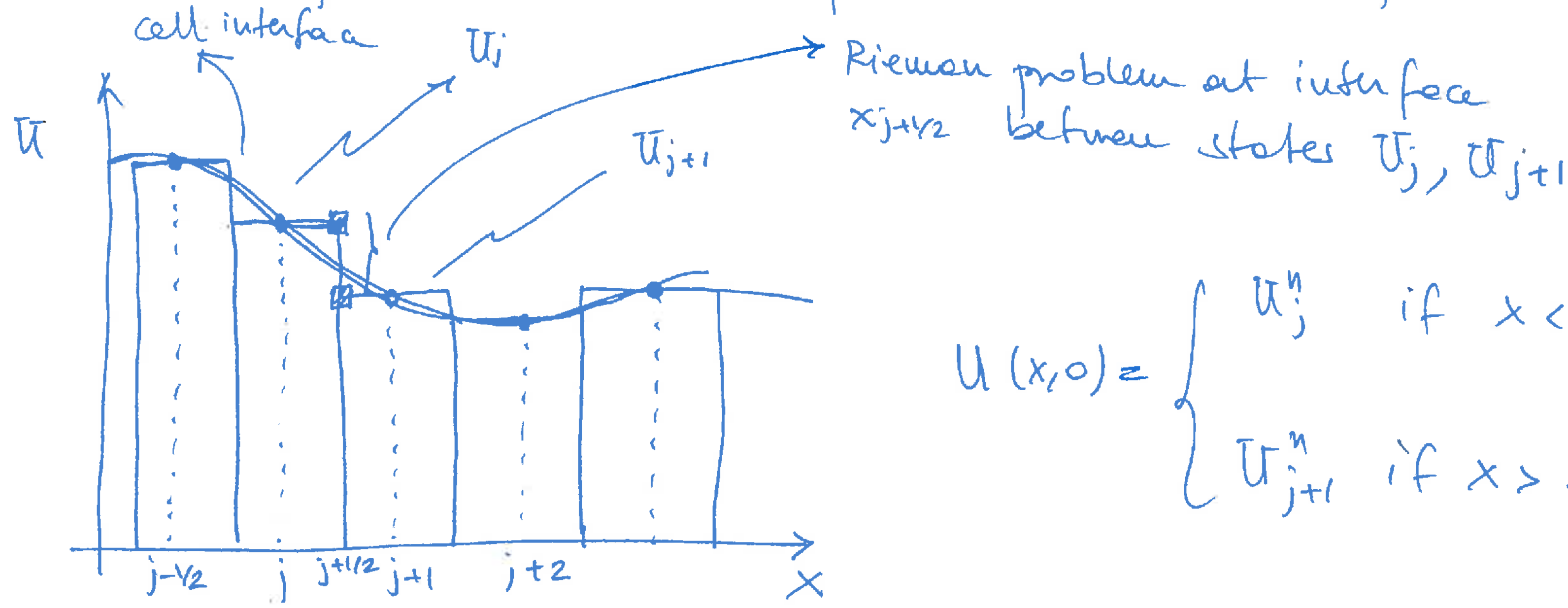
$$(v_{12})_0 = \frac{v_1 - v_2}{1 - v_1 v_2}$$

$$(v_{12})_0 > (\tilde{v}_{12})_{2\mathcal{S}} \quad : \quad \text{2 shocks}$$

$$(\tilde{v}_{12})_{\mathcal{S}\mathcal{R}} < (v_{12})_0 \leq (\tilde{v}_{12})_{2\mathcal{S}} \quad : \quad \text{1 shock, 1 RW}$$

$$(\tilde{v}_{12})_{2\mathcal{R}} < (v_{12})_0 \leq (\tilde{v}_{12})_{\mathcal{S}\mathcal{R}} \quad : \quad \text{2 RWs}$$

there is a class of upwind methods that goes under the name of Godunov methods as these originate around Godunov's original idea of considering a piecewise constant representation of a given quantity as a series of initial states of local Riemann problems.



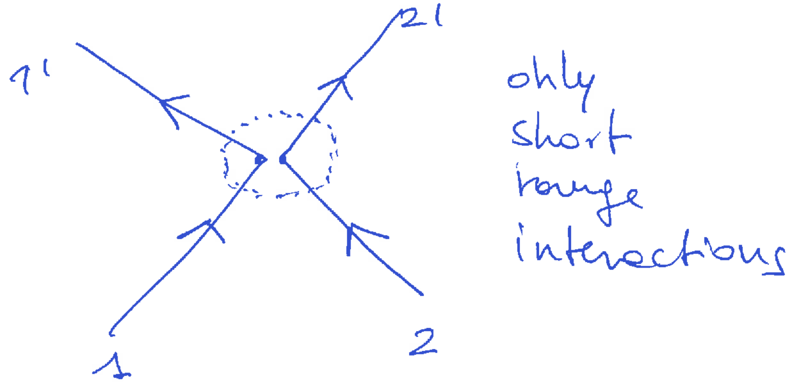
$$U(x,0) = \begin{cases} U_j^n & \text{if } x < x_{j+1/2} \\ U_{j+1}^n & \text{if } x > x_{j+1/2} \end{cases}$$

From hydrodynamics to magnetohydrodynamics

We have so far assumed that our fluid is globally and locally neutral and that the fluid constituents interact only via local collisions taking place when the particles are very close.

Trajectories are therefore straight lines up to scattering point, returning to be straight after scattering.

However, this description ceases to be valid if the particles in the fluid are locally charged, eg when the constituents are partially or fully ionized.



This state is much more common than it may appear and indeed most of the matter in the universe is in ionized state. In this case it is possible to talk of a "plasma" : ie a fluid composed by molecules and neutral atoms, in which there is also a fraction of free electrons and heavier ions. If matter is in thermal equilibrium, this state naturally develops for temperatures $\underline{T \gtrsim 10^4 K}$, but much smaller temperatures are sufficient if the matter is out of equilibrium; energies $\underline{k_B T \sim 10^{-2} eV}$ are sufficient to trigger ionization, thus with temperatures as low as

$$T \sim \frac{10^{-2} eV}{k_B} \simeq \frac{10^{-2} \cdot 10^{-12} \text{erg}}{10^{-16} \text{erg}\, K^{-1}} \simeq 10^{+2} K$$

Plasma dynamics is intrinsically different from the one in a neutral fluid in that the particles' motions are dictated not only by <u>local</u> collisions, but also by the <u>global collective motions</u> resulting from the <u>long range</u> EM fields, some of which are further generated by the global motions.

Before entering in the details of the collective plasma behaviours, it is instructive to understand how charged particle motion can be modified by the presence of large scale electric and magnetic fields, ie what is also referred to as "orbit theory".

Consider therefore a particle of mass m, charge q moving at velocity $\vec{v}$ in an electric field $\vec{E}$ and magnetic field $\vec{B}$. If the motion is non-relativistic ($v \ll 1$) the particle's dynamics is regulated by the Lorentz force

$$\vec{F}_L = m\frac{d\vec{v}}{dt} = m\dot{\vec{v}} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \qquad (1)$$

Note: an accelerated charge will radiate and modify its momentum. We will ignore these losses.

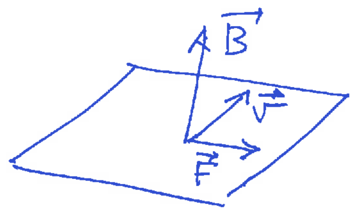
A large set of behaviours is possible as a result of the properties of $\vec{E}$ and $\vec{B}$ and we will investigate them in some of the most important cases, with a crescendo of complexity.

- <u>Gyration</u>

The simplest dynamics is obtained when $\vec{E}=0$ in (1), which then reduces to

$$\vec{F}_L = m\dot{\vec{v}} = q\left(\vec{v}\times\vec{B}\right)$$

: Lorentz force is $\perp$ to $\vec{v}$ and $\vec{B}$



Taking a dot product yields

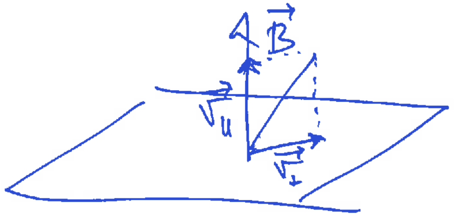
$$\vec{v}\cdot m\dot{\vec{v}} = q\left(\vec{v}\cdot\vec{v}\times\vec{B}\right) = 0 \quad \Longrightarrow$$

$$m\frac{d}{dt}\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0$$

In other words, the force is <u>conservative</u> and does not change the kinetic energy $T := \frac{1}{2} m v^2$ of the particle (once again neglecting radiative energy losses).

Without loss of generality, it is possible to split the velocity vector in a component parallel and orthogonal to $\vec{B}$

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}, \quad \text{where} \quad \vec{v}_{\parallel} = \frac{\vec{v} \cdot \vec{B}}{|\vec{B}|} \; ; \quad \vec{v}_{\perp} \cdot \vec{B} = 0$$



The Lorentz force (2) is then written as

$$m \dot{\vec{v}}_{\parallel} + m \dot{\vec{v}}_{\perp} = q \left[(\vec{v}_{\parallel} + \vec{v}_{\perp}) \times \vec{B} \right] = q (\vec{v}_{\perp} \times \vec{B})$$

$\vec{v}_{\parallel} \times \vec{B} = 0$

from which we conclude that

$$m \dot{\vec{v}}_{\parallel} = 0 \implies \boxed{\vec{v}_{\parallel} = \text{const}}$$

the particle moves with a uniform velocity along B-field

splitting in components one deduces that and

$$m\dot{\vec{v}}_{\perp} = q\,(\vec{v}_{\perp} \times \vec{B}) \qquad (3)$$

Assume for simplicity $\vec{B}$ is uniform and directed along z-axis

$B^i = (0, 0, B^z)$ with $B^z = \text{const.}$ $\Longrightarrow$

$v_{\parallel}^{i} = (0, 0, v^z)$; $v_{\perp}^{i} = (v^x, v^y, 0)$

Hereafter I will ignore differences between covariant and contravariant indices.

Equation (3) can then be written as

$$m\dot{v}_x = q\,\epsilon^{1jk} v_j B^k = q\,\epsilon^{123} v_2 B_3 = q\, v_y B_z, \qquad (4)$$

$$m\dot{v}_y = q\,\epsilon^{2jk} v_j B^k = q\,\epsilon^{213} v_1 B_3 = -q\, v_x B_z. \qquad (5)$$

Taking an additional time derivative and assuming static B field yields the following equations

$$\ddot{v}_x = \frac{q}{m}\dot{v}_y B_z \overset{(5)}{=} -\frac{q^2}{m^2} v_x B_z^2 = -\omega_c^2 v_x$$

$$\ddot{v}_y = -\frac{q}{m}\dot{v}_x B_z \overset{(4)}{=} -\frac{q^2}{m^2} v_y B_z^2 = -\omega_c^2 v_y$$

where $\boxed{\omega_c := \frac{qB}{m}}$: gyrofrequency or cyclotron frequency

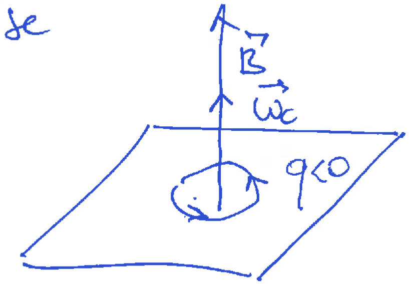
Note that $\omega_c \propto q, B, m^{-1}$: electrons have longer frequency than ions for the same B field and charge

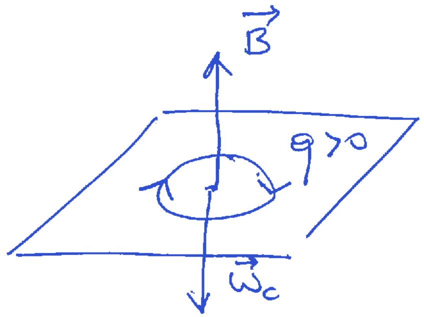
$$(6)\begin{cases} \ddot{v}_x + \omega_c^2 v_x = 0 \\ \ddot{v}_y + \omega_c^2 v_y = 0 \end{cases} \Longrightarrow$$ motion is harmonic in xy plane (circular orbits)

$$(7)\begin{cases} v_x = \frac{v_\perp}{\sqrt{2}} e^{i\omega_c t} \\ v_y = i\frac{v_\perp}{\sqrt{2}} e^{i\omega_c t} \end{cases}$$ where $v_\perp = v_x + i v_y$ and $v_\perp^2 = v_x^2 + v_y^2$

It's possible to define a gyro~~freq~~frequency vector

$\vec{\omega}_c = -\frac{q\vec{B}}{m}$, so that electrons rotate counter-clockwise and ions clock-wise





We can also calculate the radius of this gyration since the Lorentz force acts as a centripetal force

$$-m\frac{v_\perp^2}{r} = q(\vec{v}\times\vec{B}) = q\,v_\perp B^2 \Rightarrow \left(v_\perp = \omega_c r_L\right)$$

$$\boxed{r_L = \frac{m v_\perp}{|q|B} = \frac{v_\perp}{\omega_c}}$$

this is called the <u>Lormor radius</u> or gyroradius

↳ note that electrons and protons have Lormor radii which differ by their masses $\Longleftrightarrow$ $r_L|_{e^-} \sim 10^{-3}\, r_L|_p$

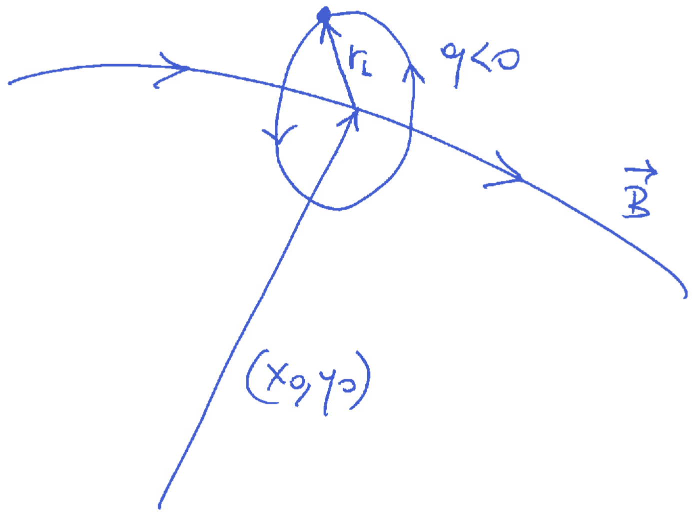
Let's look again at the equations of motion (6) and (7)

$$(8)\quad \begin{cases} v_x = \dfrac{v_\perp}{\sqrt{2}} e^{i\omega_c t} = \dot{x} \\[2mm] v_y = i\dfrac{v_\perp}{\sqrt{2}} e^{i\omega_c t} = \dot{y} \end{cases}$$

