

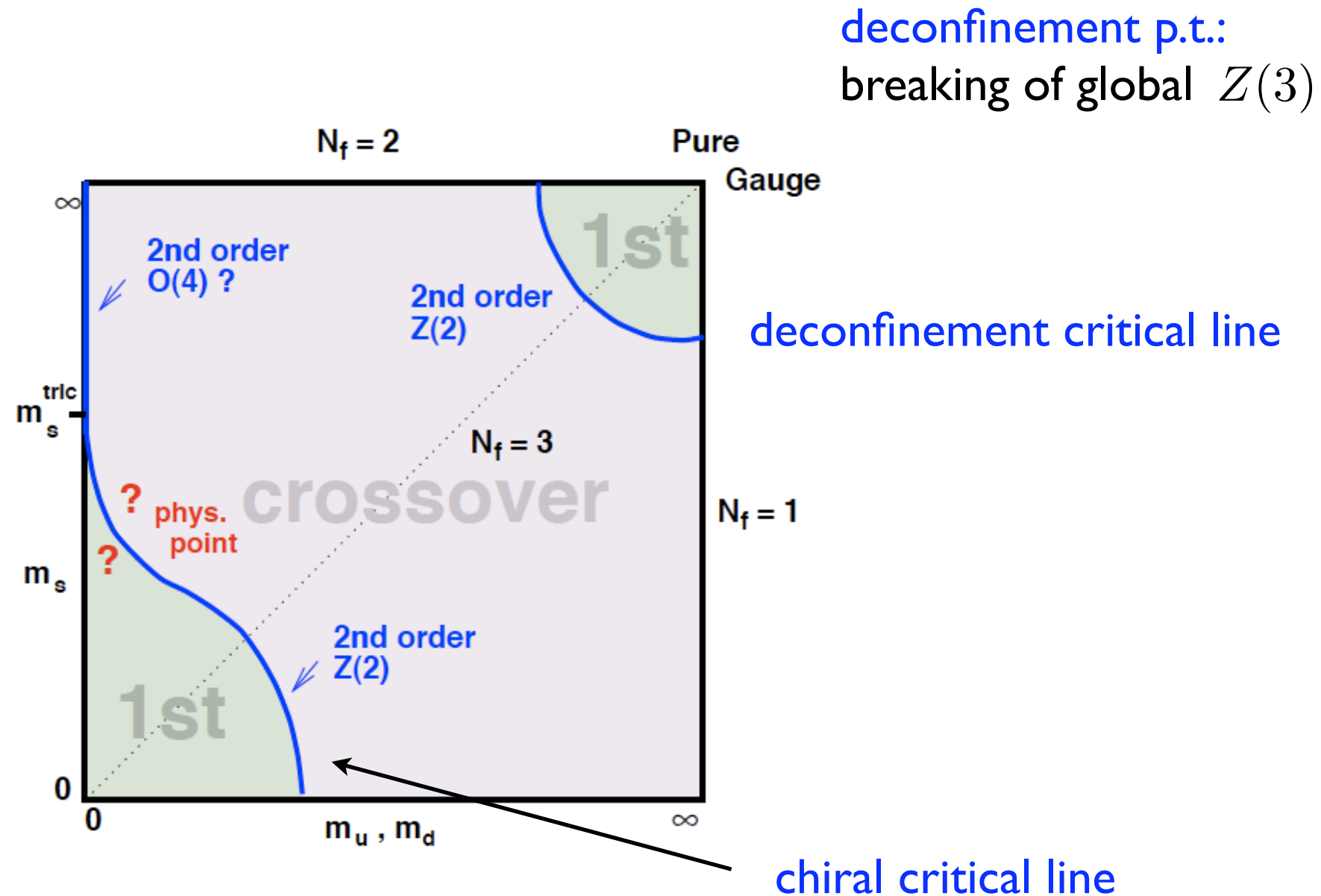
Lecture IV:

Owe Philipsen



- The thermal phase transition at zero density
- Lattice QCD at finite temperature and density
- Towards the QCD phase diagram

The order of the thermal phase transition, $\mu = 0$



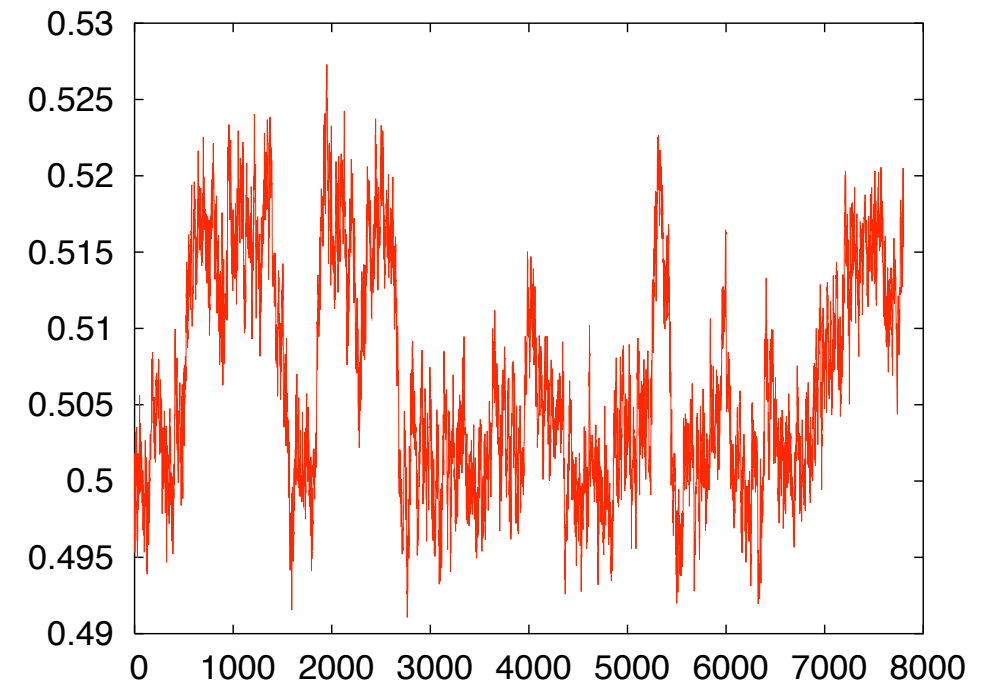
chiral p.t.
restoration of global

$$SU(2)_L \times SU(2)_R \times U(1)_A$$

↑
anomalous

Very difficult!

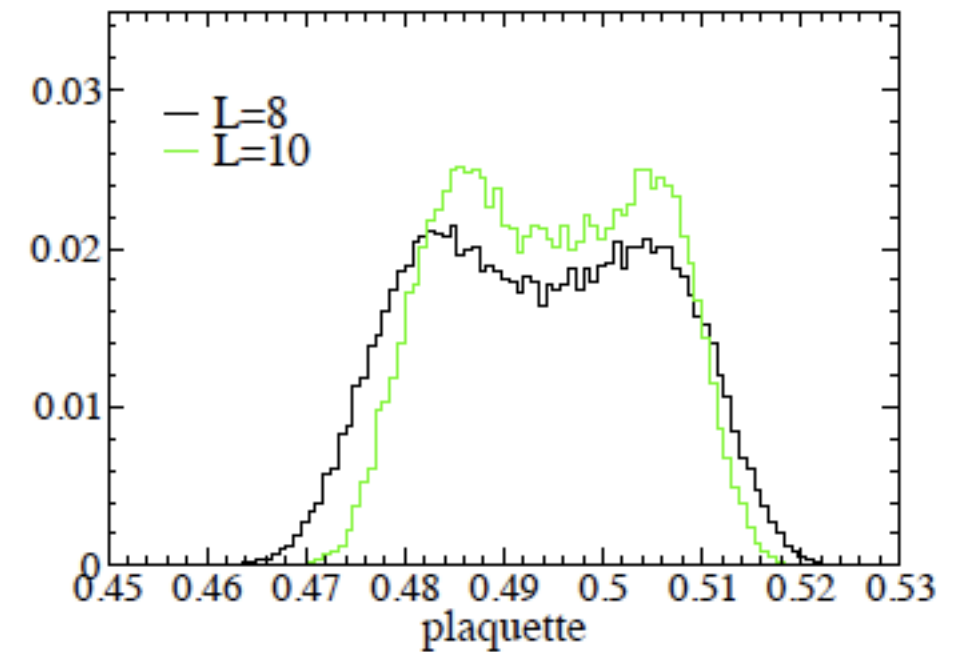
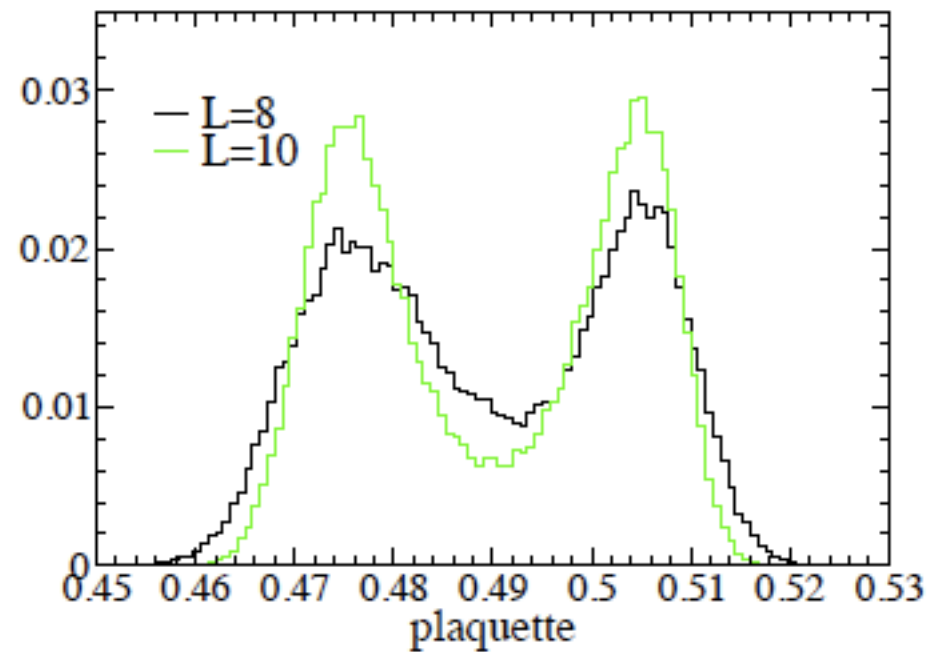
Monte Carlo history,
plaquette near phase boundary



Distribution:

first-order

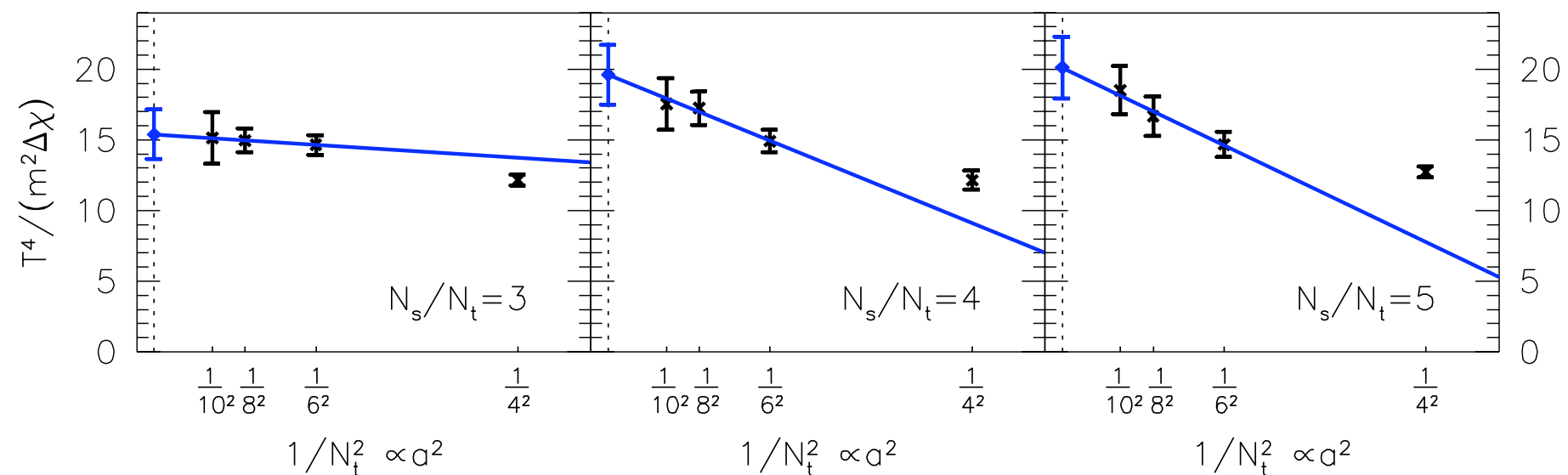
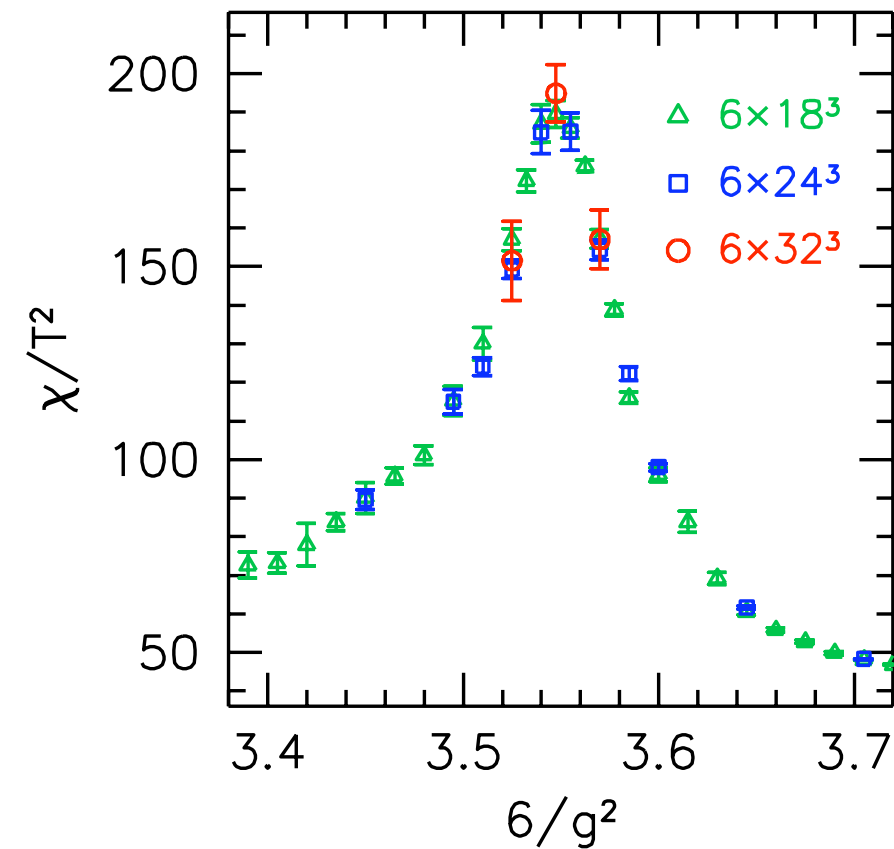
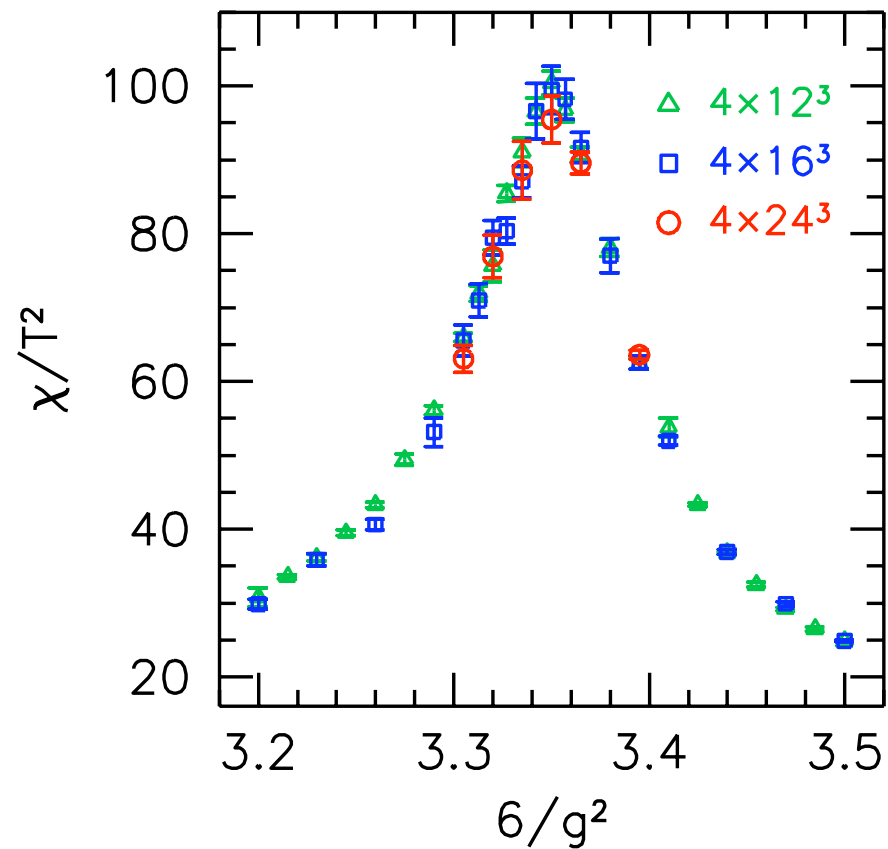
crossover



The nature of the transition for phys. masses

Aoki et al. 06

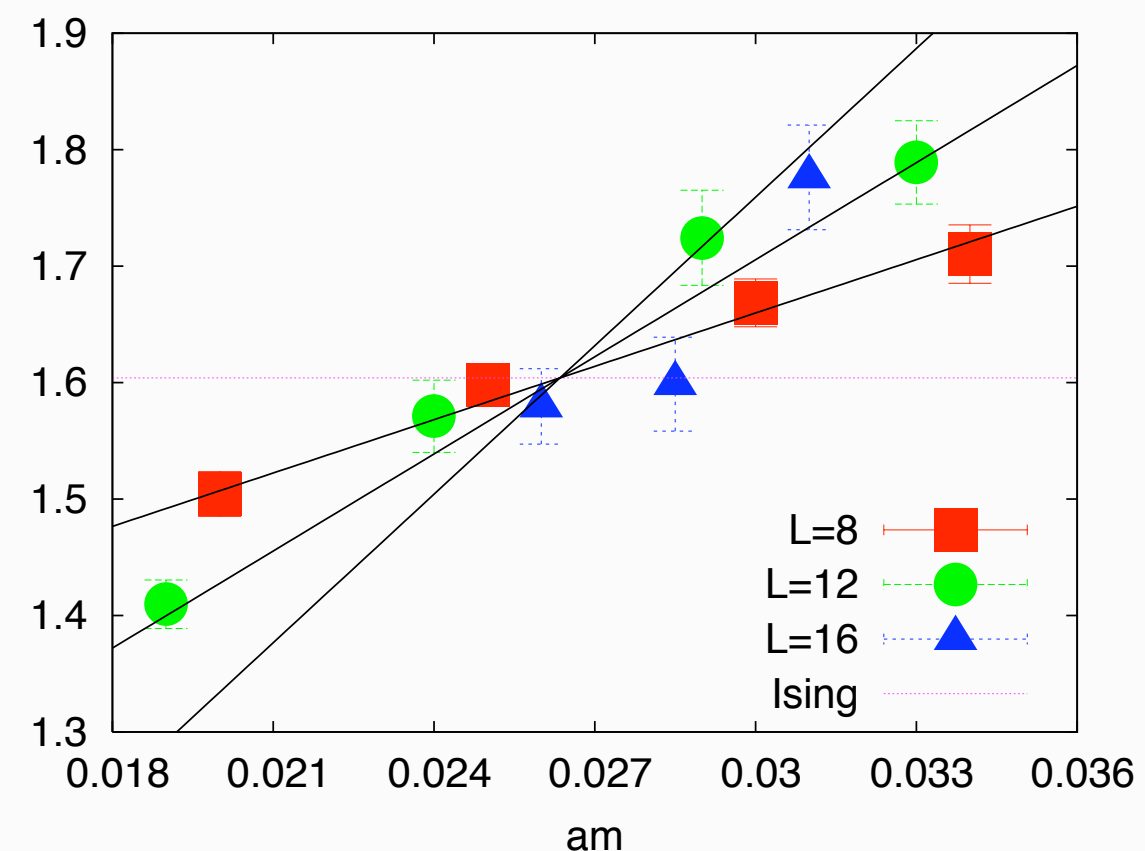
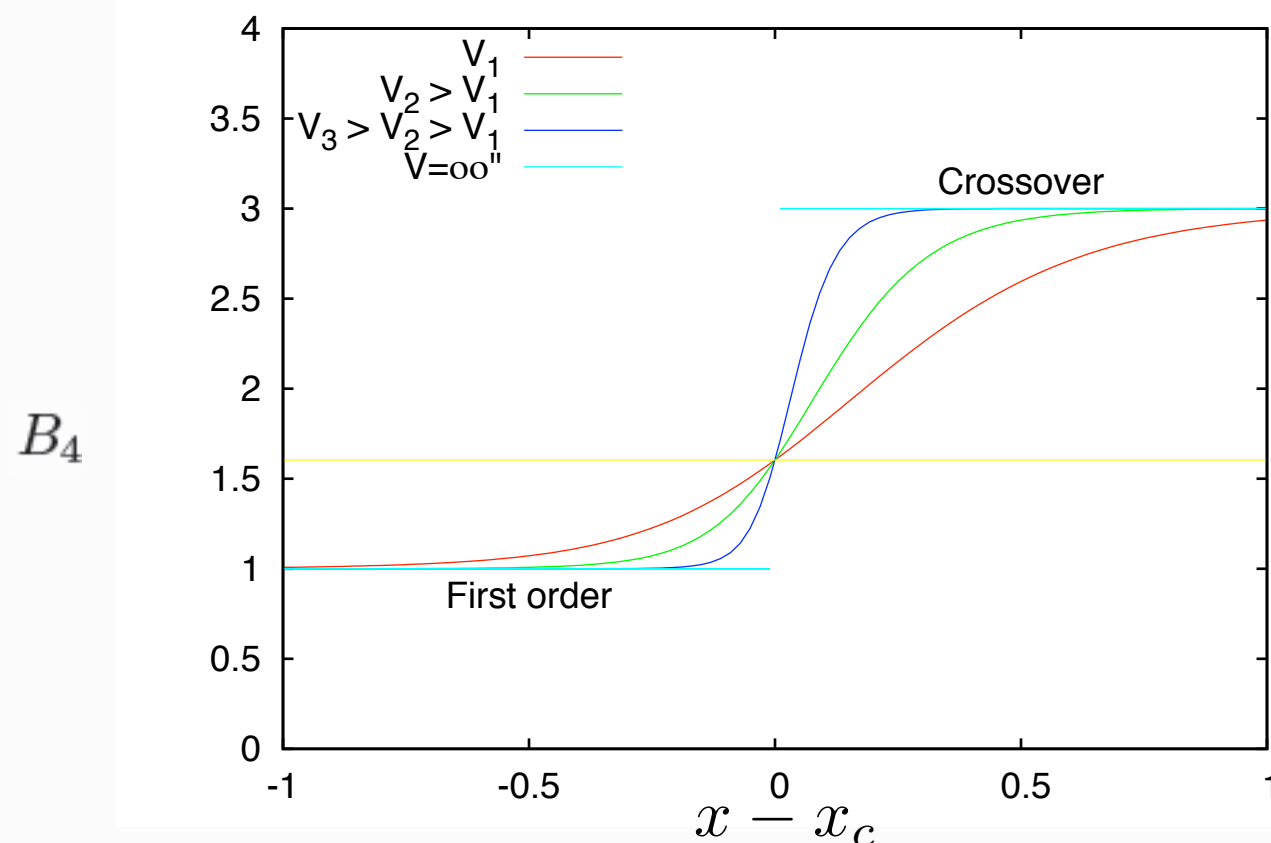
...in the staggered approximation...in the continuum...**is a crossover!**



How to identify the order of the phase transition

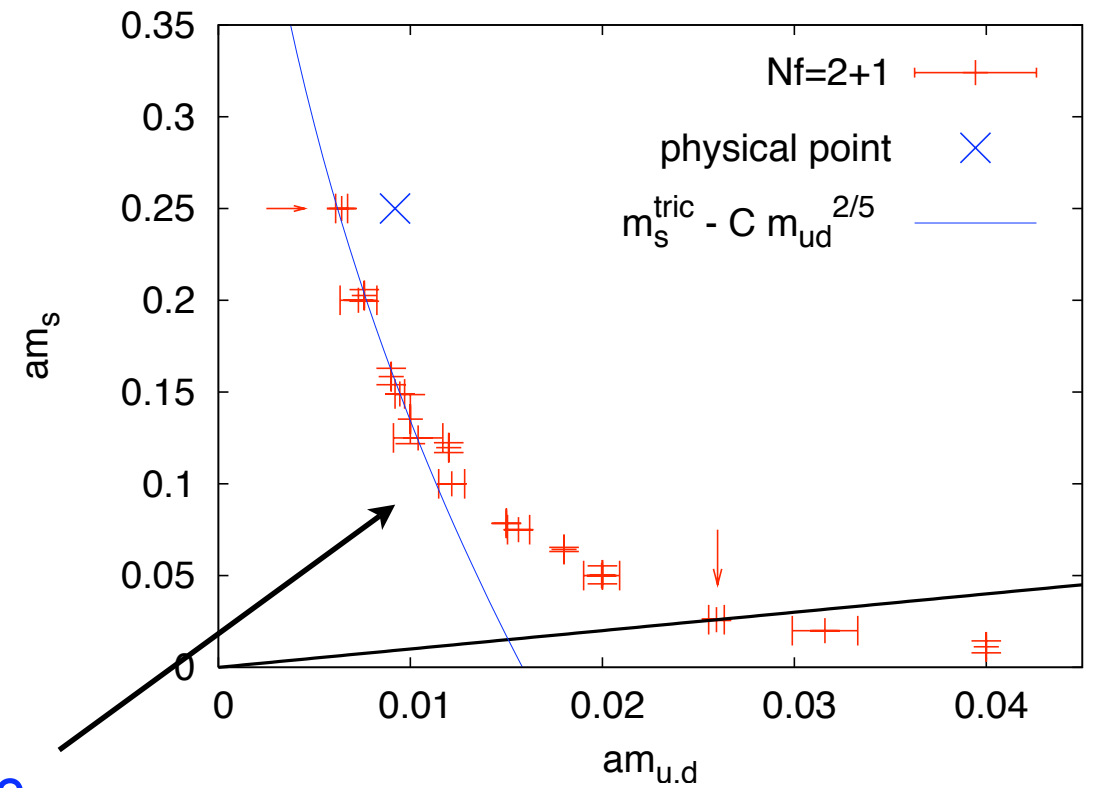
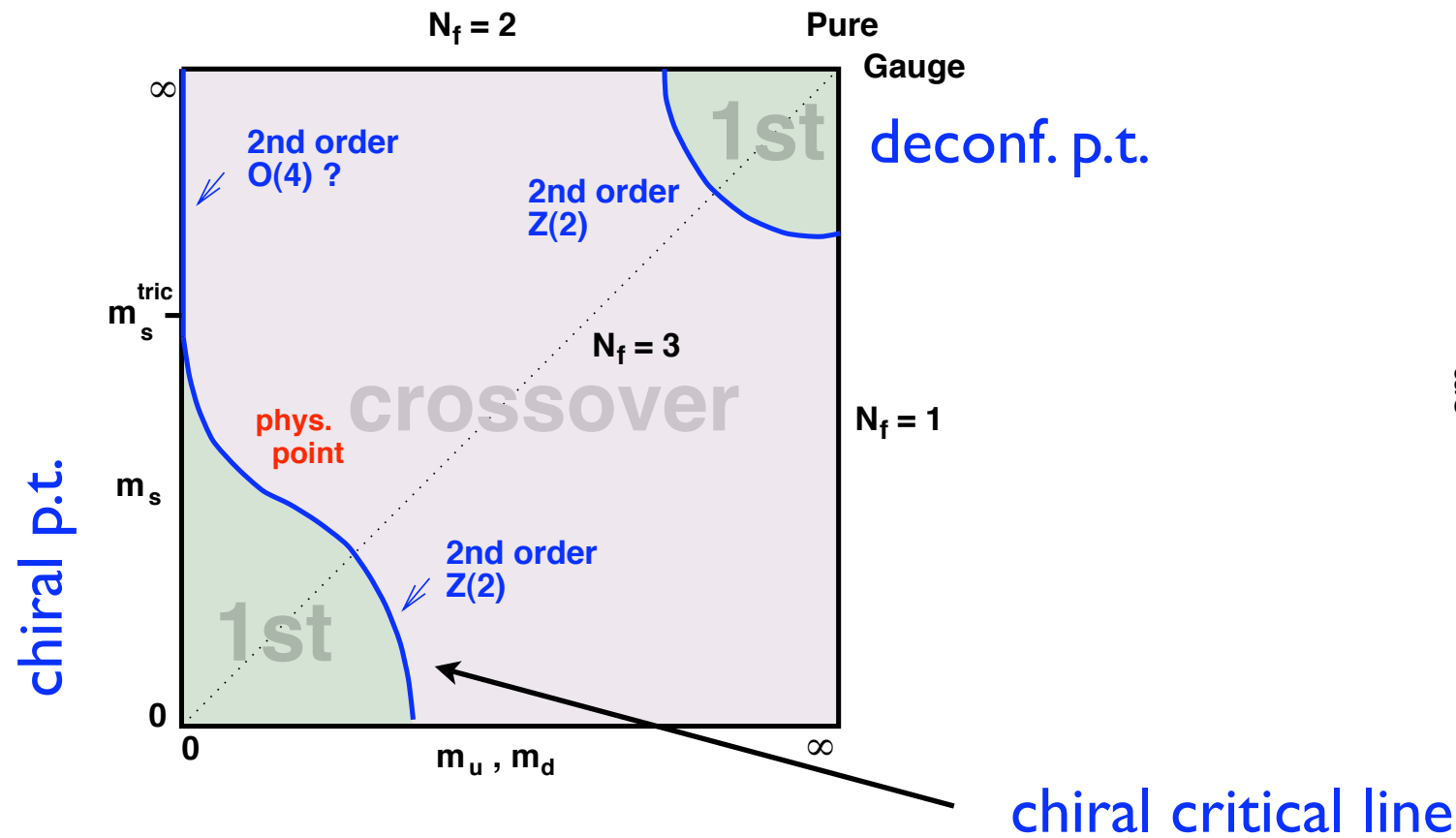
$$B_4(\bar{\psi}\psi) \equiv \frac{\langle (\delta\bar{\psi}\psi)^4 \rangle}{\langle (\delta\bar{\psi}\psi)^2 \rangle^2} \xrightarrow{V \rightarrow \infty} \begin{cases} 1.604 & \text{3d Ising} \\ 1 & \text{first - order} \\ 3 & \text{crossover} \end{cases}$$