Integrating yields

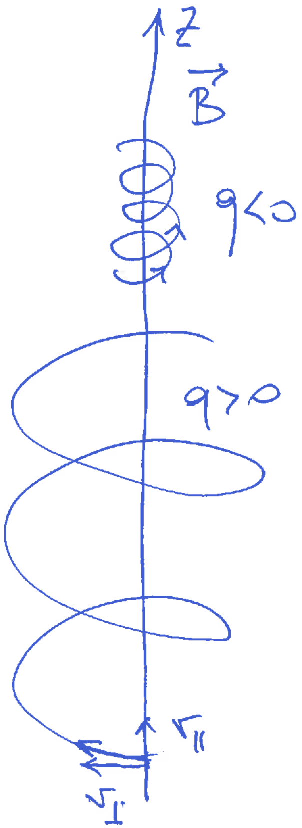
$r_L = v_\perp / \omega_c$

$$(9)\quad \begin{cases} x - x_0 = -i\dfrac{v_\perp}{\sqrt{2}} \dfrac{e^{i\omega_c t}}{\omega_c} = \dfrac{r_L}{\sqrt{2}} \sin(\omega_c t) \\[2mm] y - y_0 = \pm \dfrac{v_\perp}{\sqrt{2}} \dfrac{e^{i\omega_c t}}{\omega_c} = \dfrac{r_L}{\sqrt{2}} \cos(\omega_c t) \end{cases}$$

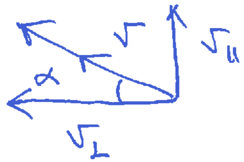


This the motion of a circular orbit around the guiding centre (x_0, y_0) with frequency ω_c

In addition to the gyration motion perpendicular to $\vec{B}$, there is also a constant-velocity motion along $\vec{B}$, which is going to be zero if zero initially. Hence the motion is generically a <u>helical motion</u>. The guiding center has integral of motion



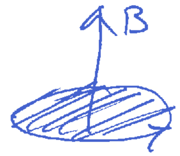
$$\vec{r}_g = \underbrace{x_0\hat{x} + y_0\hat{y}}_{\vec{v}_\perp} + (z_0 + v_\parallel t)\hat{z}$$



α : pitch angle $\qquad \alpha = tg^{-1}\left(\dfrac{v_\parallel}{v_\perp}\right)$ $\square$

A charged particle gyrating produces a current and a current produces a magnetic field. Hence, we can associate a magnetic moment to the particle

$$\mu = \underbrace{I}_{\text{current}}\ \underbrace{A}_{\text{area}}$$



$A = \pi r_L^2$

$$I = q\cdot(\text{no of orbits per unit time}) = q \cdot \frac{\omega_c \Delta t}{2\pi}\frac{1}{\Delta t} = \frac{q\,\omega_c}{2\pi}$$

$r_L = v_\perp/\omega_c$

$$\mu = \frac{q\,\omega_c}{2\pi}\,\pi r_L^2 = \boxed{\frac{m v_\perp^2}{2B} = \mu}$$

note that the information on q is lost: only $v_\perp^2$ and B matter. ▫

– Let's now consider that $\vec{E} \neq 0$ but that both $\vec{E}$ and $\vec{B}$ are uniform; the Lorentz force is again

$$m\dot{\vec{v}} = q(\vec{E} + \vec{v}\times\vec{B})$$

Let $B^i = (0, 0, B^z)$; $E^i = (E^x, 0, E^z)$

Then the equations of motion are

$$m\dot{v}_z = qE_z \implies v_z = \frac{qE}{m}t + v_{z,0}$$: acceleration along z

$$m\dot{v}_x = qE_x \pm \omega_c v_y$$

$$m\dot{v}_y = 0 \pm \omega_c v_x$$

Taking an additional time derivative yields

$$\ddot{v}_x = \left(\frac{q}{m}E_x\right)^{\cdot} \pm \omega_c \dot{v}_y = 0 \pm \omega_c(\pm\omega_c v_x) = \omega_c^2 v_x \implies v_x = \frac{v_\perp}{\sqrt{2}}e^{i\omega_c t}$$

$$\ddot{v}_y = \pm\omega_c \dot{v}_x = \pm\omega_c\left(qE_x \pm \omega_c v_y\right) = \pm\omega_c qE_x + \omega_c^2 v_y$$

E, B const in time

$$= -\omega_c^2\left(\frac{E_x}{B_z} + v_y\right)$$

$$\ddot{v}_y = \left(v_y + \frac{E_x}{B_z}\right)^{\cdot\cdot} = -\omega_c^2\left(v_y + \frac{E_x}{B_z}\right)$$

$$\implies v_y + \frac{E_x}{B_z} = \pm i\frac{v_\perp}{\sqrt{2}}e^{i\omega_c t} \implies v_y = \pm i\frac{v_\perp}{\sqrt{2}}e^{i\omega_c t} - \frac{E_x}{B_z}$$

In summary, the presence of an electric field leads to an additional velocity orthogonal to the electric field (v_y ; $E_y=0$)

$$\begin{cases} v_x = \frac{v_\perp}{\sqrt{2}} e^{i\omega_c t} \\ v_y = \pm i \frac{v_\perp}{\sqrt{2}} e^{i\omega_c t} - \frac{E_x}{B_z} \end{cases} \quad (10)$$

What's this drift velocity?

$\square$

To compute it lets write $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$, then the Lorentz eq. splits into

$$m\dot{\vec{v}}_\parallel = q\vec{E}_\parallel$$

$$m\dot{\vec{v}}_\perp = q\left(\vec{E}_\perp + \vec{v}_\perp \times \vec{B}\right)$$

Furthermore, decompose $\vec{v}_\perp$ in a time varying part (due to gyration) and a constant part (due to the drift), ie

$$\vec{v}_\perp = \vec{v}_d + \vec{v}_g(t)$$

$$\dot{\vec{v}}_\perp = \dot{\vec{v}}_y = \frac{q}{m}\left[\vec{E}_\perp + \vec{v}_d \times \vec{B} + \vec{v}_y \times \vec{B}\right]$$

It's possible to separate in the expression above the time dependent part from the time independent one.

$$\vec{E}_\perp + \vec{v}_d \times \vec{B} = 0 \implies \vec{E}_\perp = -\vec{v}_d \times \vec{B} \quad ; \quad (\times\ B)$$

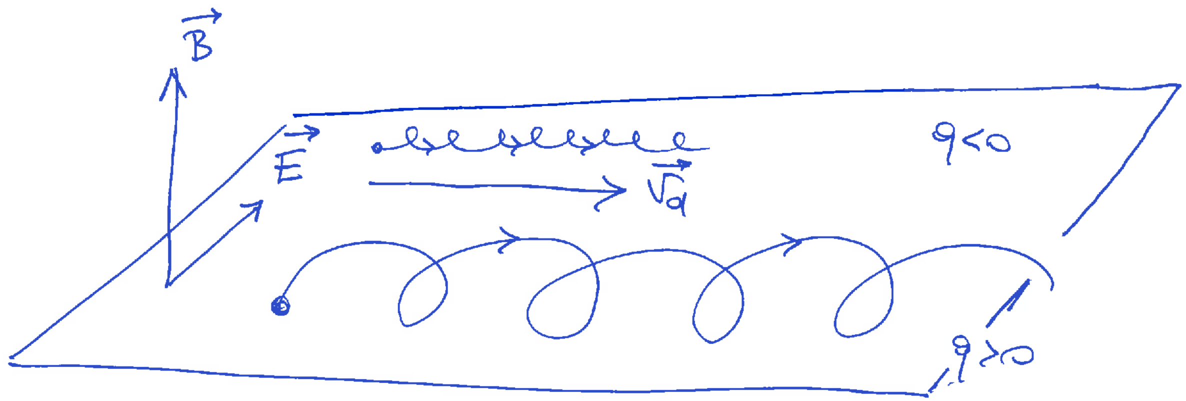
$$(\vec{E}_\perp \times \vec{B}) = \vec{E} \times \vec{B} = -(\vec{v}_d \times \vec{B}) \times \vec{B} = \vec{B} \times (\vec{v}_d \times \vec{B})$$

$$\odot$$

$$= B^2 \vec{v}_d - \vec{B}(\vec{v}_d \cdot \vec{B}) \qquad \vec{v}_d \cdot \vec{B} = 0$$

$$= B^2 \vec{v}_d \implies$$

(11) $$\boxed{\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2}}$$

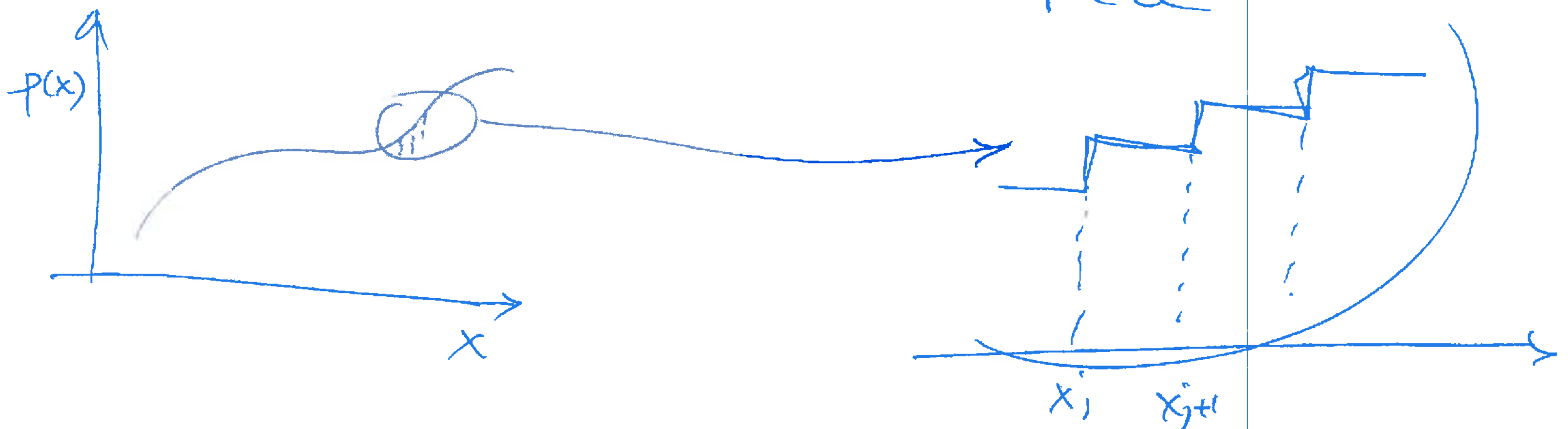


$\odot$

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}}$$

Recap

- Riemann problem represents building block of many algorithms for the solution of the hydrodynamic eqs.
These methods are called HRSC and exploit the fact that the discretization of the fluid variables leads to local RPs at each cell interface



- Plasma : fluid of particles that are locally charged but globally neutral. Such plasmas can be created

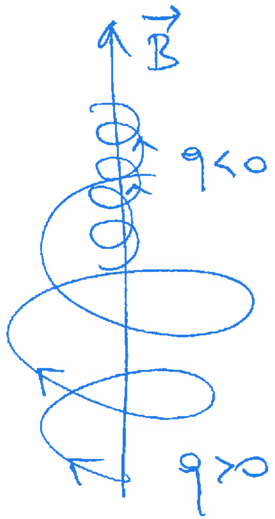
when the fluid constituents are partially or fully ionized.
If in thermal equilibrium, plasmas can be produced for temperatures $T \gtrsim 10^4$ K

- Main difference is that fluid properties are not entirely described by <u>collisions</u>, but that global collective motions are possible in response to large scale electric and magnetic fields.

→ "Orbit theory": study of isolated charge dynamics in global magnetic and electric fields. Governing eq. is the Lorentz eq

$$m\dot{\vec{v}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

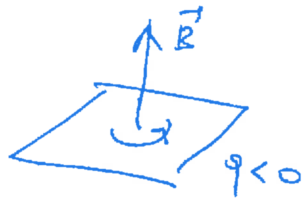
- Gyration: if $\vec{E}=0$, the generic motion of a charged particle is helical around a magnetic field line.



$$\vec{v} = \vec{v}_\parallel + \vec{v}_\perp \quad \text{where} \quad \vec{v}_\parallel = \frac{\vec{v}\cdot\vec{B}}{|\vec{B}|} \; ; \quad \vec{v}_\perp \cdot \vec{B} = 0$$

$$\left.\begin{array}{l} v_\parallel = \text{const.} \\ |\vec{v}_\perp| = \text{const.} \end{array}\right\} \text{kinetic energy is conserved.}$$

$\vec{v}_\perp \neq$ const $\quad \vec{v}_\perp$ gyrates along $\vec{B}$ at gyration frequency $\vec{\omega}_c = -\dfrac{q\vec{B}}{m}$

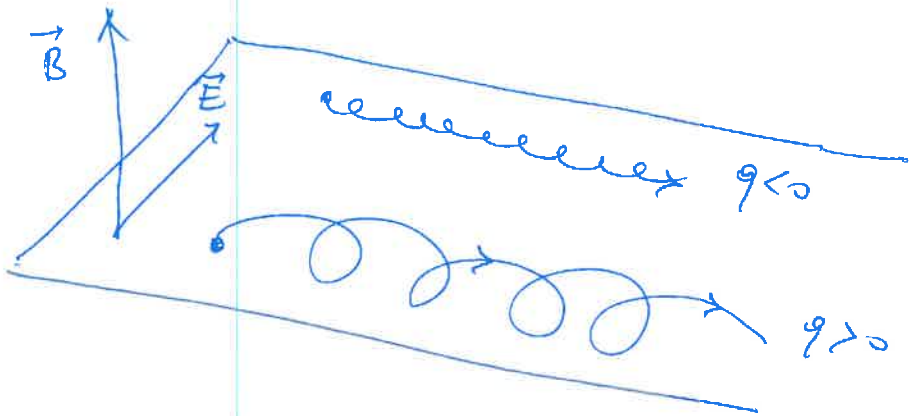


- the radius of the gyration will be the Larmor radius

$$r_L = \frac{m v_\perp}{|q|B} = \frac{v_\perp}{\omega_c}$$

- If $\vec{E} \neq 0$, particles will be accelerated and the Larmor radius will be a function of position. As a result there will be a secular drift with velocity

$$\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2}$$

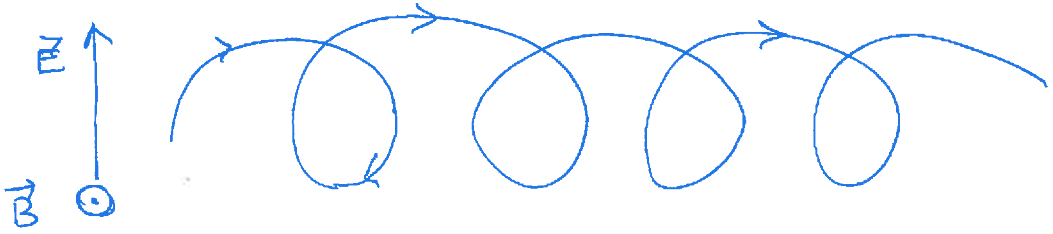


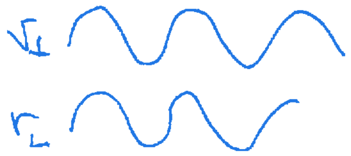
$\vec{v}_d$ is perpendicular to both $\vec{E}$ and $\vec{B}$ and is called the "E-cross-B" drift velocity