$\mu = 0$: $B_4(m, L) = 1.604 + bL^{1/\nu}(m - m_0^c), \quad \nu = 0.63$



parameter along phase boundary, $T = T_c(x)$

Order of p.t., arbitrary quark masses $\mu = 0$



● physical point: crossover in the continuum

Aoki et al 06

● chiral critical line on $N_t = 4, a \sim 0.3$ fm

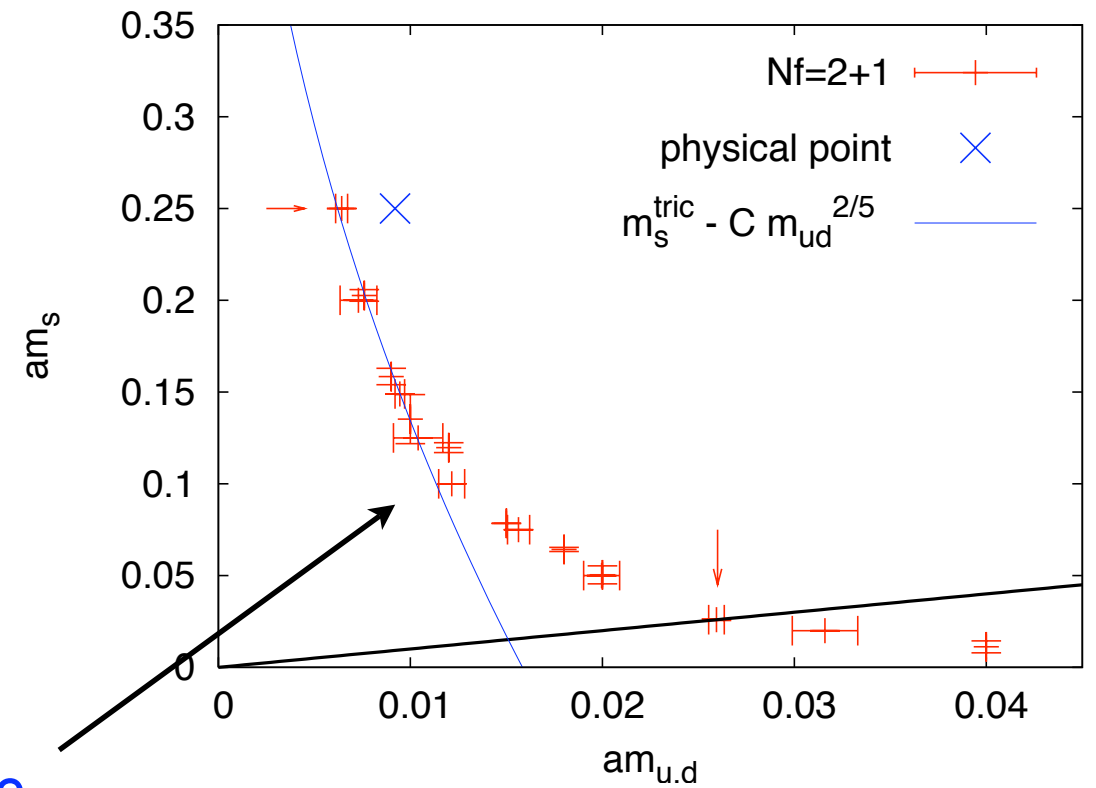
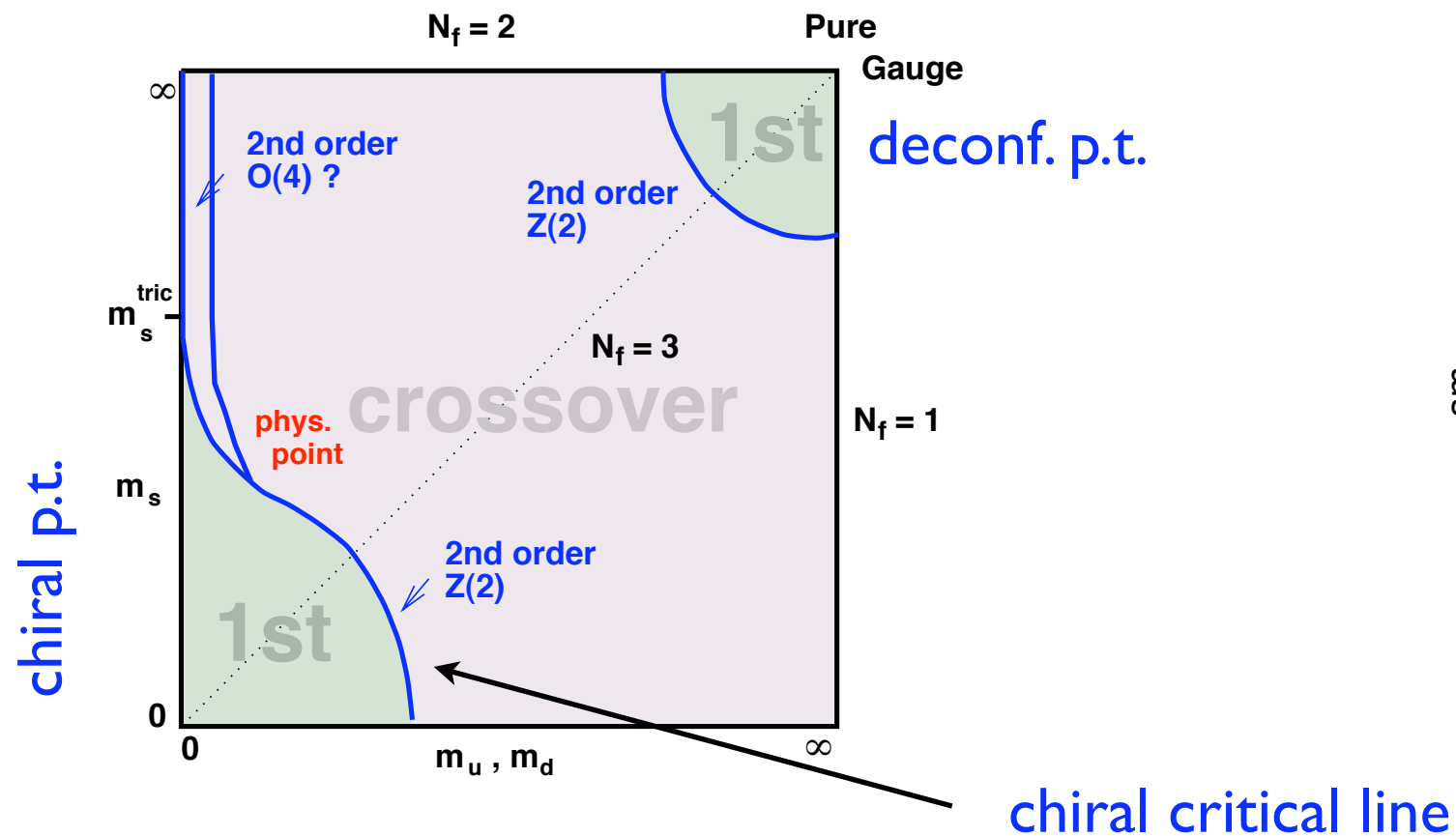
de Forcrand, O.P. 07

● consistent with tri-critical point at $m_{u,d} = 0, m_s^{\text{tric}} \sim 2.8T$

● But: $N_f = 2$ chiral $O(4)$ vs. 1st still open
 $U_A(1)$ anomaly!

Di Giacomo et al 05, Kogut, Sinclair 07
Chandrasekharan, Mehta 07
Cossu et al. 12, Aoki et al. 12

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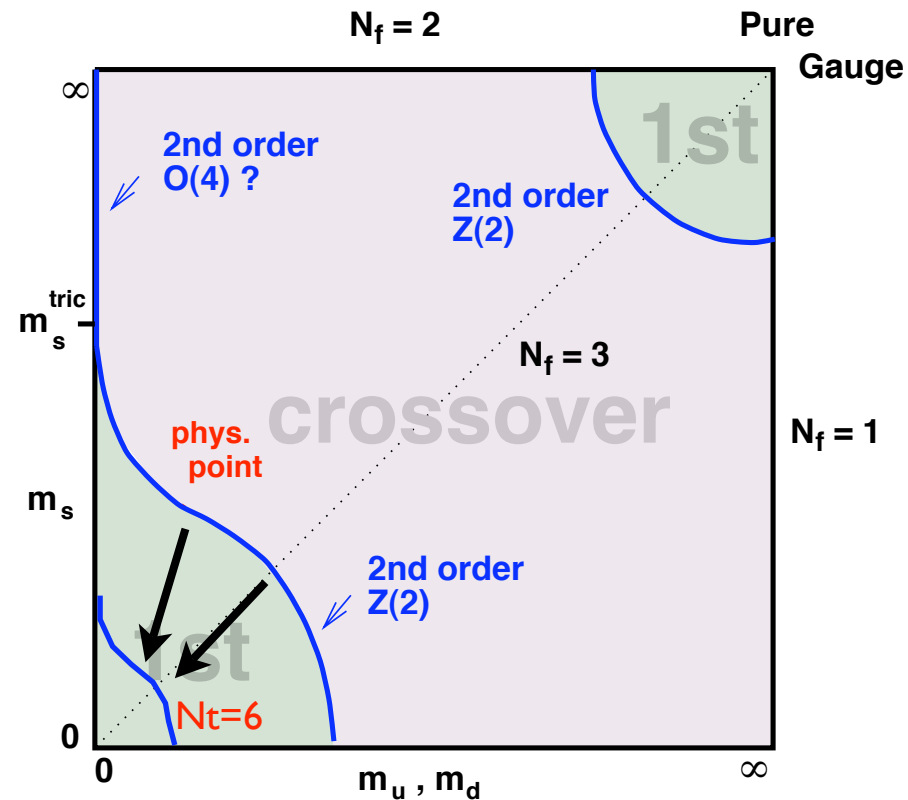
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Di Giacomo et al 05, Kogut, Sinclair 07
Chandrasekharan, Mehta 07
Cossu et al. 12, Aoki et al. 12

Towards the continuum: $N_t = 6, a \sim 0.2 \text{ fm}$



$$\frac{m_\pi^c(N_t = 4)}{m_\pi^c(N_t = 6)} \approx 1.77 \quad N_f = 3$$

de Forcrand, Kim, O.P. 07
Endrödi et al 07

First order region shrinks drastically, continuum limit not yet known...

N.B.: for fixed masses in physical units the order of the p.t. depends on the cut-off!

Lattice QCD at finite baryon density

$$Z = \hat{\text{Tr}} e^{-(H - \mu Q)}, \quad Q = \int d^3x \bar{\psi}(x) \gamma_0 \psi(x) = \int d^3x \psi^\dagger(x) \psi(x)$$

Quark number and chemical potential: $Q = B/3, \mu = \mu_B/3$

Necessary for real world applications: heavy ion collisions, nuclear matter, compact stars,...

Behaviour under charge conjugation: $C = \gamma_0 \gamma_2 \quad \gamma_\mu = \gamma_\mu^\dagger, \{\gamma_5, \gamma_\mu\} = 0$

$$A_\mu^C = -A_\mu^*, \quad \psi^C = \gamma_0 \gamma_2 \bar{\psi}^T, \quad \bar{\psi}^C \gamma_0 \psi^C = -\bar{\psi} \gamma_0 \psi \quad \text{sign flip in } Q!$$



$\mu > 0$: net baryon number

$\mu < 0$: net anti-baryon number

Exact symmetry of the continuum grand canonical partition function:

$$\begin{aligned} Z(\mu) &= \int DA^C D\bar{\psi}^C D\psi^C \exp - \left[S_g^C + S_f^C(\mu = 0) - \mu \int_0^{1/T} dx_0 Q^C \right] \\ &= \int DA D\bar{\psi} D\psi \exp - \left[S_g + S_f(\mu = 0) + \mu \int_0^{1/T} dx_0 Q \right] = Z(-\mu) \end{aligned}$$

Lattice implementation, naive: $S_f[M(\mu)] = S_f[M(0)] + a\mu \sum_x \psi(x) \gamma_0 \psi(x)$

Introduces divergence, which is absent at zero density: **failure!** $\epsilon = \frac{1}{V} \frac{\partial}{\partial(\frac{1}{T})} \ln Z \xrightarrow{a \rightarrow 0} \infty$

Another symmetry broken by the discretisation!

Continuum fermion number like current coupling to (imaginary) gauge field:

$$j^0 = \bar{\psi} \gamma^0 \psi \quad \mu Q = -ig \int d^3x A_0 j_0 \quad \text{with} \quad A_0 = i \frac{\mu}{g}$$

Effectively part of covariant derivative, “gauged” $U(1)$, protects against renormalisation

Lattice implementation: lattice covariant derivative with external gauge field

$$U_{0,\text{ext}} = e^{iagA_0} = e^{-a\mu}$$

Wilson fermions:

$$S_f^W = a^3 \sum_x \left(\bar{\psi}(x)\psi(x) - \kappa \left[e^{a\mu} \bar{\psi}(x)(r - \gamma_0)U_0(x)\psi(x - \hat{0}) + e^{-a\mu} \bar{\psi}(x + \hat{0})(r + \gamma_0)U_0^\dagger(x)\psi(x) \right] \right. \\ \left. - \kappa \sum_{j=1}^3 \left[\bar{\psi}(x)(r - \gamma_j)U_j(x)\psi(x + \hat{j}) + \bar{\psi}(x + \hat{j})(r + \gamma_j)U_j^\dagger(x)\psi(x) \right] \right)$$

(Discretisation not unique, only continuum limit)

Now use $\det(\not{D}(U^\dagger) + m + \gamma_0\mu) = \det(\not{D}(U) + m - \gamma_0\mu) \quad S_g[U^\dagger] = S_g[U]$



$$Z(\mu) = Z(-\mu)$$

The sign problem

Dirac operators satisfy
(continuum, Wilson, staggered,...)

$$(\not{D} + m)^\dagger = \gamma_5 (\not{D} + m) \gamma_5$$

With complex chemical potential:

$$\gamma_5 (\not{D} + m - \gamma_0 \mu) \gamma_5 = (-\not{D} + m + \gamma_0 \mu) = (\not{D} + m + \gamma_0 \mu^*)^\dagger$$



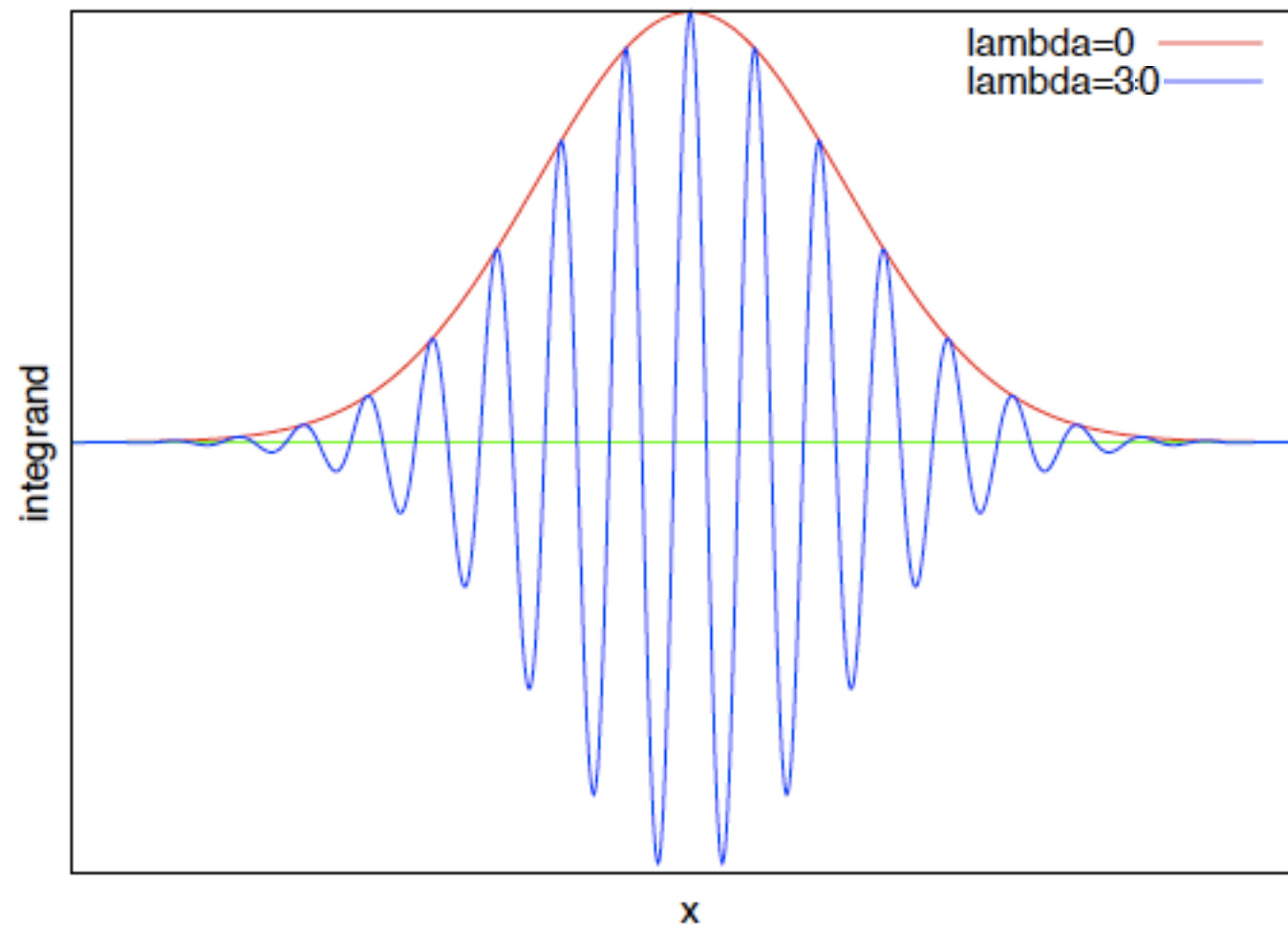
$$\det(\not{D} + m - \gamma_0 \mu) = \det^*(\not{D} + m + \gamma_0 \mu^*)$$

“Sign problem” of QCD

- Complex measure cannot be used for MC importance sampling
- After integration over gauge fields the partition function is real!
- Generic for systems with anti-particles, necessary for physics!

1 dim. illustration

- Example: $Z(\lambda) = \int dx \exp(-x^2 + i\lambda x)$



- $Z(\lambda)/Z(0) = \exp(-\lambda^2/4)$: exponential cancellations

Example: Polyakov loop

$$\langle \dots \rangle_g = \int DU \dots \exp -S_g[U]$$

$$\langle \text{Tr} L \rangle = e^{-\frac{F_Q}{T}} = \langle \text{Re Tr} L \text{ Re det } M - \text{Im Tr} L \text{ Im det } M \rangle_g$$

$$\langle (\text{Tr} L)^* \rangle = e^{-\frac{F_{\bar{Q}}}{T}} = \langle \text{Re Tr} L \text{ Re det } M + \text{Im Tr} L \text{ Im det } M \rangle_g$$

Static quarks and anti-quarks must have different free energy at finite density!

Sign problem expresses
property under C-conjugation!

$$\det(\not{D} + m - \gamma_0 \mu) \xrightarrow{C} \det(\not{D} + m + \gamma_0 \mu)$$

Fixes:

- Cluster algorithms find configs. with conjugate determinant works for particular Hamiltonians, but not QCD
- Simulation with Langevin algorithms (no importance sampling)
Only proven to work for real actions, but work for some complex models

Special cases without sign problem

Imaginary chemical potential:

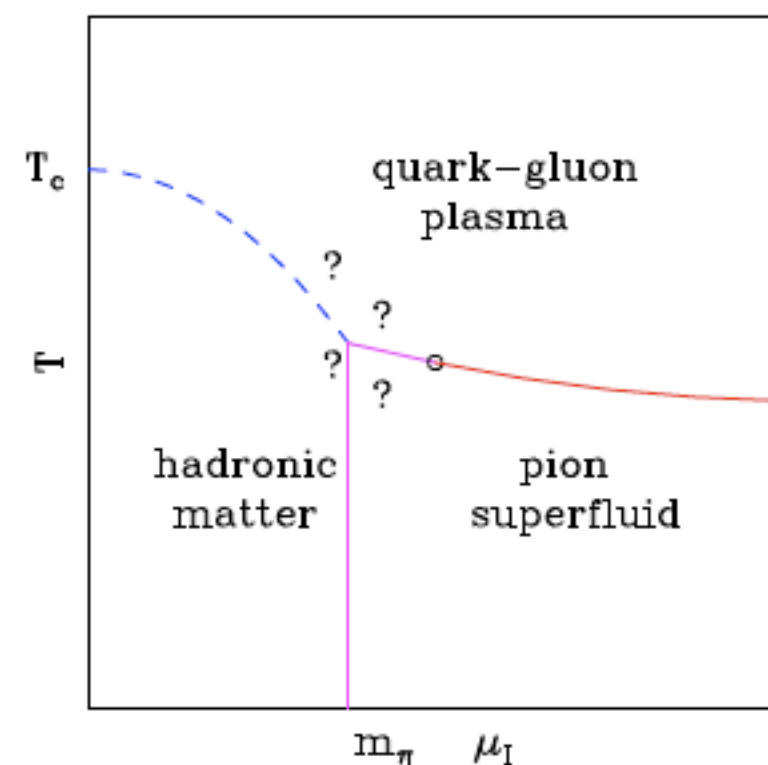
$$\det(\not{D} + m - \gamma_0 \mu) = \det^*(\not{D} + m + \gamma_0 \mu^*) \quad \text{real for} \quad \mu = i\mu_i, \mu_i \in \mathbb{R}$$

Two flavours, finite isospin chemical potential:

$$\mu_u = -\mu_d \equiv \mu_I$$

$$\begin{aligned} & \det(\not{D} + m - \gamma_0 \mu_I) \det(\not{D} + m + \gamma_0 \mu_I) \\ &= |\det(\not{D} + m - \gamma_0 \mu_I)|^2 \geq 0 \end{aligned}$$

$N_f=2$ QCD at finite isospin density



Two colours, SU(2) QCD:

$$S[\not{D} + m - \gamma_0 \mu] S^{-1} = [\not{D} + m - \gamma_0 \mu^*]^*$$

$$S = C \gamma_5 \sigma^2 \quad S T^a S^{-1} = -T^{a*}$$

real reps.

Approximate methods to evade the sign problem: Reweighting

Based on exact relation:

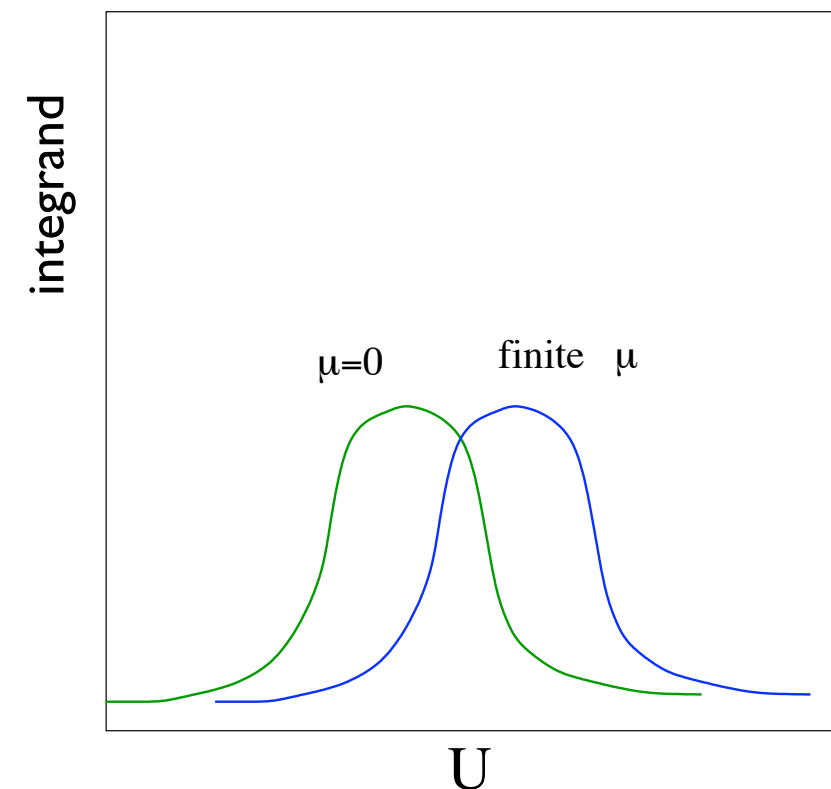
$$\begin{aligned} Z(\mu) &= \int DU \det M(\mu) e^{-S_g[U]} = \int DU \det M(0) \frac{\det M(\mu)}{\det M(0)} e^{-S_g[U]} \\ &= Z(0) \left\langle \frac{\det M(\mu)}{\det M(0)} \right\rangle_{\mu=0} . \end{aligned}$$

I. Numerically difficult, signal exponentially suppressed with volume

$$\frac{Z(\mu)}{Z(0)} = \exp -\frac{F(\mu) - F(0)}{T} = \exp -\frac{V}{T} (f(\mu) - f(0))$$

II. Overlap problem, because of importance sampling

With increasing difference the most frequent configs. are increasingly unimportant



Finite density by Taylor expansion

Taylor expansion of the pressure around zero density:

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu}{T}\right)^{2n} \equiv \Omega(T, \mu)$$

$$c_0(T) = \frac{p}{T^4}(T, \mu = 0), \quad c_{2n}(T) = \frac{1}{(2n)!} \left. \frac{\partial^{2n} \Omega}{\partial (\frac{\mu}{T})^{2n}} \right|_{\mu=0}$$

The coefficients can be computed at zero density!