- Gyrating particle produces current and hence a magnetic moment

$$\mu = IA = \frac{q\omega_c}{2\pi} \cdot \pi r_L^2 = \frac{m v_\perp^2}{2B}$$

- Notes

* 



gyration velocity varies with position $\Longrightarrow$
Larmor radius also varies position

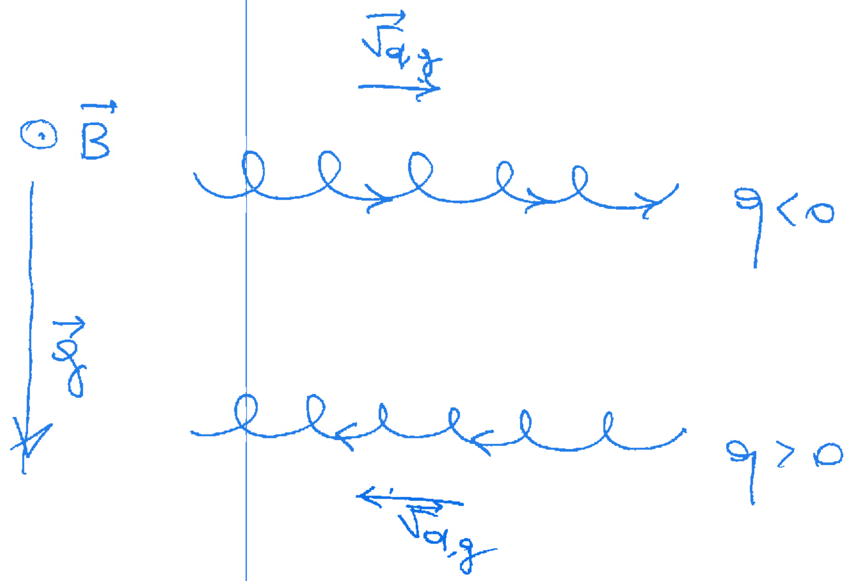
Hence, drift can be seen as the consequence of a varying Larmor radius!

* drift does not depend on charge: electrons and protons drift in the same direction but at different speeds $\Longrightarrow$ <u>no net current</u> is developed (at 1st order).

* Because the electric field just exerts a force $\vec{F} = q\vec{E}$, a drift will be present for <u>any external force</u> $\frac{\vec{F}}{q}$, eg gravity.

$$\vec{v}_{d,g} = \frac{\vec{F} \times \vec{B}}{q B^2} = \frac{m\vec{g} \times \vec{B}}{q B^2} ;$$

Note that there is now a dependence on the charge: $\Rightarrow$ net current develops.



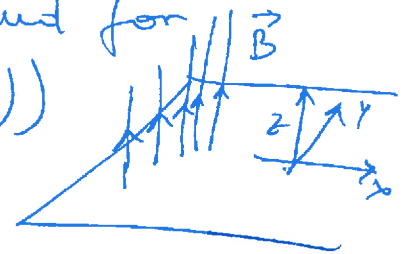
$\square$

Non-uniform magnetic field

Another possible drift develops if the magnetic field is not uniform. This is rather intuitive. B-field bends charged-particles trajectories with a Larmor radius $r_L \propto B^{-1}$!

If Larmor radius is longer where B is smaller (and viceversa), then there will be a drift qualitatively similar to the one seen for $\vec{E}$ field.

Let's derive the expression of the "grad-B" drift and for simplicity we can assume $\vec{E} = 0$; $B^i = (0, 0, B_z(y))$



Let $L := \dfrac{|\vec{B}|}{\nabla|\vec{B}|}$: characteristic lengthscale of variation of the B-field

then orbit theory is still adequate as long as $\boxed{r_L \ll L}$

Lorentz force

$$m\dot{\vec{v}} = \vec{F} = q(\vec{v} \times \vec{B}) \Rightarrow \begin{cases} F_x = q v_y B_z \\ F_y = -q v_x B_z \\ F_z = 0 \end{cases} \qquad (12)$$

$$\frac{dB_z}{dy} \sim \frac{B_z}{L} \ll \frac{B_z}{r_L} \Rightarrow r_L \frac{dB_z}{dy} \ll B_z$$

In other words, we can Taylor expand $B_z(y)$ around $y = 0$

$$B_z(y) = B_z(0) + y\frac{dB_z}{dy} = B_0 + yB_z' + O(y^2)$$

$\Rightarrow$ (12) $\Leftrightarrow$

$$v_x = r_L \omega_c \cos(\omega_c t) = v_\perp \cos(\omega_c t)$$

$$(13) \begin{cases} F_x = q v_y (B_0 + y B'_z) \\ F_y = -q v_x (B_0 + y B'_z) \end{cases}$$

We have seen that the motion around the guiding center at $x_0 = 0 = y_0$ is

$$\begin{cases} x = r_L \sin(\omega_c t) \\ y = r_L \cos(\omega_c t) \end{cases} \Longrightarrow \begin{array}{l} v_x = v_\perp \cos(\omega_c t) \\ v_y = \pm v_\perp \sin(\omega_c t) \end{array}$$

and inserting these in the expressions above we get

$$(14) \begin{cases} F_x = -q v_\perp \sin(\omega_c t) \left[B_0 \pm r_L \cos(\omega_c t) B'_z \right] \\ F_y = -q v_\perp \cos(\omega_c t) \left[B_0 \pm r_L \cos(\omega_c t) B'_z \right] \end{cases}$$

Because we are interested in secular motions (drifts) we can average expression for one period

$$\langle \psi \rangle := \frac{1}{\tau} \int_0^\tau \psi(t)\, dt$$

$$\langle F_x \rangle = -q v_\perp \left[B_0 \langle \cancelto{0}{\sin(\omega_c t)} \rangle \pm r_L \langle \cancelto{0}{\sin(\omega_c t)\cos(\omega_c t)} \rangle B'_z \right]$$

$$= 0$$

$$\langle F_y \rangle = -q v_\perp \left[B_0 \langle \cancelto{0}{\cos(\omega_c t)} \rangle \pm r_L \langle \cos^2(\omega_c t) \rangle B'_z \right]$$

$$= \mp q \frac{v_\perp r_L}{2} B'_z \qquad \text{since} \quad \langle \cos^2(\omega_c t) \rangle = \frac{1}{2}$$

I recall that in the case of the $E \times B$ drift, the guiding centre drift velocity was

$$\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2} = \frac{\vec{F} \times \vec{B}}{q B^2} \qquad \vec{F} : \text{general external force}$$

Hence

$$\hat{e}_x = \hat{e}_y \times \hat{e}_z$$

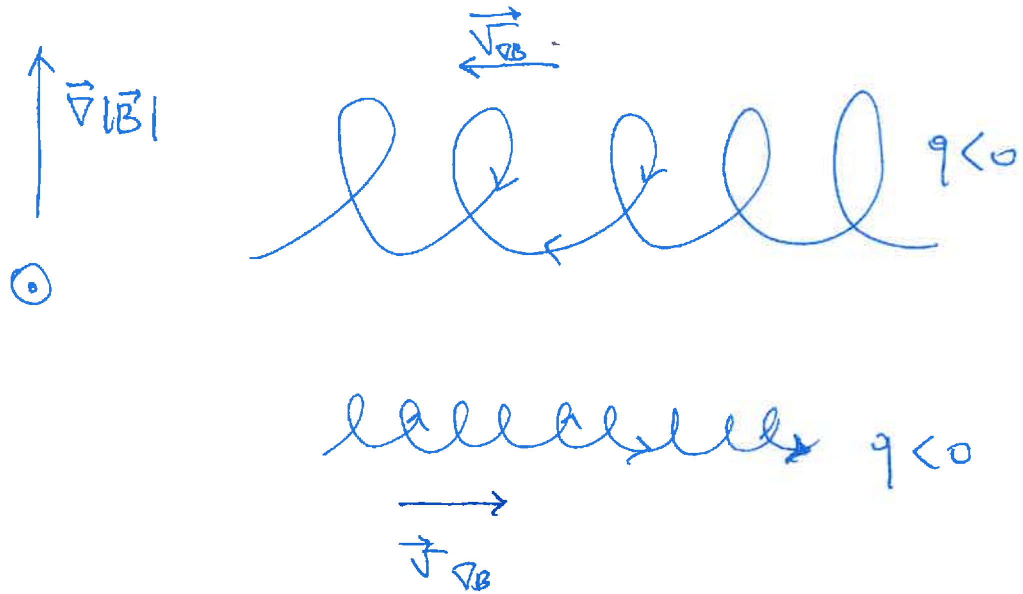
$$\vec{v}_{\nabla B} = \frac{1}{q} \frac{\langle F_y \rangle \hat{e}_y \times B_z \hat{e}_z}{B_z^2} = \mp \frac{1}{q} q \frac{v_\perp r_L}{2} B'_z \frac{B_z}{B_z^2} \hat{e}_x = \mp \frac{v_\perp r_L}{2 B_z} B'_z \hat{e}_x$$

More in general

$$\boxed{\vec{v}_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\vec{B} \times \vec{\nabla}|\vec{B}|}{B^2}} \qquad (16)$$

Note :

- $\vec{v}_{\nabla B}$ depends on charge : (+) for $q>0$ and (−) for $q<0$
- $\vec{v}_{\nabla B} \propto r_L$ which is different for electrons and ions
- differential motion of particles induces currents

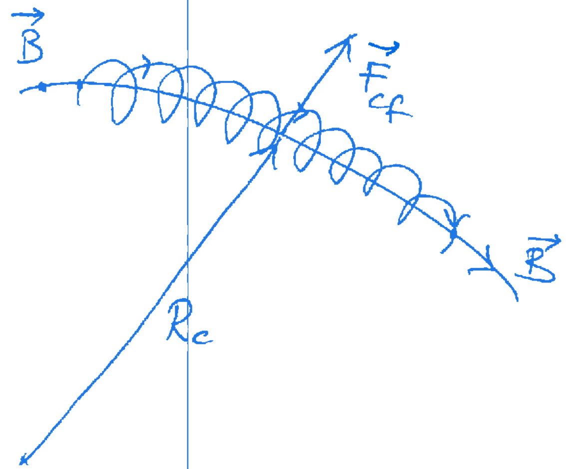


Curvature drift

We have so far idealized the B-field to be in one spatial direction only. In general, however, the magnetic field lines will be curved and the gyration along them will lead to an additional "curvature" drift. In practice the gyrating particle will experience a centrifugal force

$$\vec{F}_{cf} = \frac{m v_{\parallel}^2}{R_c} \hat{e}_r \quad \text{where}$$

R_c is the "curvature" radius of the bent magnetic-field line



The resulting drift will be

$$(15) \quad \boxed{\vec{v}_{cf} = \frac{1}{q} \frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{m v_{\parallel}^2}{q R_c} \frac{\hat{e}_r \times \vec{B}}{B^2}}$$

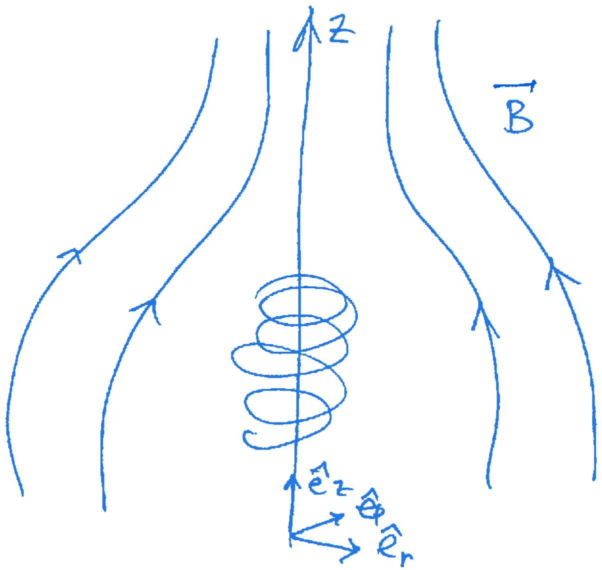
For the example made above, the drift will be different for $q>0$ and $q<0$ and in/out of the page.

Magnetic mirroring and magnetic traps (bottles)

Another important aspect of orbit theory, ie of the kinematics of charged particles in magnetic fields, is represented by the possibility of confining charged particles.

Consider a cylindrical coordinate system (r, ϕ, z) and an axisymmetric magnetic field $B^i = (B^r, 0, B^z)$ such that

$\partial_\phi = 0$



$$\vec{\nabla} \cdot \vec{B} = 0 \iff \frac{1}{r} \partial_r (r B^r) + \partial_z B^z = 0 \implies$$

$$\partial_r (r B^r) = -r \partial_z B^z$$

$$\int_0^r \partial_r (r B^r) = r B^r \Big|_0^r = r B^r = -\int_0^r r \partial_z B^z \, dr$$

assume $\partial_z B^z \hat{=}$ const $= \partial_z B^z \big|_{r=0}$

$\partial_z B^z$ does not vary much with z

$$= -\frac{1}{2} r^2 \partial_z B^z \Big|_{r=0}$$

$$\Rightarrow \boxed{B^r = -\frac{1}{2} r \, \partial_z B^z \Big|_{r=0}} \propto r$$: this provides link between the two components (only one needs to be used).

The components of the Lorentz force $m\dot{\vec{v}} = \vec{F}$ are

$$F_r = q\big(\underbrace{v_\phi B_z}_{(\mathrm{I})} - v_z \cancel{B_\phi}^{0}\big)$$

$$F_\phi = q\big(\underbrace{-v_r B_z}_{(\mathrm{II})} + \underbrace{v_z B_r}_{(\mathrm{III})}\big)$$

$$F_z = q\big(v_r \cancel{B_\phi}^{0} - \underbrace{v_\phi B_r}_{(\mathrm{IV})}\big)$$

$\Rightarrow$ (I), (II) standard Larmor gyration

(III) vanishes for $r \to 0$ $(B \cong B^z \, ; \, B^r \simeq 0)$ and provides radial drift $(v_d \propto F \times B \sim F_\phi B_z)$

(IV) : relevant component

$$F_z = -v_\phi B_r \overset{B_r = -\frac{1}{2} r B_z'}{=} q v_\phi \frac{1}{2} r \, \partial_z B^z \Big|_{r=0} = \mp q \frac{v_\perp r_L}{2} \partial_z B^z$$: mirror force

$v_\phi = \mp v_\perp \; ; \; r = r_L$

This is a force in the z-direction which can accelerate or stop a charge.