Other physical quantities follow:

$$\frac{n}{T} = \frac{\partial \Omega}{\partial (\frac{\mu}{T})} = 2c_2 \frac{\mu}{T} + 4c_4 \left(\frac{\mu}{T}\right)^3 + \dots,$$

$$\frac{\chi_q}{T^2} = \frac{\partial^2 \Omega}{\partial (\frac{\mu}{T})^2} = 2c_2 + 12c_4 \left(\frac{\mu}{T}\right)^2 + 30c_6 \left(\frac{\mu}{T}\right)^4 + \dots$$

No sign problem, but need small μ/T

Higher coeffs. increasingly difficult:

$$\frac{\partial \langle O \rangle}{\partial \mu} = \left\langle \frac{\partial O}{\partial \mu} \right\rangle + N_f \left(\left\langle O \frac{\partial \ln \det M}{\partial \mu} \right\rangle - \langle O \rangle \left\langle \frac{\partial \ln \det M}{\partial \mu} \right\rangle \right)$$

QCD at imaginary chemical potential

No sign problem; general idea:

Observables have definite symmetry,
even or odd in chemical potential

$$\langle O \rangle(\mu_i) = \sum_{k=1}^N c_k \left(\frac{\mu_i}{T} \right)^{2k}$$

- Simulate left side without further systematic error
- Check if fit to low order polynomial is possible $\mu/T < 1$
- Analytic continuation trivial (in the absence of singularities) $\mu_i \rightarrow -i\mu_i$

General considerations:

Partition function is periodic $Z = \hat{\text{Tr}} e^{-\frac{(H - i\mu_i Q)}{T}}$

Is this a healthy theory?

Yes! Recall

$$\mu Q = -ig \int d^3x A_0 j_0 \quad \text{with} \quad A_0 = i \frac{\mu}{g}$$

Equivalent to theory in real external field!

Periodicity non-trivial:

Chemical potential can be absorbed by boundary conditions

$$Z^{(1)}(i\mu_i) = \int DU \det M(0) e^{-S_g}, \quad \text{b.c.: } \psi(\tau + N_\tau, \mathbf{x}) = -e^{i\frac{\mu_i}{T}} \psi(\tau, \mathbf{x})$$

Consider the topological gauge trafo $g'(\tau + N_\tau, x) = e^{-i\frac{2\pi n}{N}} g'(\tau, \mathbf{x})$

Measure and action are invariant, hence

$$Z^{(2)}(i\mu_i) = \int DU \det M(0) e^{-S_g}, \quad \text{b.c.: } \psi(\tau + N_\tau, \mathbf{x}) = -e^{-i\frac{2\pi n}{N}} e^{i\frac{\mu_i}{T}} \psi(\tau, \mathbf{x})$$

$$Z^{(2)}\left(i\frac{\mu_i}{T} + i\frac{2\pi n}{N}\right) = Z^{(1)}\left(i\frac{\mu_i}{T}\right)$$

Both partition fcns. related by gauge trafo, **identical!**

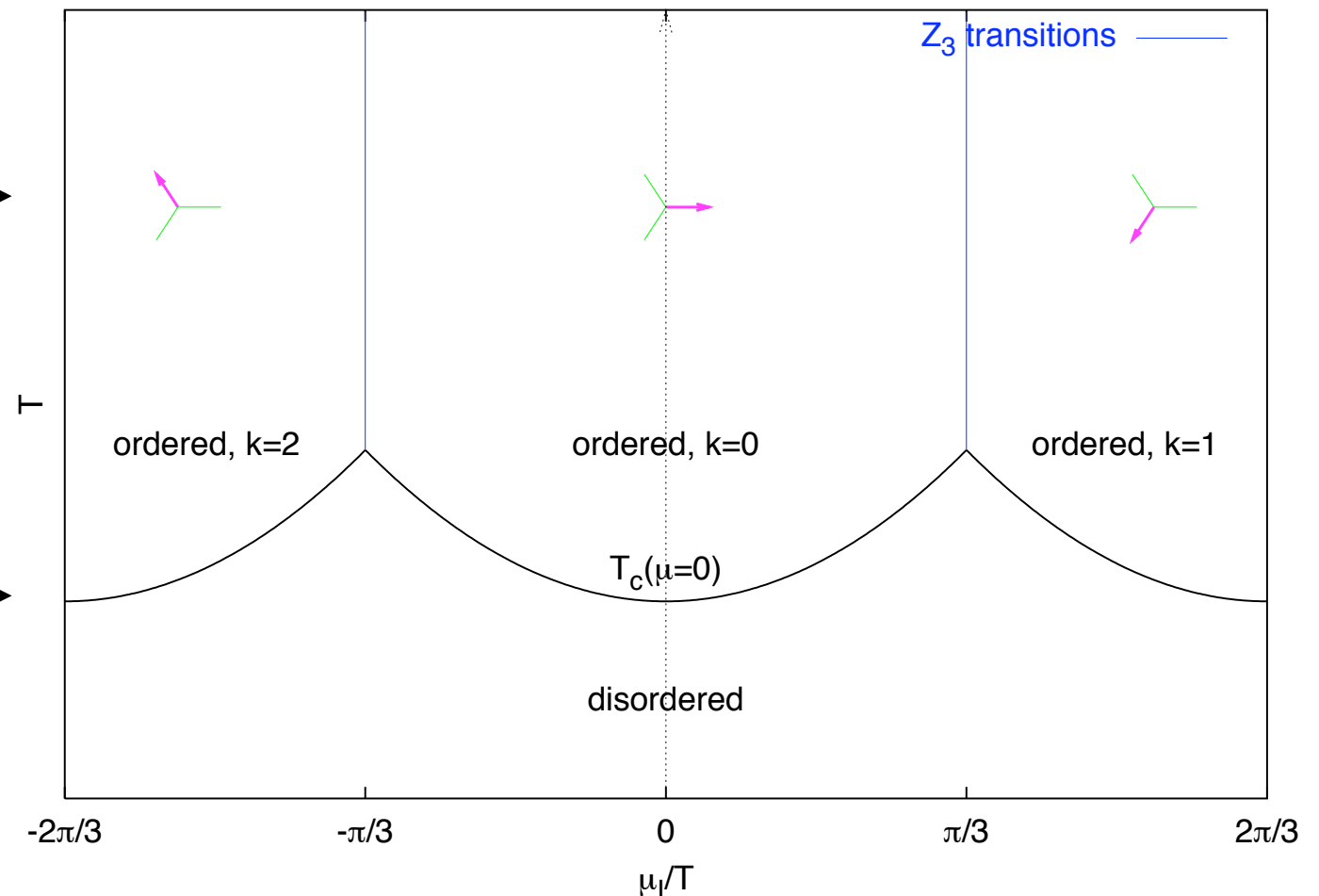
Roberge-Weiss symmetry:

$$Z\left(i\frac{\mu_i}{T} + i\frac{2\pi n}{N}\right) = Z\left(i\frac{\mu_i}{T}\right)$$

The phase diagram at imaginary chemical potential

Phase of Polyakov loop $\longrightarrow$

Analytic continuation
of chiral/deconfinement
transition, depends on
 N_f , quark masses $\longrightarrow$



Roberge-Weiss: $Z(3)$ transitions are first order for large T (perturbation theory)
crossover for small T (strong coupling limit)

analytic continuation within:

$$|\mu|/T \leq \pi/3 \Rightarrow \mu_B \lesssim 550 \text{ MeV}$$

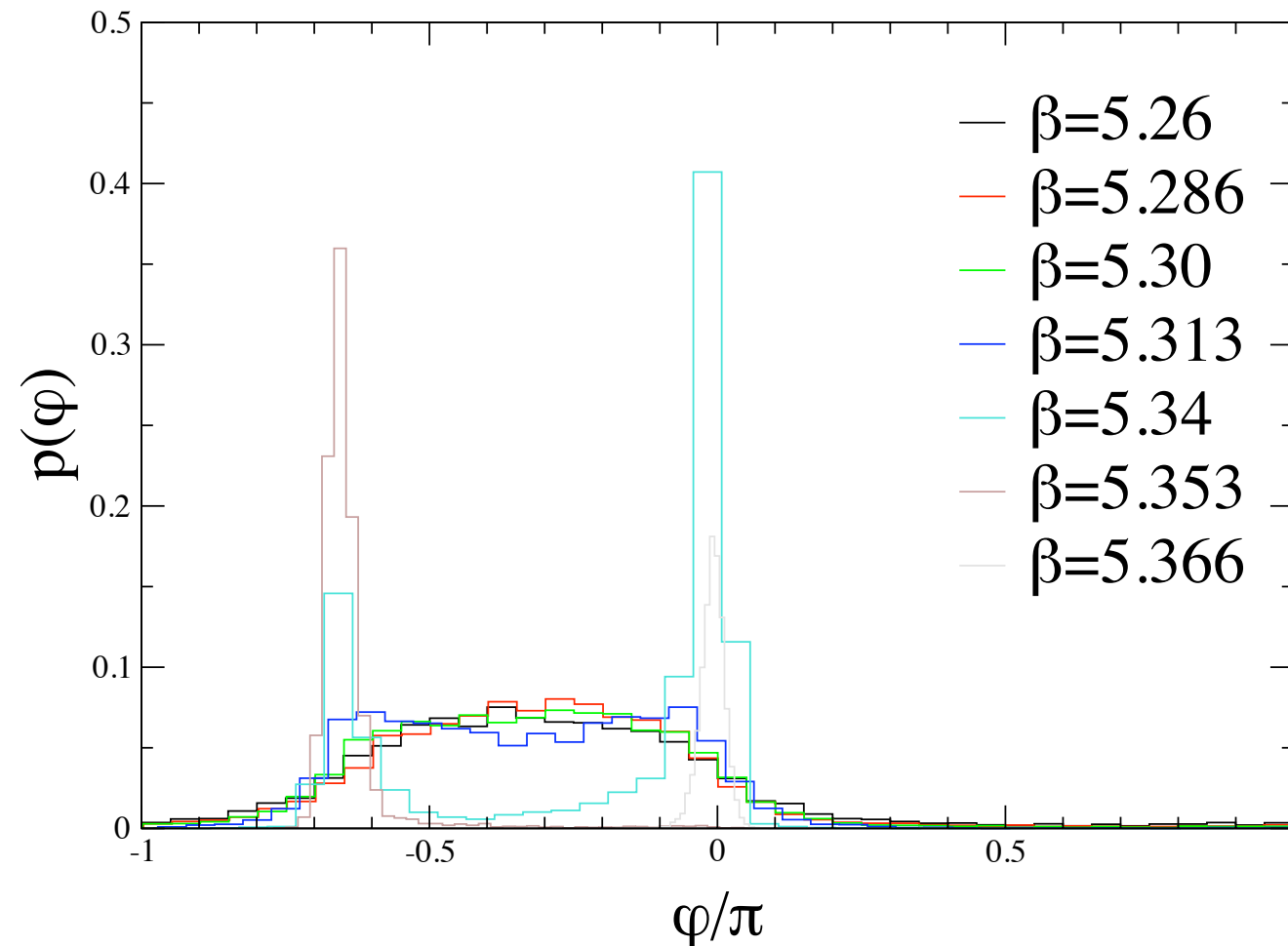
Limited by singularity (phase transition)
closest to $\mu = 0$

The $Z(3)$ transition numerically

Nf=2: de Forcrand, O.P. 02

Nf=4: D'Elia, Lombardo 03

Sectors characterised by phase of Polyakov loop: $\langle L(x) \rangle = |\langle L(x) \rangle| e^{i\varphi}$



Low T: crossover

High T: first order p.t.

The canonical partition function

Is related to the grand canonical partition function at imaginary chemical potential by Fourier transform:

$$\begin{aligned}
 \tilde{Z}(\bar{Q}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\left(\frac{\mu_i}{T}\right) e^{-i\frac{\mu_i}{T}\bar{Q}} Z\left(i\frac{\mu_i}{T}\right) \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\left(\frac{\mu_i}{T}\right) e^{-i\frac{\mu_i}{T}\bar{Q}} Z\left(i\frac{\mu_i}{T} + i\frac{2\pi}{N}\right) \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\left(\frac{\mu_i}{T}\right) e^{-i\left(\frac{\mu_i}{T} - \frac{2\pi}{N}\right)\bar{Q}} Z\left(i\frac{\mu_i}{T}\right) \quad \rightarrow \quad \frac{\bar{Q}}{N} = n, \quad n = 0, 1, \dots
 \end{aligned}$$

$$\delta(\bar{Q} - Q) = \frac{1}{2\pi} \int d\left(\frac{\mu_i}{T}\right) \exp\left(i\frac{\mu_i(\bar{Q} - Q)}{T}\right)$$

$$\rightarrow \tilde{Z}(\bar{Q}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\left(\frac{\mu_i}{T}\right) e^{-i\frac{\mu_i}{T}\bar{Q}} \text{Tr}\left(e^{-\frac{(H - i\mu_i Q)}{T}}\right) = \text{Tr}\left(e^{-\frac{H}{T}} \delta(\bar{Q} - Q)\right) = Z_C(\bar{Q})$$

Canonical partition function

Equivalence in thermodynamic limit

Fugacity expansion:

$$\begin{aligned} Z(V, T, \mu) &= \sum_{B=1}^{\infty} e^{\frac{\mu_B B}{T}} Z_C(V, T, B) \\ Z(T, \mu) &\stackrel{V \rightarrow \infty}{\equiv} \int_{-\infty}^{\infty} d\rho e^{V N \rho \frac{\mu}{T}} Z_C(T, \rho) & B = \rho V \\ &= \lim_{V \rightarrow \infty} \int_{-\infty}^{\infty} d\rho e^{-\frac{V}{T} (f(\rho) - \mu N \rho)} \end{aligned}$$

Conversion back to chemical potential: $\mu(\rho) = \frac{1}{N} \frac{\partial f(\rho)}{\partial \rho}$ (saddle point)

- Simulate grand canonical partition function at imaginary chemical potential
- Numerical Fourier transform: sign problem re-enters, restricted to small volumes
- But no truncation of Taylor series or reweighting

Towards the QCD phase diagram

Analyticity of the (pseudo-)critical line

Recall definition by peak of susceptibilities:

$$\chi_{max} = \chi(\beta_c, m_f, \mu)$$

Implicit definition of pseudo-critical line

$$\beta_c(m_f, \mu)$$

Implicit function theorem:

For analytic susceptibility, also the implicitly defined pseudo-critical coupling is analytic (always true on finite V!)

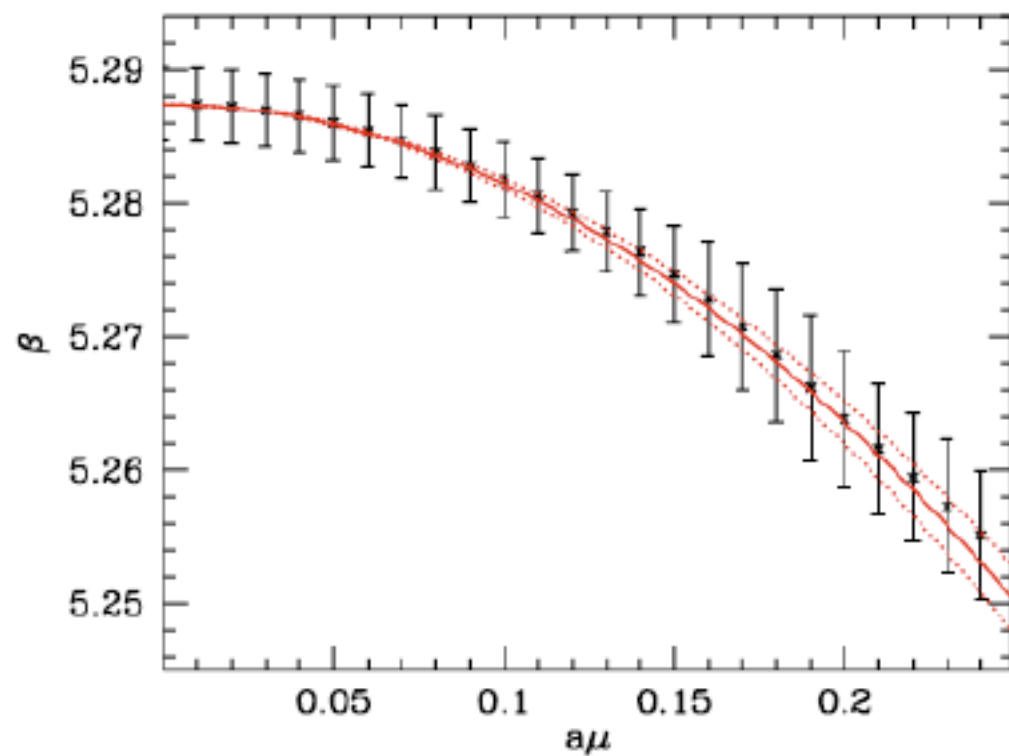
$$\beta_c(m_f, \frac{\mu}{T}) = \sum_n b_{2n}(m_f) \left(\frac{\mu}{T}\right)^{2n}$$


$$\frac{T_c(m_f, \mu)}{T_c(m_f, 0)} = 1 + t_2(m_f) \left(\frac{\mu}{T}\right)^2 + t_4(m_f) \left(\frac{\mu}{T}\right)^4 + \dots$$

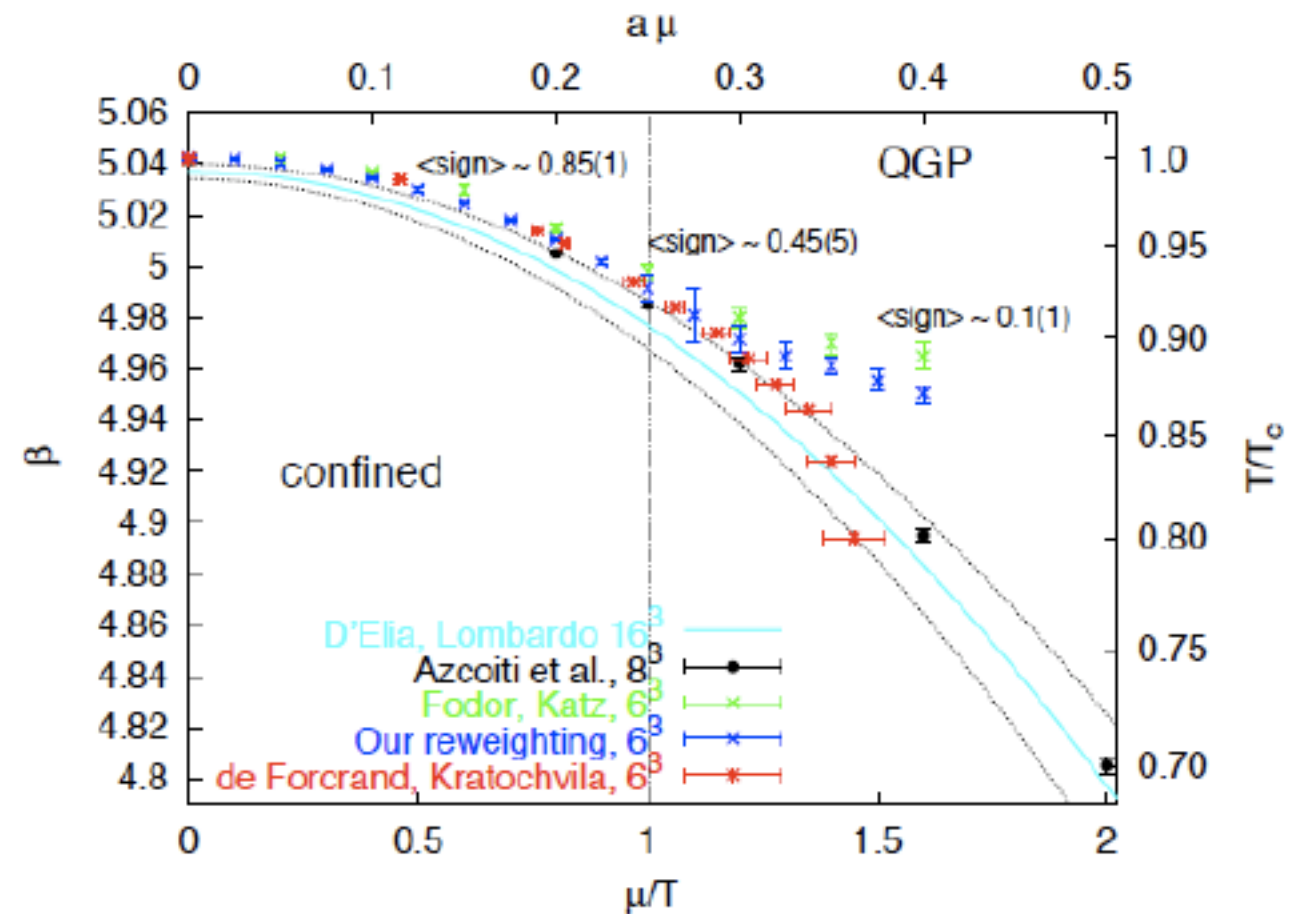
- Accessible to all methods discussed for sufficiently small chemical potential
- Crosscheck, in particular between Taylor coefficients and imaginary chem. pot.

Test of methods: comparing $T_c(\mu)$

Reweighting vs. imag. μ (FK, FP)



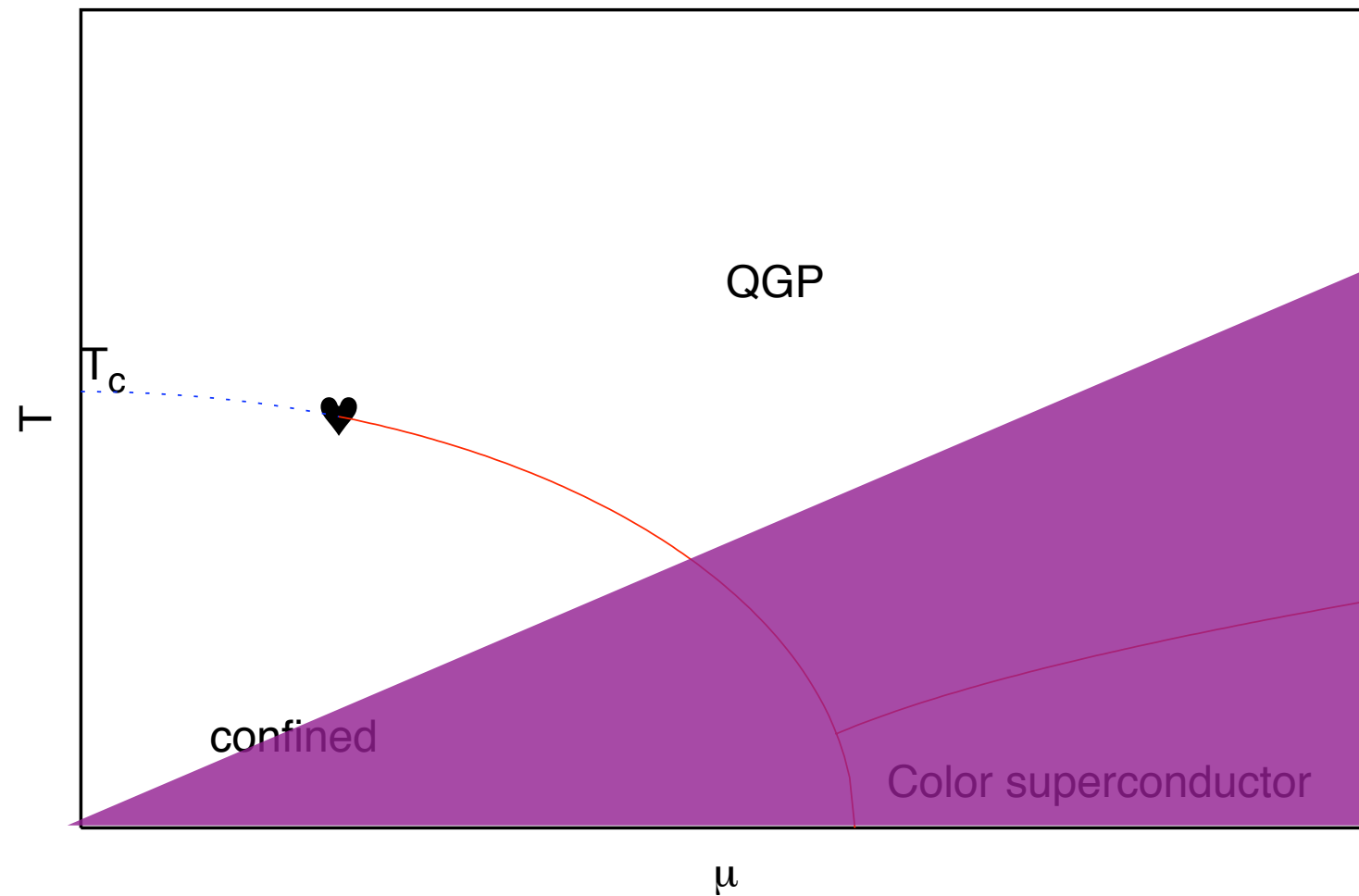
Rew., imag. μ , canonical ensemble ...



All agree on $T_0(m, \mu)$!!!

$(\mu/T \lesssim 1)$

The calculable region of the phase diagram



- **need** $\mu/T \lesssim 1$ ($\mu = \mu_B/3$)
- Upper region: equation of state, screening masses, quark number susceptibilities etc. under control

Pseudo-critical temperature

- Curvature of crit. line from Taylor expansion
2+1 flavours, Nt=4, 8 improved staggered
- Extrapolation to chiral limit **assuming**
O(4), O(2) scaling of magn. EoS
- $\kappa(\bar{\psi}\psi) = 0.059(2)(4)$