Recap

$$r_L = \frac{v_\perp}{\omega_c} \qquad \omega_c = \frac{\pm qB}{m} \qquad \mu = \frac{m v_\perp^2}{2B}$$

$$F_z = \mp \frac{q v_\perp r_L}{2} \partial_z B^z = \mp \frac{q v_\perp^2}{2\omega_c} \partial_z B^z = -\frac{m v_\perp^2}{2B} \partial_z B^z = -\mu\, \partial_z B^z$$

More in general $$\boxed{F_\parallel = -\mu \frac{dB}{ds} = -\mu\, \vec{\nabla}_\parallel |\vec{B}|} \qquad (17)$$

$$\boxed{\vec{\nabla}_\parallel |\vec{B}| := \frac{\vec{B}\cdot\vec{\nabla}}{|\vec{B}|}(|\vec{B}|)}$$

Eq. (17) allows us to derive an important conserved quantity: μ

$$F_\parallel = m\dot{v}_\parallel = -\mu \frac{dB}{ds} \;\overset{/\cdot v_\parallel}{\Longrightarrow}\; m v_\parallel \dot{v}_\parallel = -\mu v_\parallel \frac{dB}{ds} = -\mu \frac{ds}{dt}\frac{dB}{ds} = -\mu \frac{dB}{dt}$$

$$= \frac{1}{2}\frac{d}{dt}\left(m v_\parallel^2\right)$$

In other words: $$\frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2\right) = -\mu \frac{dB}{dt}; \qquad (18)$$

on the other hand the kinetic energy needs to be conserved $\Longleftrightarrow$

$$\frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2\right) = -\frac{1}{2}\frac{d}{dt}\underbrace{\left(\frac{1}{2} m v_\perp^2\right)}_{\mu B} \overset{(18)}{=} -\frac{1}{2}\mu \frac{dB}{dt}$$

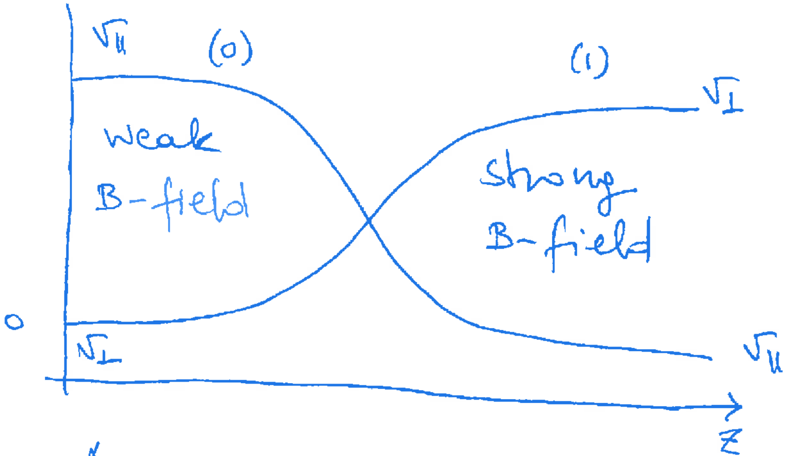
$$\Leftrightarrow \quad \frac{d}{dt}(\mu B) = \mu \frac{dB}{dt} \Rightarrow B \frac{d}{dt}\mu = 0 \Rightarrow \boxed{\mu = \text{const}} = \frac{m v_\perp^2}{2B}$$

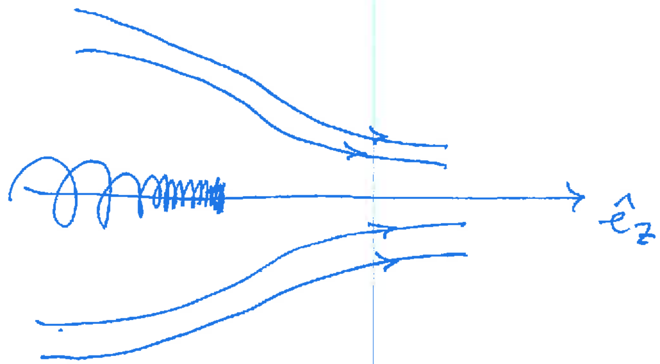
first adiabatic invariant of the particle orbit

Let's explore the consequences of $\dot{\mu} = 0$

As the particle moves into a region of stronger/weaker B, the gyration velocity $v_\perp$ has to increase/decrease to keep $\mu = \text{const}$

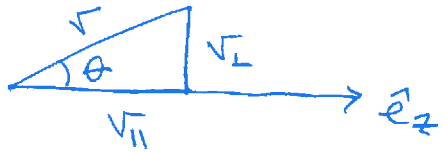
At the same time, conservation of kinetic energy implies that if $v_\perp$ increases, $v_\parallel$ has to decrease





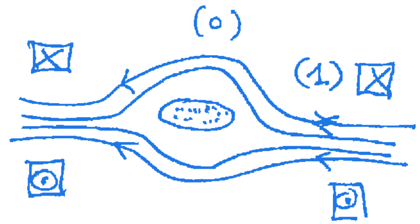
As $v_\parallel \to 0$, the particle slows down till $v_\parallel$ change sign and the particle is reflected. This is a <u>magnetic mirror</u>

Note that this is not a perfect mirror since particles near the z-axis have $v_\perp \ll 1$ and reflection is not efficient.



$$\sin\theta = \frac{v_\perp}{v} \sim \frac{v_\perp}{v_\parallel}$$

In the example above $\frac{1}{2} m \frac{v_{\perp,0}^2}{B_0} \overset{\mu_0 = \mu_1}{=} \frac{1}{2} m \frac{v_{\perp,1}^2}{B_1}$;



If at (1) $v_{\parallel,1}^2 = 0 \Rightarrow v_{\perp,1}^2 = v_1^2 = v_0^2 \Rightarrow$

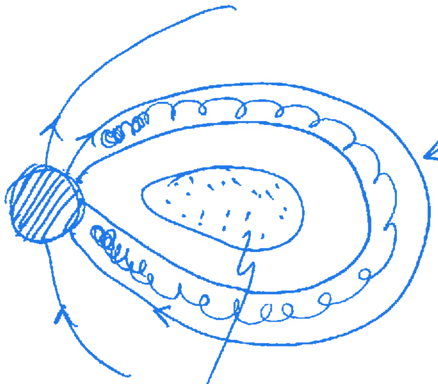
$$\frac{B_0}{B_1} = \frac{v_{\perp,0}^2}{v_{\perp,1}^2} = \frac{v_{\perp,0}^2}{v_0^2} = \sin^2\theta$$: loss-cone angle

In other words, in the mirror particles with pitch angle $< \theta = \sin^{-1}\left(\frac{B_0}{B_1}\right)^{1/2}$ will be lost (loss-cone). The ratio $\frac{B_1}{B_0} = R_m$: mirror ratio

Applications

- plasma experiments where particles need to be confined

- plasma dynamics in Earth's magnetosphere



particle
motion
in magnetosphere

trapped particles : van Allen belt

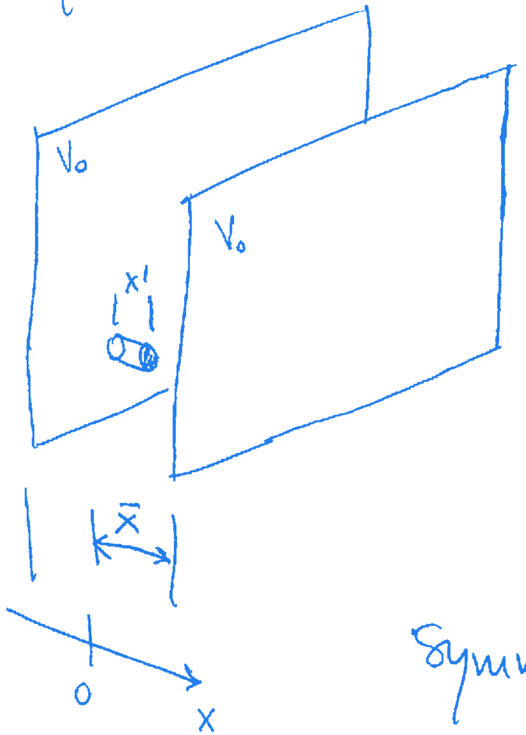
coronal loops
in the sun

particle motion
and acceleration (electric fields)

PLASMAS

Let's go back to the concept of plasma and study some of its main properties: plasma frequency, Debye length, plasma criteria

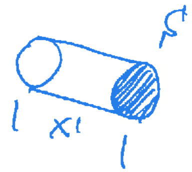
Let's first define in a more quantitative way the condition under which collective effects dominate over random thermal effects. Consider two infinite plates that are plane parallel.



Let n be the charge density between the plates. In this example V_0 is the large-scale potential determining the motion in the plasma and the region between the plates a region of the plasma where thermal effects can be important.

Symmetry requires that $E(x=0) =: E(0) = 0$

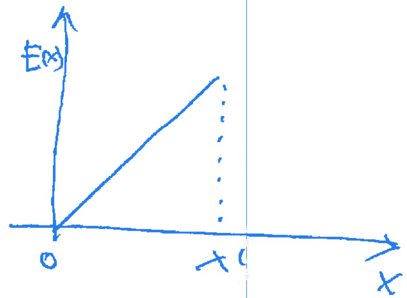
Apply Gauss' theorem to a cylinder of height x' and base S



$$\int \vec{\nabla}\cdot\vec{E}\, dV = \int_0^{x'} E\, ds = S\left(E(x') - E(0)\right) = 4\pi \int \rho\, dV = qn\, 4\pi S x'$$

$$S\, E(x') = 4\pi\, qn\, S\, x' \Rightarrow$$

$$E(x') = 4\pi q n x'$$



The potential V will be the integral of such field, so that

$$V(x') - V(0) = \int_0^{x'} E(x)\, dx = \int_0^{x'} 4\pi q n x\, dx = 2\pi x'^2 q n \Rightarrow$$

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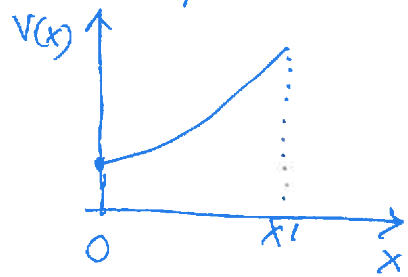


The energy necessary to move a charge q from the middle plane is

$$q\left(V(0)-V(x')\right) = 2\pi q^2 n x^2$$

This energy should be compared with the typical energy of the charge if in thermal equilibrium at T ie

$$q\left(V(0)-V(x')\right) \gtrsim \frac{1}{2} k_B T \iff$$

$$2\pi q^2 n x^2 \gtrsim \frac{1}{2} k_B T \iff$$

$$x \gtrsim \lambda_D := \left(\frac{k_B T}{4\pi q^2 n}\right)^{1/2}$$: Debye length.

λ_D represents the maximum lenght at which thermal effects (or microscopic effects due to short-range interactions) can play a role. At scales longer than λ_D plasma effects dominate

$L \gtrsim \lambda_D$: plasma effects

$L \lesssim \lambda_D$: thermal effects (collisional)

Recap

- orbit theory shows that in general charged particle motion in large-scale EM will lead to secular drifts, ie variations of gyration motion on large timescales.

  * if $\vec{E} \neq 0$; $\vec{v}_d = \dfrac{\vec{E} \times \vec{B}}{B^2}$; $\vec{v}_d \perp$ to $\vec{E}$ and $\vec{B}$; insensitive to charge

  * if external force is present $\vec{v}_F = \dfrac{\vec{F} \times \vec{B}}{q B^2} \overset{\vec{F} = m\vec{g}}{=} \dfrac{m\vec{g} \times \vec{B}}{q B^2}$

  * if $B \neq$ const. and $r_L \ll L := \dfrac{|\vec{B}|}{|\vec{\nabla}|\vec{B}||}$: B-field scale-height

$$\vec{v}_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\vec{B} \times \vec{\nabla}|\vec{B}|}{B^2} \quad : \quad \vec{v}_{\nabla B} \perp \text{ to } \vec{B} \text{ and } \vec{\nabla}|\vec{B}|$$

* if B-field lines are curved particles will experience centrifugal force $\Rightarrow$ drift

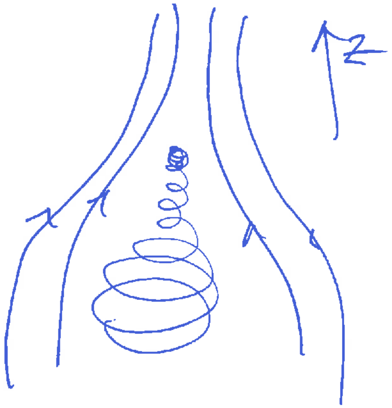
$$\vec{v}_{ct} = \frac{1}{q} \frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{m v_\parallel^2}{q R_c} \frac{\hat{e}_r \times \vec{B}}{B^2}$$

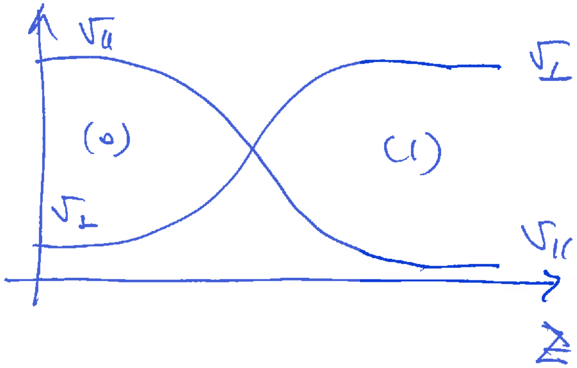
R_c : gyration radius ($\neq$ Larmor radius)

- particles gyrating produce current and hence magnetic moment $\mu = \frac{m v_\perp^2}{2B}$; plasma invariant ie $\frac{d}{dt} \mu = 0$

- the velocity of a particle along the B-field can vary depending on the strength of B-field.

If B-field is suitably shaped to increase, particles can be reflected in a so-called magnetic mirror





conservation of kinetic energy and magnetic moment reduce $v_\parallel$ where $v_\perp$ is large: reflection!

- Magnetic mirrors have a loss cone of angle $\theta_c = \sin^{-1}\left(\frac{B_0}{B_1}\right)$ particles with $\theta < \theta_c$ will be transmitted $\Rightarrow$ large jumps lead to high reflectivity.

- Van Allen belts are example of magnetic mirrors

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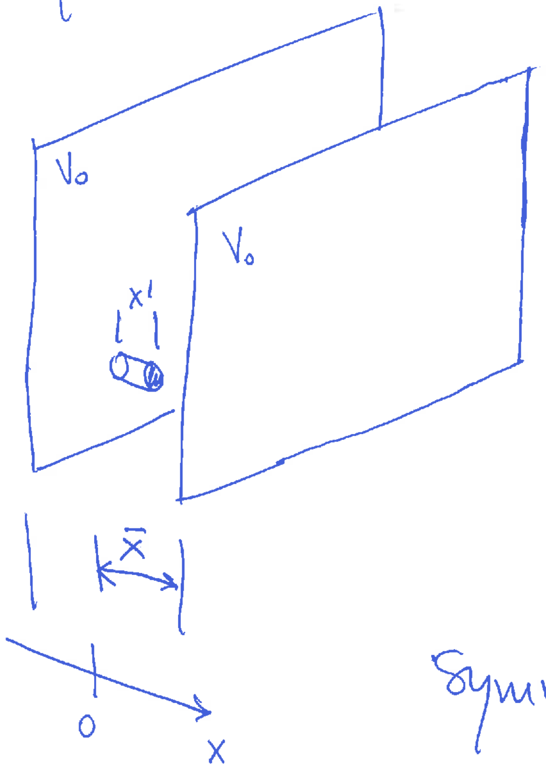

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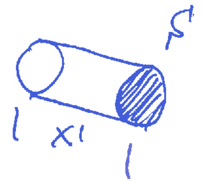

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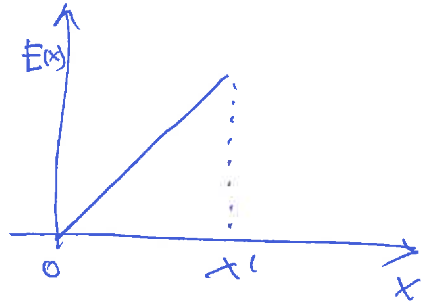

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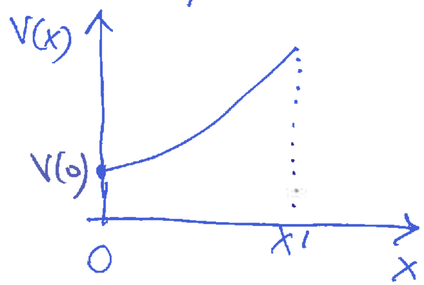

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λ_D represents the maximum lenght at which thermal effects (or microscopic effects due to short-range interactions) can play a role. At scales longer than λ_D plasma effects dominate

$L \gtrsim \lambda_D$: plasma effects

$L \lesssim \lambda_D$: thermal effects (collisional)

If the charges are represented by electrons, then

$$\lambda_D = 740 \left(\frac{k_B T}{eV}\right)^{1/2} \left(\frac{cm^{-3}}{n}\right) cm$$

Eg $T \sim 10^4$ K ; $k_B T \sim 1$ eV ; $n = 1$ (as in interstellar medium)

$\boxed{\lambda_D \sim 700 \text{ cm}}$: Debye length much smaller typical astronomical lengthscale : plasma everywhere in astrophysics. □

- Plasmas tend to be electrically neutral even if there are free charges in thermal motion. This is because if a charge excess is produced locally, then particles of the opposite charge would soon be accelerated towards this excess by the resulting field.

For this to happen the thermal energy should be larger than the electrostatic energy jumps in the plasma, 223

$$\Delta V = -2\pi\left(\Sigma_i Z_i n_i - n_e\right)\lambda_D^2$$

$$-e\,\Delta V \ll \frac{k_B T_e}{2} \iff -2e\,\Delta V \ll k_B T_e \iff$$

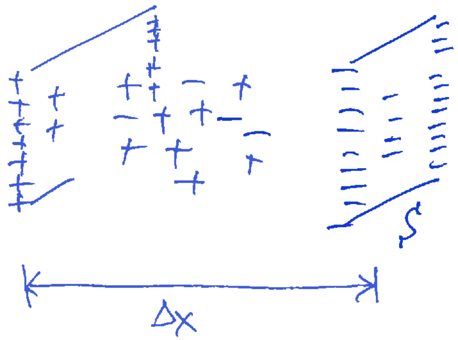
$$k_B T = 4\pi e^2 n_e \lambda_D^2$$

$$4\pi e\left(\Sigma_i Z_i n_i - n_e\right)\lambda_D^2 \ll k_B T = 4\pi e^2 n_e \lambda_D^2 \iff$$

$$\boxed{\left(\Sigma_i Z_i n_i - n_e\right) \ll n_e}$$ plasmas are quasi neutral

□

- Plasma is not static and is always vibrating as a result of the thermal dispacement of electrons. Let initially $n_i = n = n_e$



Suppose of displacing the electrons of a length Δx, so that a net charge q is produced

$$q = n_e e \Delta x S = -E S \Rightarrow E = -n_e e \Delta x$$

The electric force that the free charges will experience is

eg electrons

$$m_e \Delta\ddot{x} = qE = -n_e e^2 \Delta x \Rightarrow$$

$$\Delta\ddot{x} = -\omega_p^2 \Delta x \qquad \boxed{\omega_p^2 := \frac{n_e e^2}{m_e}} \quad \text{plasma frequency}$$

This is the frequency at which the plasma vibrates

$f_p = \frac{\omega_p}{2\pi} \sim 9000\,(n_e)^{1/2}$ Hz : interstellar plasma vibrates at 9 kHz

(see exercise)

Clearly this is possible only if the time between collisions, τ_c, is longer than the plasma oscillation period, ie if

$$\boxed{\frac{\tau_p}{\tau_c} \ll 1} \quad : \text{plasma criterion}$$

□

The Debye length also marks an important length scale namely the length scale below which net charge is not zero

Indeed it is possible to think that each ion will sorround itself of a "cloud" of electrons that will neutralize its charge and "shield" it from the other charges.

Let's calculate this. Consider a neutral plasma with $n_i = n = n_e$

Add now a charge $Q > 0$. This will generate a potential

$$\phi'(r) = \frac{1}{4\pi}\frac{Q}{r}$$

attracting electrons, thus changing the potential, which will become

$$\nabla^2\phi = -\rho_q = -e(n_e - n_i) \quad (*)$$

charge density unbalance ($Z_i = 1$)

$$n_e(r) = n\, e^{-e\phi(r)/k_B T_e}$$

: Maxwell-Boltzmann distribution of charge density produced by ϕ. Replacing in $(*)$ we obtain (note that $n_e \neq n$, while $n = n_i$)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\phi\right) = -ne\left(e^{-e\phi/k_B T} - 1\right)$$

$e^x \simeq 1 + x$

$$\simeq -ne\left(1 - \frac{e\phi}{k_B T_e} - 1\right) \sim \frac{ne^2\phi}{k_B T_e} = \frac{\phi}{\lambda_D^2}$$

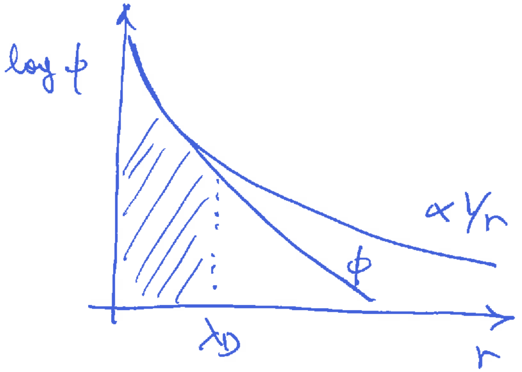
$$\Longleftrightarrow$$

$$\frac{d}{dr}\left(r^2\phi\right) \simeq \frac{r^2\phi}{\lambda_D^2}$$

whose solution is

$$\boxed{\phi(r) = \frac{q}{4\pi r}\, e^{-r/\lambda_D}}$$

: in other words, the standard potential fall off $\frac{1}{r}$ is screened exponentially on a scale λ_D by the electrons



$$\lambda_D = \left(\frac{k_B T}{n e^2}\right)^{1/2}$$

- size of cloud increases with temperature (electrons can move further away from ions)
- size of cloud decreases with numer density (more electrons are available to populate cloud and screening is achieved more easily).

From λ it is possible to define the plasma parameter

$$\Lambda := 4\pi n \lambda_D^3 \quad \propto \quad \text{no. of electrons in the Debye sphere}$$

$$\sim 10^6 \frac{T_e^{3/2}}{n^{1/2}}$$

$\Lambda \ll 1$ strongly coupled plasma (cold and dense, eg NSs)

$\Lambda \gg 1$ weakly coupled plasma (hot and diluted, eg space plasma)

The plasma criterion $\frac{\tau_p}{\tau_c} \ll 1$ can also be written as

$\frac{\nu_p}{\nu_c} \gg 1$; $\nu_c = n \sigma v \simeq n\sigma \left(\frac{k_B T}{m}\right)^{1/2} \sim n e^2 \left(\frac{k_B T}{k_B^4 T^4}\right)^{1/2} \sim \frac{n e^2}{(k_B T)^{3/2}}$

collision freq.

$\sigma \sim \frac{e^2}{k_B^2 T^2}$

$\nu_p = \omega_p / 2\pi$

$$\frac{\nu_p}{\nu_c} = \frac{\omega_p / 2\pi}{e^2 n (k_B T)^{-3/2}} \Rightarrow \nu_c \simeq \frac{\omega_p}{2\pi} \frac{(k_B T)^{3/2}}{n e^2} = \left(\frac{n e^2}{m_e}\right)^{1/2} \frac{(k_B T)^{3/2}}{n e^2}$$

$$= \left(\frac{\cancel{n} e^{\cancel{2}} k_B^3 T^3}{m_e n^{\cancel{2}} e^{\cancel{4}\,2}}\right)^{1/2} =$$

$$= \frac{(k_B T)^{3/2}}{(m_e n e^2)^{1/2}}$$

so that $\frac{\nu_p}{\nu_c} \gg 1 \iff$

$$\boxed{\nu_c \gg \frac{(k_B T)^{3/2}}{(m_e n e^2)^{1/2}}}$$ plasma criterion

Single fluid plasmas : MHD

A common approximation is that of magnetohydrodynamics, in which electrons and ions are assumed to be locked in their motion$^{\odot}$ and provides a good approximation for highly conducting fluids, including low density plasmas.

In a two-fluids plasma the equations of conservation of mass and momentum are given by

$$\partial_t n_j + \vec{\nabla}\cdot(n\vec{v}_j) = 0 \qquad j = e, i \quad \text{for electrons and ions}$$

$$m_j n_j \cdot (\partial_t \vec{v}_j + v^i \nabla_i \vec{v}_j) = -\vec{\nabla}' p_{ij}^{\circledast} + q_j n_j (\vec{E} + \vec{v}_j \times \vec{B}) + \vec{P}_{kj}$$

collision term: expresses mom. variat. of j after collisions with k

pressure tensor

Lorentz force

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \quad ; \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \quad ; \quad \vec{\nabla}\times\vec{B} = \vec{J} + \partial_t\vec{E}$$

$\odot$ plasma is highly ionized and electrons and ions are forced to be tightly coupled by frequent collisions.

where $\vec{J} := n_e q_e \vec{v}_e + n_i q_i \vec{v}_i = n_e q_e (\vec{v}_e - \vec{v}_i)$ $\quad \boxed{n_i \simeq n_e \,;\ q_e = -q_i}$

center of mass velocity $\quad \vec{v} := (n_e m_e \vec{v}_e + n_i m_i \vec{v}_i)/\rho_m = n_e (m_e \vec{v}_e + m_i \vec{v}_i)/n_i m_i$

$\rho_m := n_e m_e + n_i m_i \simeq n_i m_i = \rho \quad \boxed{m_e \ll m_i}$

$= (m_e \vec{v}_e + m_i \vec{v}_i)/m_i \sim \vec{v}_i$: $\boxed{\text{center-of-mass velocity coincides with ion velocity}}$

$P = P_e + P_i$

$\rho_e = n_e q_e + n_i q_i \cong n_e (q_e + q_i) \simeq 0$

As a result the (single fluid) MHD equations can be derived after considering the sum of the two conservation equations of mass and momentum, ie

$$\partial_t \rho_m + \vec{\nabla} \cdot (\rho_m \vec{v}) = 0$$

$$\rho\, \partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\cdot\underline{P} + n_e q_e \vec{E} + n_i q_i \vec{E} + \vec{J}\times\vec{B}$$

$\vec{P}_{ej} = -\vec{P}_{ij} \Rightarrow$ their sum vanishes

$$= -\vec{\nabla}\cdot\underline{P} + \underbrace{(n_e q_e + n_i q_i)}_{\rho_e \simeq 0}\vec{E} + \vec{J}\times\vec{B}$$

$$= -\vec{\nabla}\cdot\underline{P} + \vec{J}\times\vec{B}.$$

which are then complemented with the Maxwell equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \quad ; \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \qquad \vec{\nabla}\times\vec{B} = \vec{J} + \partial_t\vec{E} \simeq \vec{J}$$

$\square$

Let's go back to the generalized MHD eqs. for a two-fluid plasma, ie let's drop the assumptions $n_e \simeq n_i$; $q_e = q_i$

$$m_j n_j \left(\partial_t \vec{v}_j + (v^k \nabla_k \vec{v}_j)\right) = -\overrightarrow{\nabla^k P_{kj}} + q_j n_j (\vec{E} + \vec{v}_j \times \vec{B}) + \underbrace{\vec{P}_{kj}}$$

$$P_{kj}\,\vec{e}^{\,k} = P_{kj}\,\frac{\vec{E}}{|\vec{E}|}$$

The LHSs are

$$\left.\begin{aligned} m_e n_e \left(\partial_t \vec{v}_e + (v^k \nabla_k \vec{v}_e)\right) &= \cdots \quad \cdot\, q_e/m_e \\ m_i n_i \left(\partial_t \vec{v}_i + (v^k \nabla_k \vec{v}_i)\right) &= \cdots \quad \cdot\, q_i/m_i \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} q_e n_e \left(\partial_t \vec{v}_e + (v^k \nabla_k \vec{v}_e)\right) &= \ldots \text{ RHS} \\ q_i n_i \left(\partial_t \vec{v}_i + (v^k \nabla_k \vec{v}_i)\right) &= \ldots \text{ RHS} \end{aligned}\right\}$$ summing the two expressions

$$= q_e n_e \partial_t \vec{v}_e + q_i n_i \partial_t \vec{v}_i = -q_e n_e v^k \nabla_k v_e - q_i n_i v^k \nabla_k v_i + \ldots \Sigma(\text{RHS})$$

$$= \partial_t (q_e n_e \vec{v}_e + q_i n_i \vec{v}_i) \sim \Sigma(\text{RHS})$$

ignore nonlinear terms $v^k \nabla_k v$

$$= \partial_t \vec{J} = \Sigma(\text{RHS})$$

Generalized Ohm's law

($\vec{J} = \sigma \vec{E}$ Ohm's law)