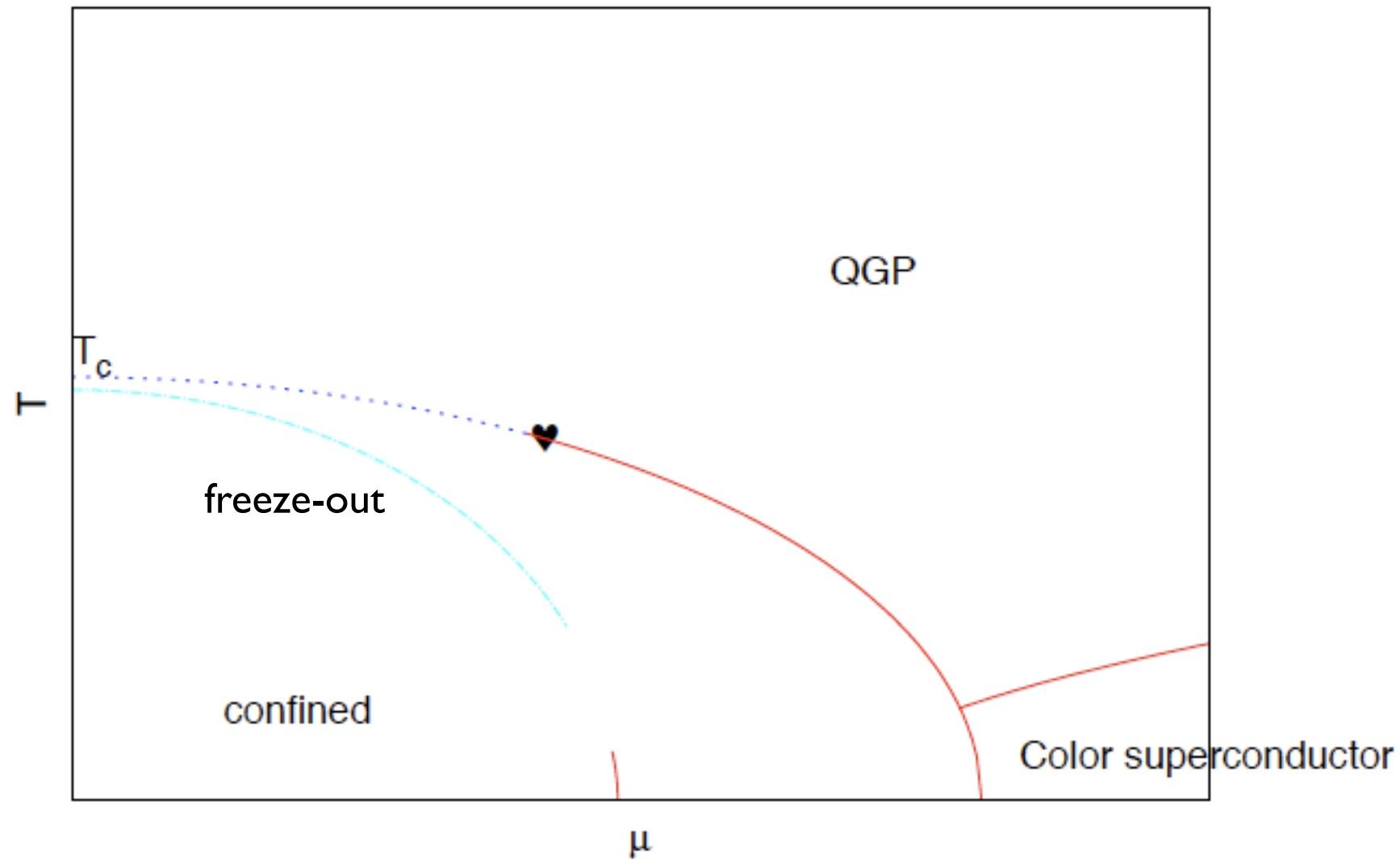
hotQCD II

Endrödi et al. II

- Curvature of crit. line from Taylor expansion
2+1 flavours, Nt=6, 8, 10 improved staggered
- Continuum extrapolation:

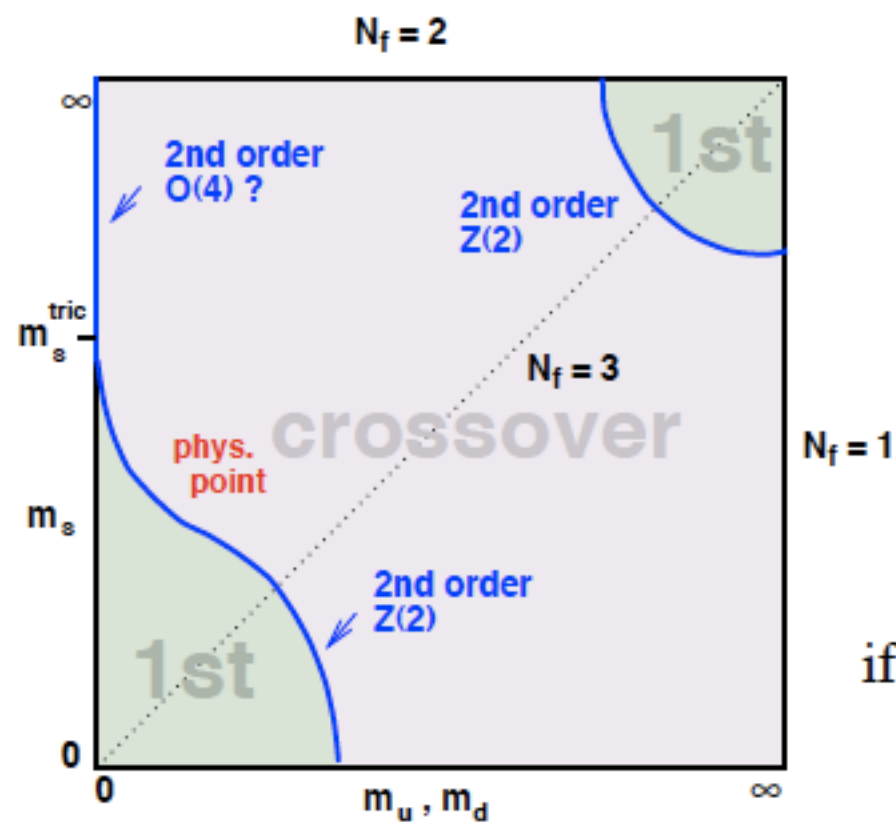
$$\kappa(\bar{\psi}\psi) = 0.059(18)$$

Comparison with freeze-out curve

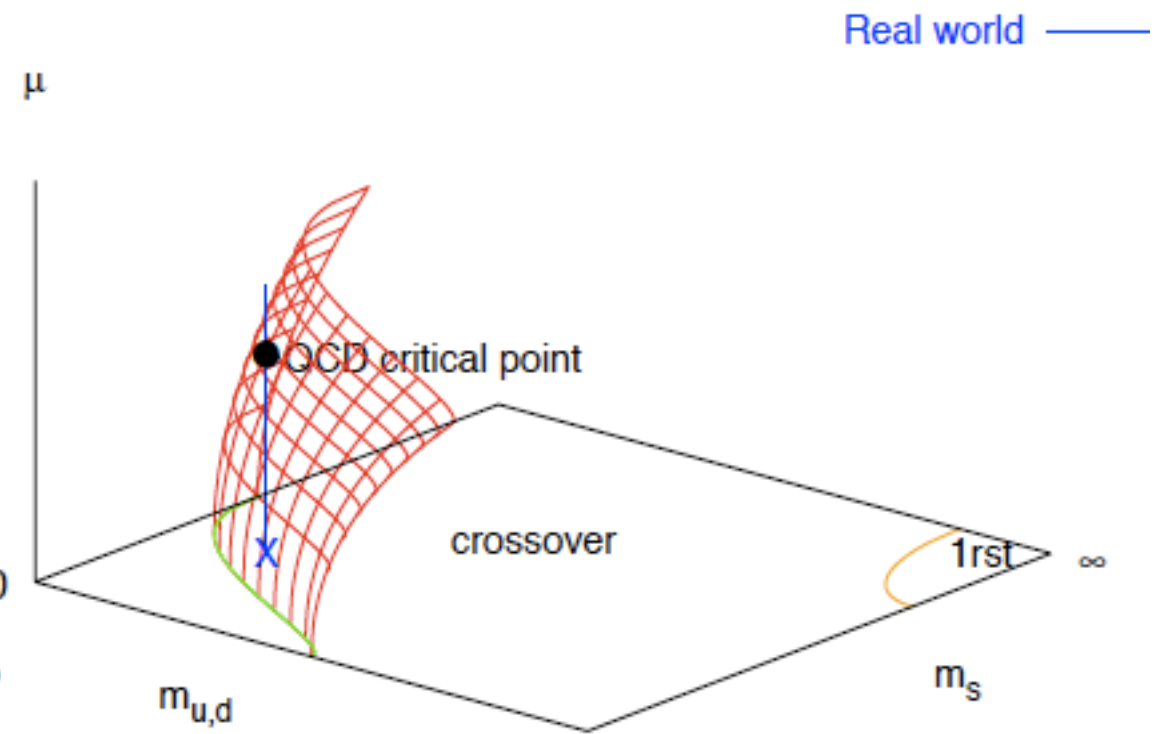


$T_c(\mu)$ considerably flatter than freeze-out curve (factor ~ 3 in $\left. \frac{d^2 T_c}{d\mu^2} \right|_{\mu=0}$)

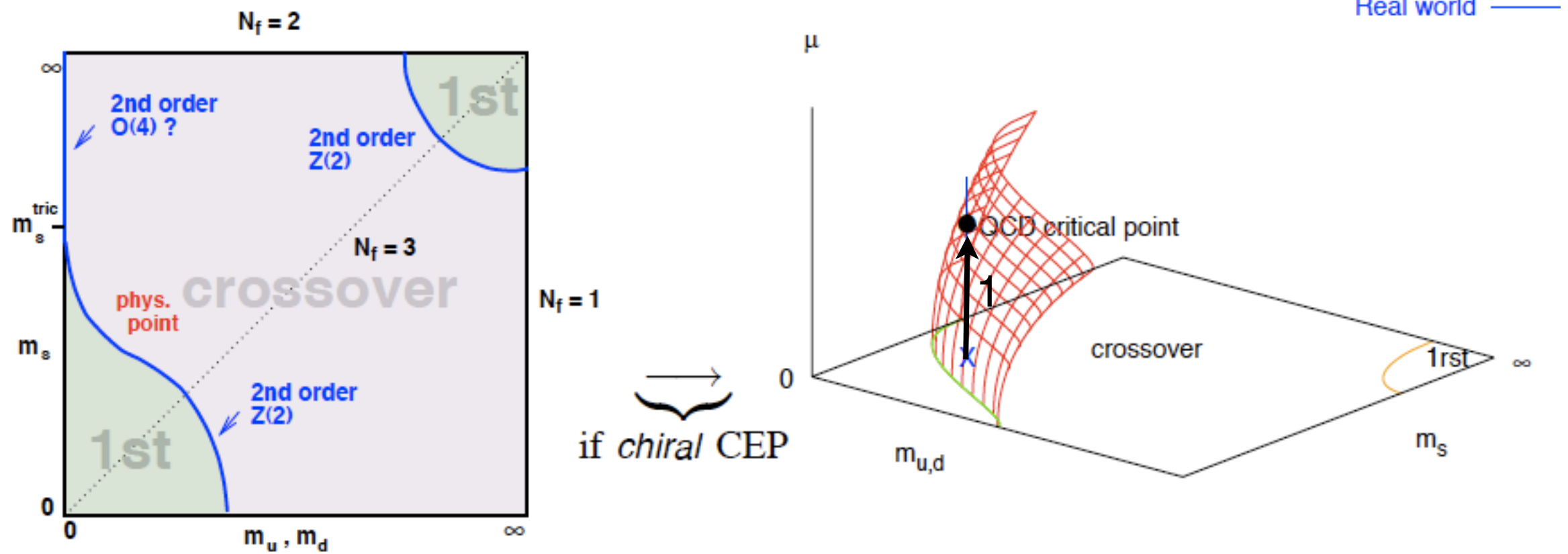
Much harder: is there a QCD critical point?



if *chiral* CEP



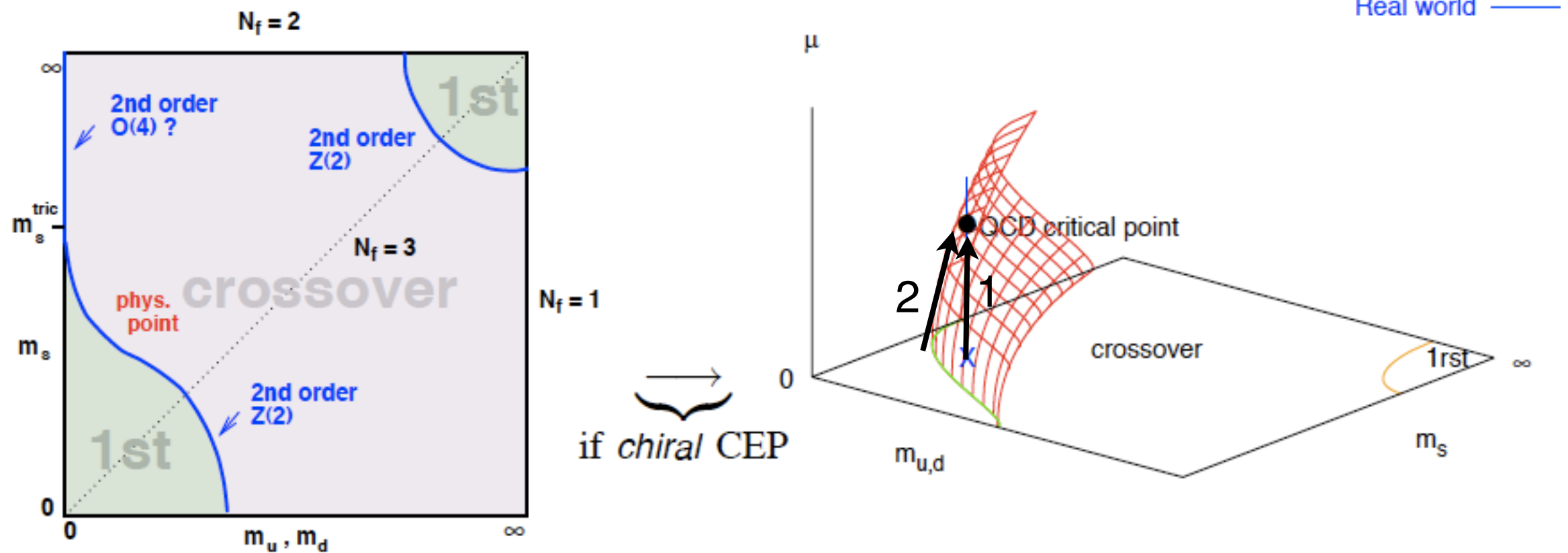
Much harder: is there a QCD critical point?



Two strategies:

1 follow **vertical line**: $m = m_{\text{phys}}$, turn on μ

Much harder: is there a QCD critical point?