A bit of algebra (see exercise) leads to

$$\partial_t \vec{J} = -\frac{q_e}{m_e}\overrightarrow{\nabla^k P_{ke}} - \frac{q_i}{m_i}\overrightarrow{\nabla^k P_{ki}} + \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)\vec{E} + \underbrace{\left(\frac{n_e q_e^2}{m_e}\vec{v}_e + \frac{n_i q_i^2 \vec{v}_i}{m_i}\right)}\times\vec{B}$$

$$+\frac{q_e}{m_e}\vec{P}_{ei} + \frac{q_i}{m_i}\vec{P}_{ie}$$

$$\left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)\vec{v} + \left(\frac{q_e}{m_e} + \frac{q_i}{m_i}\right)\vec{J}$$

where $\vec{v} := \frac{1}{\rho_m}\left(n_e m_e \vec{v}_e + n_i m_i \vec{v}_i\right)$

$$\Rightarrow$$

$$\partial_t \vec{J} = -\frac{q_e}{m_e}\overrightarrow{\nabla^k P_{ke}} - \frac{q_i}{m_i}\overrightarrow{\nabla^k P_{ki}} + \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)\left(\vec{E} + \vec{v}\times\vec{B}\right) +$$

$$+\left(\frac{q_e}{m_e} + \frac{q_i}{m_i}\right)\vec{J}\times\vec{B} + \frac{q_e}{m_e}\vec{P}_{ei} + \frac{q_i}{m_i}\vec{P}_{ie}$$

$$\boxed{\begin{aligned} &\vec{P}_{ei} = \eta\, q_e^2 n_e^2 (\vec{v}_i - \vec{v}_e) \\ &\vec{P}_{ie} = \eta\, q_e^2 n_e^2 (\vec{v}_e - \vec{v}_i) \\ &\eta : \text{resistivity (tensor)} \end{aligned}}$$

Specialize now to electrons and ions, so that $m_e \ll m_i$ and the generalized Ohm's eq. becomes

$$\partial_t \vec{J} = -\frac{q_e}{m_e}\overrightarrow{\nabla^k P_{ke}} + \frac{n_e q_e^2}{m_e}\left(\vec{E} + \vec{v}\times\vec{B}\right) + \frac{q_e}{m_e}\left(\vec{J}\times\vec{B}\right) - \frac{n_e q_e^2}{m_e}\underline{\eta}\cdot\vec{J} \qquad (*)$$

where $\underline{\eta}\cdot\vec{J} = -\frac{q_e m_e}{n_e}\nu_e^2\left(\vec{v_i} - \vec{v_e}\right)$

It's more convenient to write eq. (*) as a function of $\vec{E}$

$$\vec{E} = \underbrace{-\vec{v}\times\vec{B}}_{①} \underbrace{-\frac{\vec{J}\times\vec{B}}{n_e q_e}}_{②} + \underbrace{\frac{\overrightarrow{\nabla^k P_{ke}}}{n_e q_e}}_{③} + \underbrace{\underline{\eta}\cdot\vec{J}}_{④} + \underbrace{\frac{m_e}{n_e q_e}\partial_t\vec{J}}_{⑤}$$

① plasma motion

② Hall effect

③ ambipolar diffusion (E-field produced by pressure gradients.)

④ Ohmic losses (Joule heating) produced by resistivity

⑤ electron inertia

Let's try to recognize a familiar expression. Neglect ambipolar diffusion, pressure gradients, Hall effect; $\Longrightarrow$

$$\vec{E} + \vec{v} \times \vec{B} = \underline{\eta} \cdot \vec{J} = \frac{1}{\sigma} \vec{J}$$

$\underline{\eta} = \frac{1}{\sigma}$: isotropic resistivity; σ: conductivity

$\Longleftrightarrow$

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \quad : \text{ one fluid Ohm's law.}$$

$\square$

Recap: one fluid MHD eqs (isotropic pressure)

$$\begin{cases} \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \\ \rho\left(\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v}\right) = -\vec{\nabla} p + \vec{J} \times \vec{B} \\ \vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \\ \vec{\nabla} \times \vec{B} = \vec{J} \quad ; \quad \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \ ; \ \vec{\nabla} \cdot \vec{E} = 0 \end{cases}$$

This set is complemented by an evolution eq. for the internal energy and an EOS: $p = p(\rho, \epsilon)$

$$\rho(\partial_t \epsilon + \vec{v} \cdot \vec{\nabla} \epsilon) + p \vec{\nabla} \cdot \vec{v} = 0$$

Let's have a look at the RHS of the momentum eq.

$= \vec{J} \times \vec{B}$: magnetic force

$= (\vec{\nabla} \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \vec{\nabla}) \vec{B} - \vec{\nabla} \left(\frac{B^2}{2} \right) \Rightarrow$

$$\rho \left(\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = (\vec{B} \cdot \vec{\nabla}) \vec{B} - \vec{\nabla} \left(p + \frac{B^2}{2} \right)$$

magnetic tension — gas pressure — magnetic pressure (P_M)

$$\boxed{\beta_p := \frac{p}{P_m} = \frac{2p}{B^2}}$$

$\beta_p \gg 1$ gas dominated plasma (essentially a fluid)

$\beta_p \ll 1$ magnetically dominated plasma (magnetization effect are important)

$$\boxed{P_{tot} = p + P_m = p + \frac{B^2}{2}}$$

Induction equation

Recall that Maxwell and Ohm's law can be combined so that

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} = -\vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{1}{\sigma}\vec{\nabla}\times\vec{J}$$

$$\vec{J} = \sigma(\vec{E}+\vec{v}\times\vec{B})$$

and thus

$$\begin{aligned}
\partial_t\vec{B} &= \vec{\nabla}\times(\vec{v}\times\vec{B}) - \frac{1}{\sigma}\vec{\nabla}\times\vec{J} \\
&= \vec{\nabla}\times(\vec{v}\times\vec{B}) - \frac{1}{\sigma}\left(\vec{\nabla}\times(\vec{\nabla}\times\vec{B})\right) \\
&= \vec{\nabla}\times(\vec{v}\times\vec{B}) - \frac{1}{\sigma}\vec{\nabla}\left(\cancel{\vec{\nabla}\cdot\vec{B}}\right) + \frac{1}{\sigma}\nabla^2\vec{B}
\end{aligned}$$

$$\boxed{\partial_t\vec{B} = \vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{1}{\sigma}\nabla^2\vec{B}}$$: induction equation

advection term

diffusive term

Recap

- we have seen there is in a plasma a characteristic length

  $\lambda_D := \left(\frac{k_B T}{4\pi q^2 n}\right)^{1/2}$: Debye length : equivalence of thermal and electric potential energies

  $L > \lambda_D$: plasma effects dominate

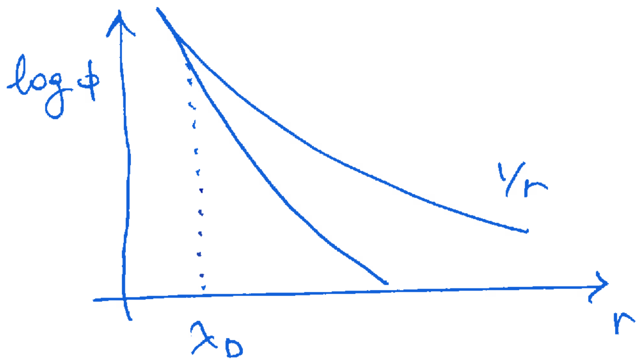
  $L < \lambda_D$: thermal effect "

- plasma is globally neutral but not locally: charge clouds
- plasma vibrates at a characteristic frequency or plasma frequency:

  $\omega_p = \left(\frac{n_e e^2}{m_e}\right)^{1/2}$: oscillation freq. of electron displaced from the charged cloud

- the presence of a charged cloud of electrons around each proton (ion) will change the electric potential

$$\phi \sim \frac{q}{r} \longrightarrow \phi(r) = \frac{q}{4\pi r} e^{-r/\lambda_D}$$



- λ_D therefore defines the lengthscale beyond which the proton (ion) is essentially (but not entirely) screened. This is also called the Debye screening

- $\Lambda = 4\pi n \lambda_D^3$: no of charges in Debye sphere

  = plasma parameter

  $\Lambda \gg 1$ weakly coupled plasma (hot and diluted)

  $\Lambda \ll 1$ strongly coupled plasma (cold and dense)

- $\Lambda \cong 10^6 \frac{T_e^{3/2}}{n_e^{1/2}}$

- in a plasma one should have $\frac{\tau_p}{\tau_c} \ll 1$: plasma criterion

in other words, the collision frequency should be smaller than the plasma freq. (collisions are not frequent)

- we have started looking at single fluid plasma

Recap

- from two fluids to one fluid MHD. We have seen that we can regard the plasma as composed of two fluids with number densities n_e, n_i, masses m_e, m_i, charges q_e, q_i.

- each fluid will conserve its energy, mass and momentum, with the possibility of exchanging the latter.

- it is therefore possible to write 2 sets of equations expressing conservation of mass, momentum and energy

$$\partial_t n_j + \vec{\nabla}\cdot(n_j \vec{v}_j) = 0 \qquad j = e, i$$

$$m_j n_j (D_t \vec{v}_j) = - \overline{\nabla^k P_{kj}} + q_j n_j (\vec{E} + \vec{v}_j \times \vec{B}) + \overline{P_{kj}}$$

pressure tensor

collision term

- to which one adds the Maxwell eqs

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \quad ; \quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho_e$$

$$\vec{\nabla}\times\vec{B} = \vec{J} \quad ; \quad \vec{\nabla}\cdot\vec{B} = 0$$

$J = n_e q_e \vec{v}_e + n_i q_i \vec{v}_i$: net current.

- the two fluids equations simplify considerably if one assumes

$$\left.\begin{array}{l} n_e \simeq n_i \\ q_e = -q_i \\ m_e \ll m_i \end{array}\right\} \Longrightarrow \begin{array}{l} \partial_t\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \\ \rho D_t\vec{v} = -\vec{\nabla}\underline{P} + \vec{J}\times\vec{B} \end{array}$$

where $\rho \simeq m_i n_i$; $\vec{v} \sim \vec{v}_i$

Hence, in addition to Maxwell eqs, the single-fluid MHD eqs have an important additional term in the Lorentz force $\vec{J}\times\vec{B}$

- Ohm's law is very simple for standard conductors

$$\vec{E} = \vec{J}/\sigma$$

and becomes considerably more complex in a plasma

$$\vec{E} = \vec{J}/\sigma - \vec{v}\times\vec{B} + \begin{pmatrix}\text{ambipolar} \\ \text{diffusion}\end{pmatrix} + \begin{pmatrix}\text{Hall} \\ \text{effect}\end{pmatrix} + \begin{pmatrix}\text{displacement} \\ \text{currents}\end{pmatrix}$$

$$\sim \vec{J}/\sigma - \vec{v}\times\vec{B}$$

ie $\vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B})$: this complements the full set of MHD eqs.

- $\vec{J}\times\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{B} - \vec{\nabla}\dfrac{B^2}{2} \Longrightarrow$

$$\rho D_t\vec{v} = \underbrace{(\vec{B}\cdot\vec{\nabla})\vec{B}}_{\text{mag. tension}} - \vec{\nabla}\Big(p + \underbrace{\frac{B^2}{2}}_{\text{mag. pressure}}\Big)$$

$$p_{tot} = p + p_m = p + \frac{B^2}{2}$$

$$\beta_p = \frac{p}{p_m} = \frac{2p}{B^2} = \begin{cases} \gg 1 & \text{matter dominated} \\ \ll 1 & \text{mag. dom.}\end{cases}$$

- Lorentz force is therefore responsible for a <u>pressure-gradient</u> term (gas plus mag.) and a <u>magnetic tension</u> term

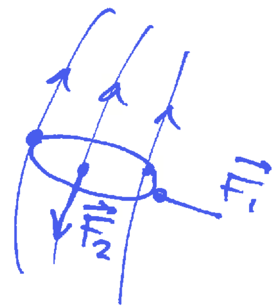
- Lorentz force can also be cast as the gradient of a magnetic stress tensor

$$\vec{J}\times\vec{B} = (\vec{\nabla}\times\vec{B})\times\vec{B} = -\vec{\nabla}\,\underline{M} \quad ; \quad M_{ij} = \frac{1}{2}\delta_{ij}B^k B_k - B_i B_j$$

$$\int_V (\vec{\nabla}\times\vec{B})\times\vec{B}\,dV = \int_\Sigma -\vec{n}\cdot\underline{M}\,ds$$

$\vec{F}_s = -\vec{n}\cdot\underline{M}$: force on volume V

$= \vec{F}_1 + \vec{F}_2 =$ (pressure gradient) + (mag. tension)



- Induction equation

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} = -\vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{1}{\sigma}\vec{\nabla}\times\vec{J} \qquad \left[\vec{J} = \sigma(\vec{E}+\vec{v}\times\vec{B})\right]$$

$$\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \frac{1}{\sigma} \nabla^2 \vec{B}$$

advection term

diffusion term (vanishing if $\sigma \to \infty$, ie ideal MHD)

—

If $\sigma \neq \infty$; $v = 0$ $\quad \partial_t \vec{B} = \frac{1}{\sigma} \nabla^2 \vec{B}$

B-field decays on timescale $\tau = \frac{4\pi L^2 \sigma}{c^2}$

It's easier to study the two terms separately. (1) Set $\vec{v}=0$ (static plasma)

$$\partial_t \vec{B} = \frac{1}{\sigma} \nabla^2 \vec{B} \quad : \text{diffusion (heat) equation}$$

which has solution $B = B_0(\vec{x})\, e^{-t/\tau_d}$ where $\tau_d = \frac{4\pi L^2 \sigma}{c^2}$ ⊙

In other words, the magnetic field diffuses exponentially on timescale τ_d; the larger the conductivity, the longer the diffusion time.

(2) $\sigma \to \infty$ ideal MHD limit, $\vec{E} = -\vec{v}\times\vec{B}$

$$\partial_t \vec{B} = \vec{\nabla}\times(\vec{v}\times\vec{B})$$

$$= \underbrace{-\vec{B}(\vec{\nabla}\cdot\vec{v})}_{\text{compression term}} \underbrace{-(\vec{v}\cdot\vec{\nabla})\vec{B}}_{\text{advection term}} + \underbrace{(\vec{B}\cdot\vec{\nabla})\vec{v}}_{\text{stretching term}}$$

⊙ Note that we have set $c=1$; if not,

$$\partial_t \vec{B} = \frac{c^2}{\sigma}\nabla^2\vec{B}$$

$$\frac{B}{\tau} = \frac{c^2}{\sigma}\frac{B}{L^2} \Rightarrow \sigma = \frac{c^2}{L^2}\tau = \tau^{-1}$$

To distinguish between the two regimes we can introduce the magnetic Reynold number

$$R_M := \frac{\tau_D}{\tau_A} = \frac{\text{diffusion timescale}}{\text{advection timescale}} = \frac{B\sigma L^2}{B\,L/v} = \sigma v L$$

$R_M \gg 1$: convection dominates, magnetic field is mostly advected

$R_M \ll 1$: diffusion dominates, magnetic field is mostly dissipated

In most astrophysical conditions $R_M \gg 1$ and this makes the ideal-MHD limit (IMHD) the most common one. An important property of the plasma dynamics in this case is the so called <u>frozen flux theorem</u>.

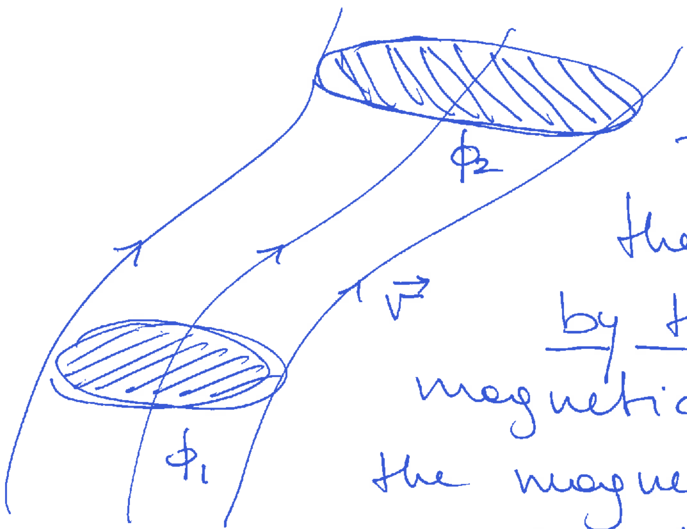
Let ϕ_B be the magnetic flux across an open surface of border Γ, ie

$$\phi_B := \oint_\Gamma \vec{B}\cdot\vec{n}\, ds$$

Then the induction eq. in the IMHD limit implies (see exercise)

$$\boxed{\frac{d}{dt}\phi_B = 0}$$

: the magnetic flux is conserved along the plasma fluid lines



$$\phi_1 = \phi_2$$

In turn, the theorem implies that the magnetic field lines cannot be crossed by the plasma, which simply advects the magnetic field. Alternatively $\vec{B} = k\vec{v}$, ie the magnetic field and the plasma velocity are parallel: studying $\vec{v}$ you know about $\vec{B}$.

MHD waves

When dealing with unmagnetized fluids we have seen that the linearized hydrodynamical equations lead to a wave equation for the perturbations, which propagate at the "speed of sound".

This remains true also for the MHD equations, but with some important differences. We will see more in detail later what are the differences and the mathematical origins of such differences. However, we can go a long way in understanding what to expect simply from first-principles considerations.

* If a global magnetic field is present, then it too can vibrate and perturbations can propagate along it. Such perturbations are just like waves in a string which

are present if the string has a tension. In such an example the velocity of propagation is proportional to the tension in the string.① Similarly, in the presence of a magnetic field there will be waves propagating along the field lines with speed proportional to the magnetic field tension (ie the B-field strength). We have just derived logically the existence of Alfvén waves.

* If a B field is present, there will be anisotropy (B is a vector field). Hence I expect that there will be waves whose speed depends on the direction of propagation relative to the B field, ie a $\vec{k}\cdot\vec{B}$ dependence.

* There will be waves that will tend to sound waves in the

① $v = (T/m/L)^{1/2} = (LT/m)^{1/2}$; T string tension limit of zero magnetic field, ie magnetosonic waves

* because of the existence of two fundamental velocities (sound, Alfven) there will be two magnetosonic waves: fast and slow depending on how they are superposed.

$\square$

Before going to linearized equations, let's recall the concepts of group and phase velocity. Let $u(x,t)$ be the solution of a wave equation

$$\square u = \left[\partial_t^2 + (\partial_x^2 + \partial_y^2 + \partial_z^2) c^2\right] u = 0$$

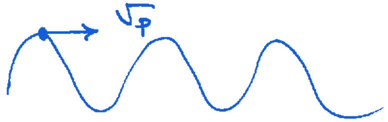
$$u = u_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad \vec{k} : \text{wave vector}$$

$$= u_0 e^{i\phi} \qquad \vec{x} : \text{position vector}$$

$$\phi := \vec{k}\cdot\vec{x} - \omega t$$

Phase is conserved if $\frac{d}{dt}\phi = 0 \iff \frac{d}{dt}(kx) = \omega \iff \dot{x} = \frac{\omega}{k} = v_p$

$\boxed{v_p := \omega/k}$ phase velocity.



If u is composed of several waves with dispersion in wave-vector space given by Δk and in frequency given by $\Delta\omega$

$$u = u_0 e^{i(\Delta k x - \Delta\omega t)} = u_0 e^{i\tilde{\phi}} \qquad \tilde{\phi} := \underbrace{\Delta k x - \Delta\omega t}_{\text{effective phase}}$$

group phase

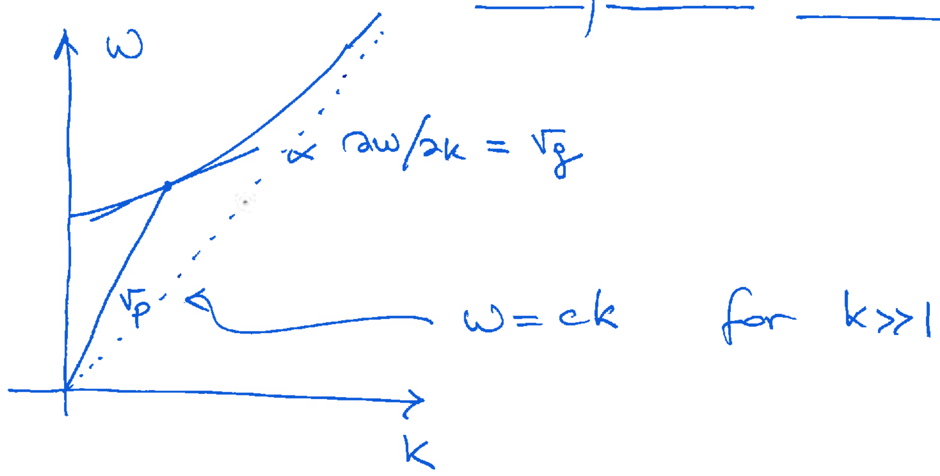
$$\frac{d}{dt}\tilde{\phi} = 0 \iff \dot{x} = \frac{\Delta\omega}{\Delta k} = v_g$$

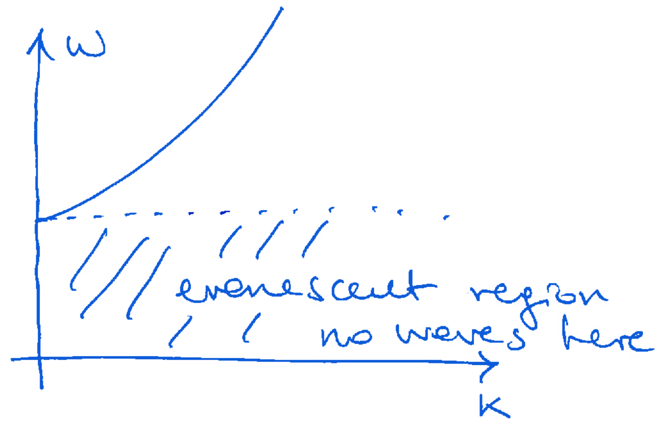
In the limit:

$$\boxed{v_g := \frac{\partial\omega}{\partial k}}$$

group velocity; it's the velocity at which energy and momentum are transferred.

$\omega^2 = \omega^2(k)$: dispersion relation ; waves propagate for $\omega^2 \geq 0$





$\square$

MHD Equations

$$\begin{cases} \partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \\ \rho\,\partial_t \vec{v} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}p + \vec{J}\times\vec{B} \\ (\partial_t + \vec{v}\cdot\vec{\nabla})\left(\dfrac{p}{\rho^\gamma}\right) = 0 \end{cases}$$

$p = k\rho^\gamma$

$\to \partial_t p + (\vec{v}\cdot\vec{\nabla})p = -\gamma p\,\vec{\nabla}\cdot\vec{v}$

Maxwell eqs.

$$\begin{cases} \partial_t \vec{B} = -\vec{\nabla}\times\vec{E} \\ \vec{\nabla}\times\vec{B} = 4\pi \vec{J} \\ \vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B}) \end{cases}$$

Assume $\sigma \to \infty$: ideal MHD $\Longleftrightarrow$ $\vec{E} = -\vec{v} \times \vec{B}$

$$\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B})$$

Express all quantities as a background and a perturbation, ie
$\vec{B}, \vec{v}, \rho, p, T$

$$\psi \to \psi_0 + \psi_1 \qquad |\psi_1| \ll |\psi_0|$$

where $\psi_1 = \psi_1(\vec{x}, t)$ while ψ_0 is uniform and time independent

$$\partial_t \psi_0 = 0 = \partial_{\vec{x}} \psi_0$$

and that the plasma is initially at rest, ie $\vec{v}_0 = 0$

$\Rightarrow$ "magnetohydrostatic equilibrium"

$$\vec{\nabla} p_0 = \vec{J}_0 \times \vec{B}_0$$

Of course this equation is trivially verified for a uniform background medium

Introducing these perturbations one obtains the following linearized[⊙] MHD eqs. (Exercise)

$$\left\{\begin{array}{ll} \partial_t \rho_1 + \vec{\nabla}(\rho_0 \vec{v}_1) = 0 & \text{continuity} \\ \rho_0 \partial_t \vec{v}_1 = -\vec{\nabla} p_1 + \frac{1}{4\pi}\left((\vec{\nabla}\times\vec{B}_0)\times\vec{B}_1 + (\vec{\nabla}\times\vec{B}_1)\times\vec{B}_0 \right) & \text{Euler eq.} \\ \partial_t p_1 + (\vec{v}_1\cdot\vec{\nabla}) p_0 = -\gamma p_0 \vec{\nabla}\cdot\vec{v}_1 & \text{energy eq.} \\ \partial_t \vec{B}_1 = \vec{\nabla}\times(\vec{v}_1\times\vec{B}_0) & \text{induction eq.} \\ \vec{\nabla}\cdot\vec{B}_1 = 0 & \text{div} B \\ p_1 = \frac{k_B}{m}(\rho_0 T_1 + \rho_1 T_0) & \text{EOS} \end{array}\right.$$

⊙ Note that I have neglected terms $O(\psi_1^2)$

We can now insert the ansatz of a plane wave

$$u = u_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

and bear in mind that in Cartesian coordinates $k_x := \vec{k}\cdot\vec{x}$

$$\partial_t u \to -i\omega u \; ; \quad \partial_t^2 u \to -\omega^2 u \; ; \quad \partial_x u \to i k_x u \; ; \quad \partial_x^2 \to -k_x^2 u$$

$$\vec{\nabla} \to i\vec{k} \; ; \quad \vec{\nabla}\cdot \to -i\vec{k}\cdot \; ; \quad \vec{\nabla}\times \to i\vec{k}\times$$

$\square$

Let's start with something we already know: pure hydrodynamics (Exercise)

$$\left.\begin{aligned} \text{continuity} &: -i\omega\rho_1 + i\rho_0(\vec{k}\cdot\vec{v}_1) = 0 \\ \text{Euler} &: -i\omega\rho_0\vec{v}_1 = -i\vec{k}p_1 \\ \text{Energy} &: -i\omega p_1 = -i\gamma p_0(\vec{k}\cdot\vec{v}_1) \end{aligned}\right\}$$

from which we deduce $\vec{v}_1 = \left(\frac{p_1}{\omega\rho_0}\right)\vec{k}$: particles move in the direction of perturbation

longitudinal wave

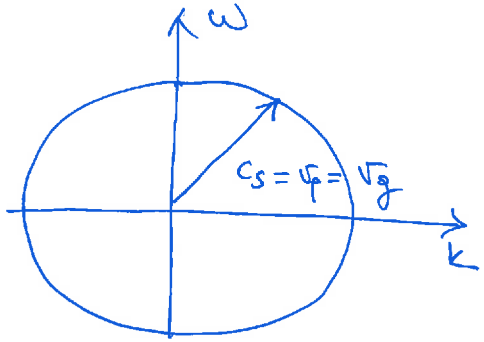
Furthermore

$$\rho_1 = \frac{\rho_0}{\omega}\left(\vec{k}\cdot\vec{v}_1\right) = \frac{\cancel{\rho_0}}{\omega}\,\frac{p_1}{\omega\,\cancel{\rho_0}}\,k^2 = \frac{p_1}{\omega^2}\,k^2 \qquad p_1 = \frac{\gamma p_0}{\omega}\left(\vec{k}\cdot\vec{v}_1\right) = \gamma p_0 \frac{\rho_1}{\rho_0}$$

$$\Longrightarrow \underbrace{\frac{k^2}{\omega^2} = \frac{1}{c_s^2}}_{\text{dispersion relation}} = \frac{\rho_1}{p_1} \qquad \text{or} \qquad \underbrace{c_s^2 = \frac{p_1}{\rho_1} = \frac{\gamma \rho_1 p_0}{\rho_1 \rho_0} = \gamma \frac{p_0}{\rho_0}}_{\text{sound speed: speed of perturbations}}$$

$$\begin{cases} \omega^2 = k^2 c_s^2 \\ v_p = \dfrac{\omega}{k} = \pm c_s \\ v_g = \dfrac{\partial \omega}{\partial k} = \pm c_s \end{cases} \qquad \omega = \pm k c_s$$

group and phase velocities coincide and are equal to the sound speed.



Sound waves $\Longleftrightarrow$ compressional waves

Let's now consider rather different waves, ie waves that do not produce any compression ($\rho_1 = 0 = p_1$): <u>incompressional waves</u>

If the background state is uniform, $\vec{\nabla} p_0 = 0 \Rightarrow$

$$\vec{J}_0 \times \vec{B} = 0 = (\vec{\nabla} \times \vec{B}_0) \times \vec{B}_0 = 0 \quad \text{if } B_0 \text{ is uniform}$$

In this case the linearized eqs reduce to

$$\vec{\nabla} \cdot \vec{v}_1 = 0 \implies i(\vec{k} \cdot \vec{v}_1) = 0 \qquad (\cdot)$$

$$\rho_0 \partial_t \vec{v}_1 = \frac{(\vec{\nabla} \times \vec{B}_1)}{4\pi} \times \vec{B}_0 \implies i\omega \rho_0 \vec{v}_1 = i \frac{(\vec{k} \times \vec{B}_1)}{4\pi} \times \vec{B}_0 \qquad (*)$$

$$\partial_t \vec{B}_1 = \vec{\nabla} \times (\vec{v}_1 \times \vec{B}_0) \implies -i\omega \vec{B}_1 = i\vec{k} \times (\vec{v}_1 \times \vec{B}_0) \qquad (**)$$

$$\vec{\nabla} \cdot \vec{B}_1 = 0 \implies i\vec{k} \cdot \vec{B}_1 = 0 \qquad (\cdot\cdot)$$

$(\cdot)$ and $(\cdot\cdot)$ reveal that $\vec{v}_1$ and $\vec{B}_1$ are anti parallel

$\vec{v}_1$ and $\vec{k}$ are orthogonal : <u>transverse wave</u>

contracting (*) with $\vec{B}_0$:

$$-\omega\rho_0 \vec{v}_1\cdot\vec{B}_0 = \left(\left(\vec{k}\times\vec{B}_1\right)\times\vec{B}_0\right)\cdot\vec{B}_0 = 0 \implies \vec{v}_1\cdot\vec{B}_0 = 0,$$

$\vec{v}_1$ and $\vec{B}_0$ are orthogonal

(**) $\iff$

$$-\omega\vec{B}_1 = \vec{k}\times\left(\vec{v}_1\times\vec{B}_0\right)$$

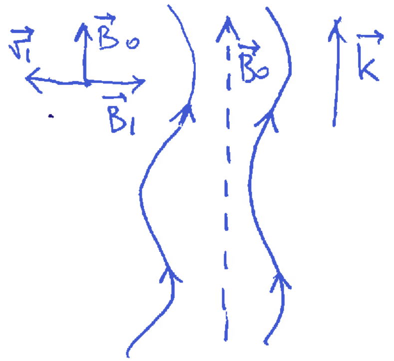
$$= \left(\vec{k}\cdot\vec{B}_0\right)\vec{v}_1 - \left(\cancel{\vec{k}\cdot\vec{v}_1}\right)\vec{B}_0 = \left(\vec{k}\cdot\vec{B}_0\right)\vec{v}_1$$

from which we deduce that $\vec{B}_1 /\!/ -\vec{v}_1$

Taking a scalar product with $\vec{B}_0$

$$-\omega\vec{B}_1\cdot\vec{B}_0 = \left(\vec{k}\cdot\vec{B}_0\right)\left(\cancel{\vec{v}_1\cdot\vec{B}_0}\right) = 0$$

$\vec{B}_1$ and $\vec{B}_0$ are orthogonal



In other words: these are incompressible transverse waves in which the magnetic field is distorted in the direction orthogonal to that of propagation.

These are the same type of were encountered in a string and are called (shear) Alfven waves. ▫

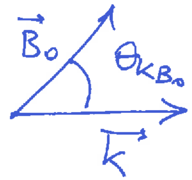
Let's calculate the dispersion relation for these waves.

Take (**) and multiply by ω and replace $\vec{v}_1 = \frac{1}{4\pi\rho_0\omega}(\vec{k}\times\vec{B}_1)\times\vec{B}_0$ (*)

$$\omega^2\cdot\vec{B}_1 = \frac{1}{4\pi\rho_0}\vec{k}\times\left[\left((\vec{k}\times\vec{B}_1)\times\vec{B}_0\right)\times\vec{B}_0\right] \underset{\vec{k}\cdot\vec{B}_1=0}{=} \frac{(\vec{k}\cdot\vec{B}_0)^2\vec{B}_1}{4\pi\rho_0}$$

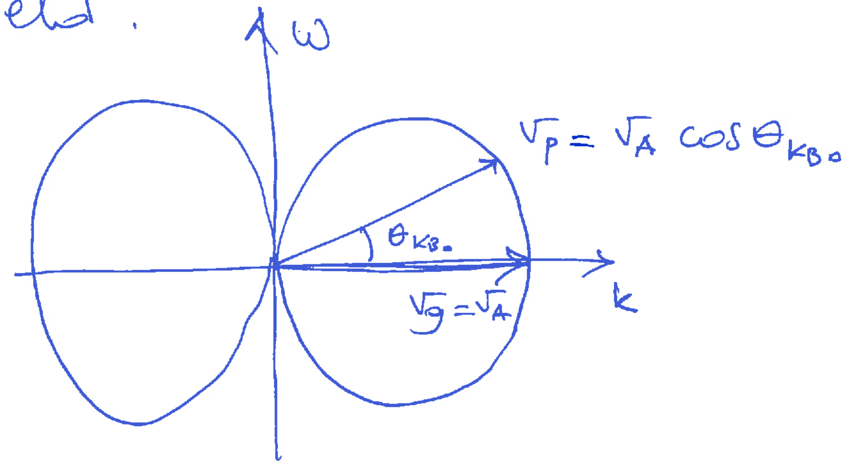
$$\Rightarrow \quad \omega^2 = \frac{(\vec{k}\cdot\vec{B}_0)^2}{4\pi\rho_0} = \frac{k^2B_0^2}{4\pi\rho_0}\cos^2\theta_{kB_0} = k^2 v_A^2\cos^2\theta_{kB_0}$$

where $\cos\theta_{kB_0} = \frac{\vec{k}\cdot\vec{B}_0}{|\vec{k}||\vec{B}_0|}$



where $v_A^2 := \frac{B_0^2}{4\pi\rho_0}$: this the Alfven velocity and is the equivalent of the propagation velocity of a wave on a string with tension $B_0^2 S/4\pi$ ⊙

Note that this speed is no longer isotropic and is actually zero if $\vec{k}\cdot\vec{B}_0 = 0$, ie for propagations orthogonal to the B-field.



$$\omega^2 = k^2 \cos^2\theta_{kB_0} V_A^2$$

$$\omega = \pm k V_A (\cos\theta_{kB_0})$$

$$V_p = \pm V_A (\cos\theta_{kB_0})$$

$$V_g = \frac{\partial\omega}{\partial k} = \pm V_A (\cos\theta_{kB_0})$$

Alfven waves have maximum speed along magnetic field lines and zero velocity across them.

In a cylindrically symmetric system, a shear Alfven wave is represented by a "torsional" wave

Consider now compressional waves, ie $p_1 \neq 0$; $\rho_1 \neq 0$

The linearized equations reduce to (see Exercise)

$$\frac{\omega^2}{V_A^2}\vec{v}_1 = k^2\cos^2\theta_{kB_0}\vec{v}_1 - (\vec{k}\cdot\vec{v}_1)\,k\cos\theta_{kB_0}\,\hat{B}_0 + \vec{k}\left[\left(1+\frac{c_s^2}{V_A^2}\right)(\vec{k}\cdot\vec{v}_1) - k\cos\theta_{kB_0}(\hat{B}_0\cdot\vec{v}_1)\right]$$

where $\hat{B}_0 := \dfrac{\vec{B}_0}{|\vec{B}_0|}$;

Collecting terms we obtain the dispersion relation

$$\omega^4 - \omega^2 k^2 (c_s^2 + V_A^2) + c_s^2 V_A^2 k^4 \cos^2\theta_{kB_0} = 0$$

with solutions

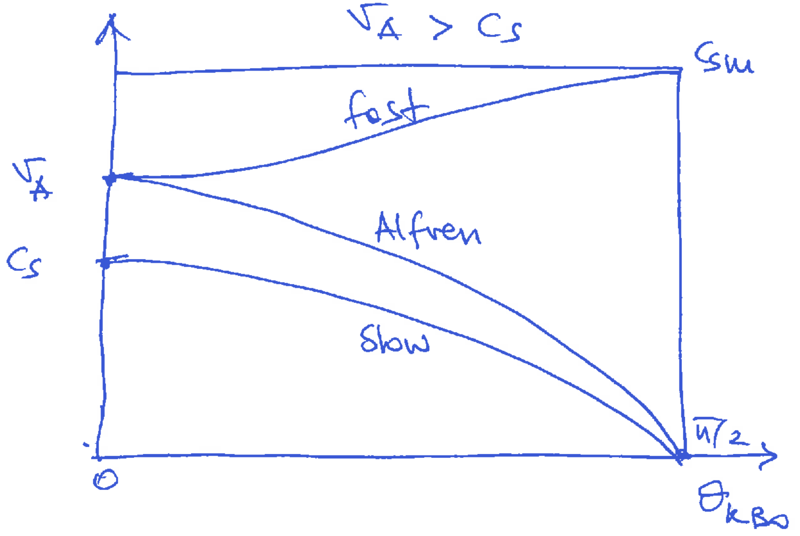
$$\frac{\omega^2}{k^2} = V_f^2 = \frac{1}{2}\left[c_s^2 + V_A^2 + \sqrt{(c_s^2+V_A^2)^2 - 4c_s^2V_A^2\cos^2\theta_{kB_0}}\right]$$ fast magnetosonic wave

$$\frac{\omega^2}{k^2} = V_s^2 = \frac{1}{2}\left[c_s^2 + V_A^2 - \sqrt{(c_s^2+V_A^2)^2 - 4c_s^2V_A^2\cos^2\theta_{kB_0}}\right]$$ slow magnetosonic wave

this is what distinguishes fast and slow

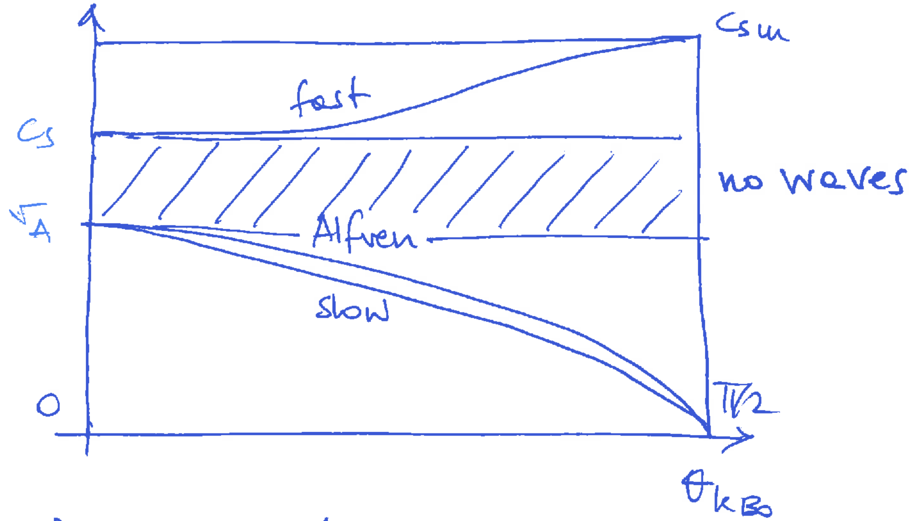
low β

$V_A > C_s$

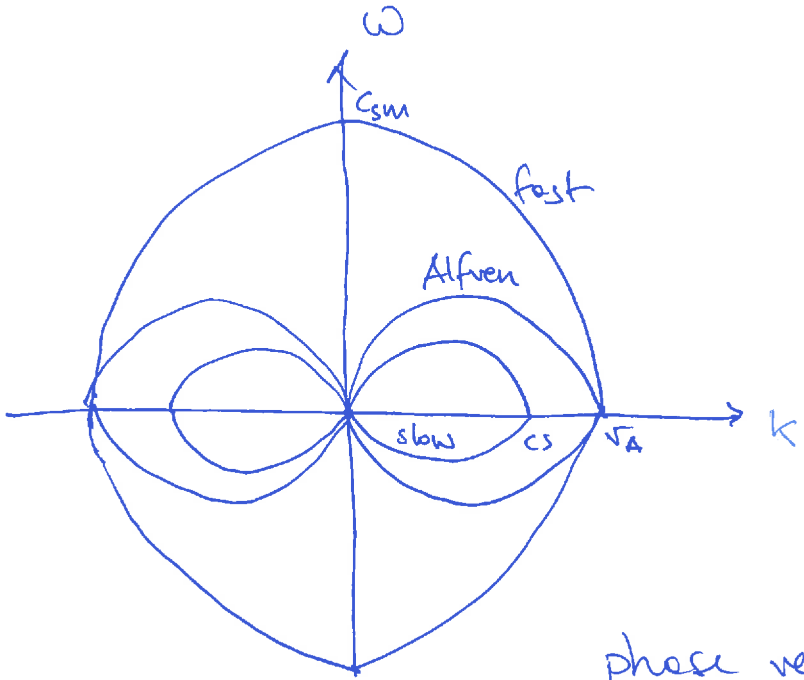


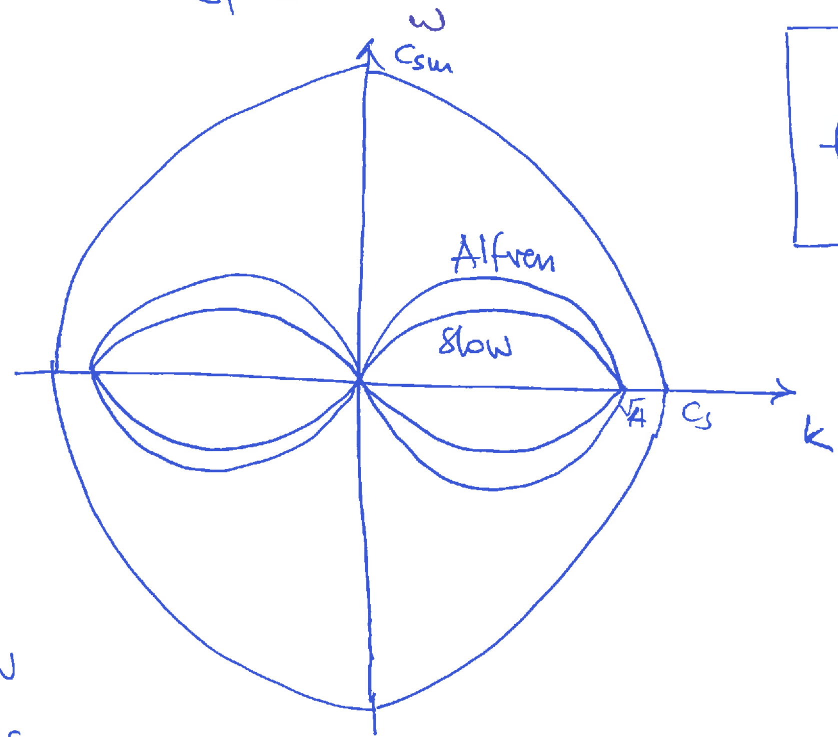
high beta

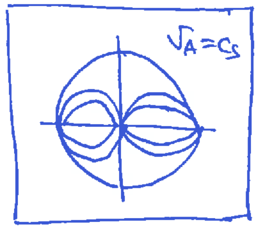
$V_A < C_s$



$C_{sm}^2 := C_s^2 + V_A^2$: magnetosonic speed







phase velocities of fast and slow magnetosonic waves

Note that $\boxed{v_A > c_s} \iff \dfrac{B_0}{\sqrt{4\pi\rho_0}} > c_s = \sqrt{\dfrac{p_0}{\rho_0}} \iff$

$$\frac{B_0}{p_0} > 1 \iff \beta_p = \frac{2p_0}{B_0} < 1 \quad \boxed{\text{low beta}}$$

Similarly $\boxed{v_A < c_s} \iff \beta_p > 1 \quad \boxed{\text{high beta}}$

Summary

Wave type	Propagation	Restoring $\beta_p < 1$ (mag. dom.)	Restoring $\beta_p > 1$ (gas dom.)
Alfven	mostly along $\vec{B}_0$	mag. tension	mag. tension
fast	isotropic	mag. pressure	gas pressure
slow	mostly along $\vec{B}_0$	gas press.	mag. tension

$\square$