Two strategies:

1 follow **vertical line**: $m = m_{\text{phys}}$, turn on μ

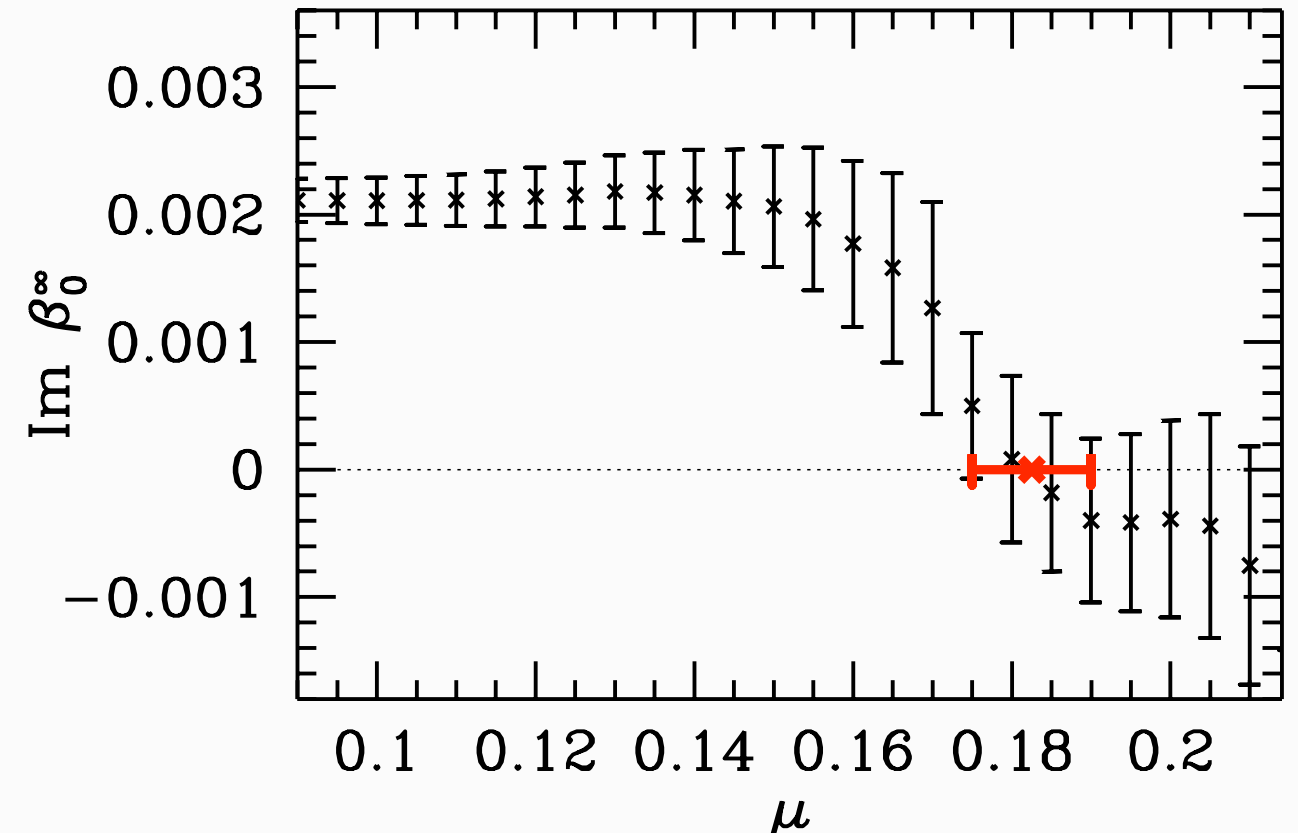
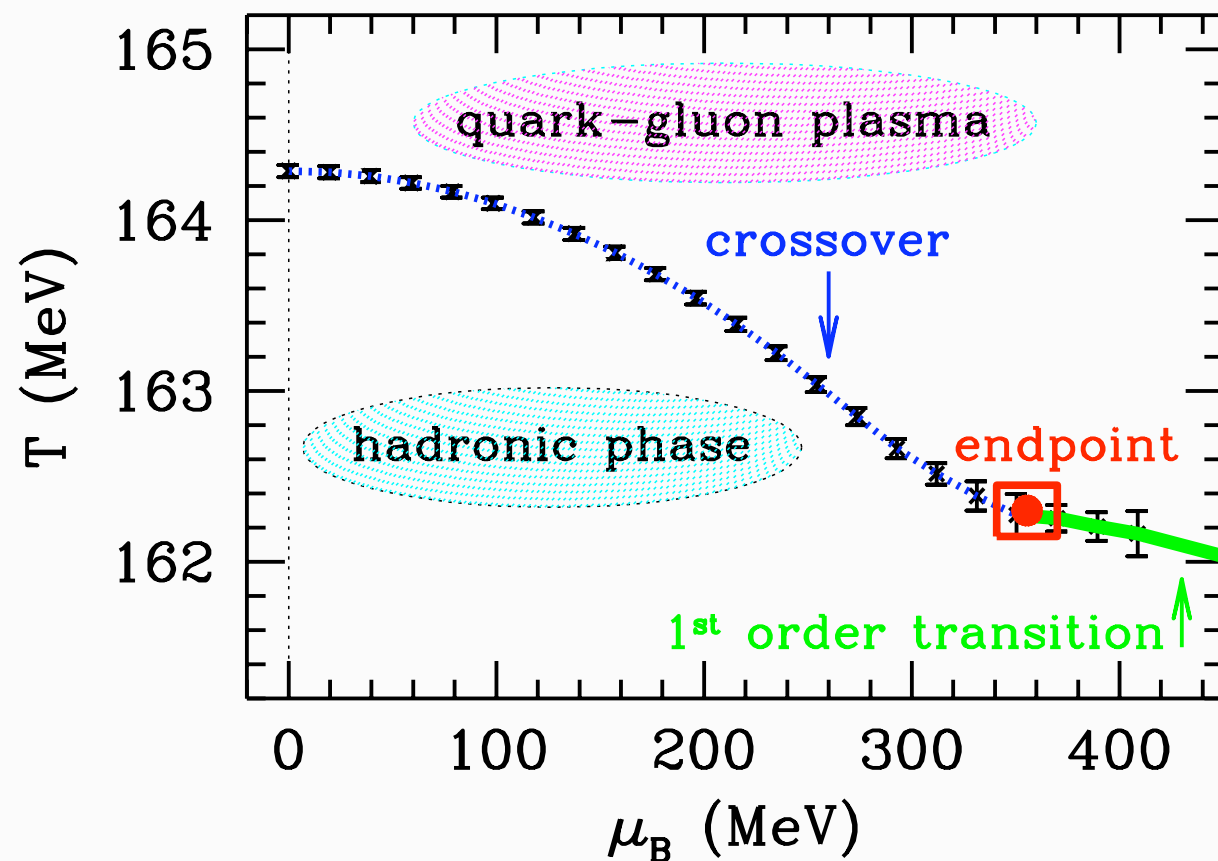
2 follow **critical surface**: $m = m_{\text{crit}}(\mu)$

Approach Ia: CEP from reweighting

Fodor, Katz 04

$N_t = 4, N_f = 2 + 1$ physical quark masses, unimproved staggered fermions

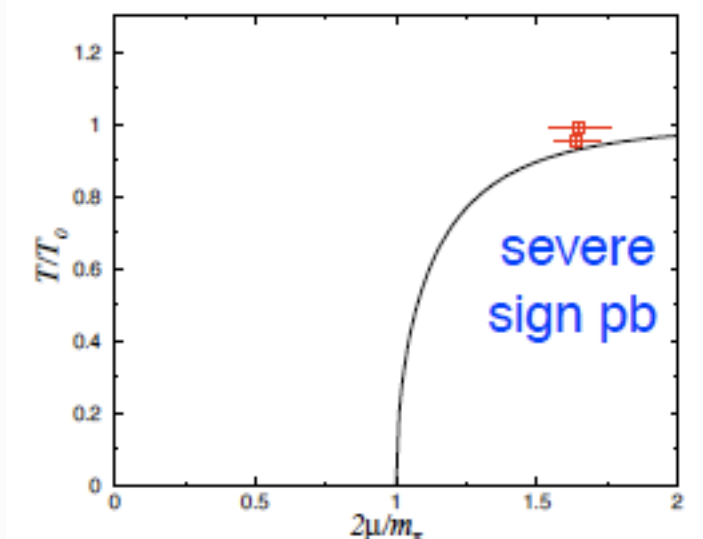
Lee-Yang zero:



$$(\mu_E^q, T_E) = (120(13), 162(2)) \text{ MeV}$$

abrupt change: caused by baryon or pion condensation?

Splittorf 05, Stephanov 08



Approach 1b: CEP from Taylor expansion

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu}{T} \right)^{2n}$$

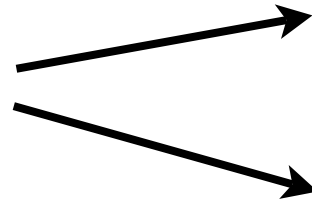
Nearest singularity=radius of convergence

$$\frac{\mu_E}{T_E} = \lim_{n \rightarrow \infty} \sqrt{\left| \frac{c_{2n}}{c_{2n+2}} \right|}, \quad \lim_{n \rightarrow \infty} \left| \frac{c_0}{c_{2n}} \right|^{\frac{1}{2n}}$$

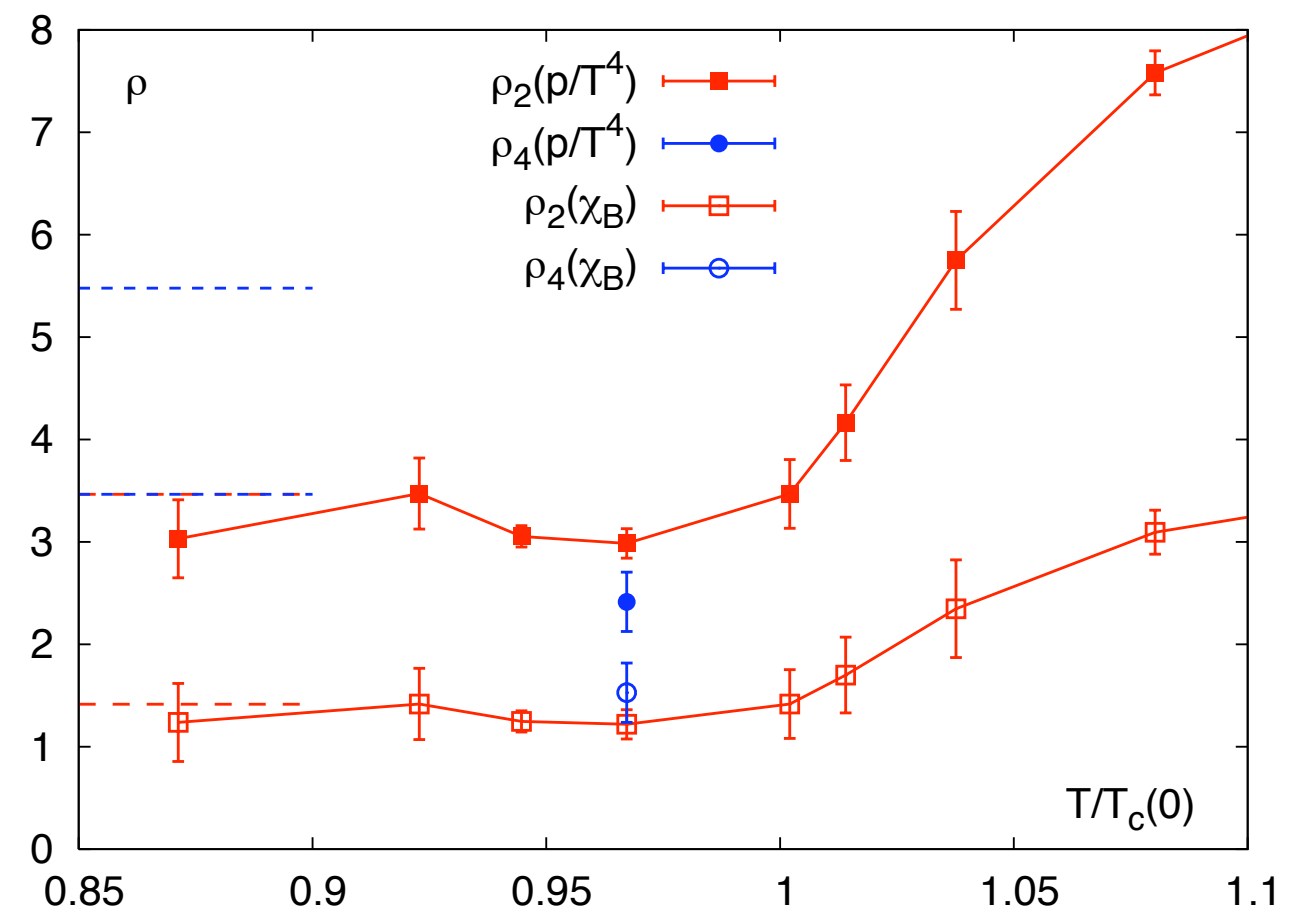
Different definitions agree only for $n \rightarrow \infty$
not $n=1,2,3,\dots$

control of systematics?

Hadron resonance gas



C.Schmidt, hotQCD 09



Radius of convergence necessary condition for CEP, but can it proof its existence?

Approach I c: canonical ensemble

$$Z_C(V, T, k) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-ik\phi} Z(V, T, \mu)|_{\mu=i\phi T}$$

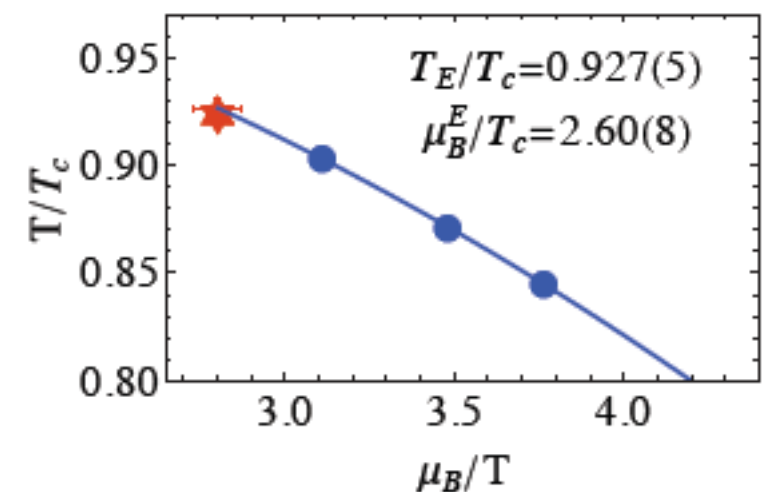
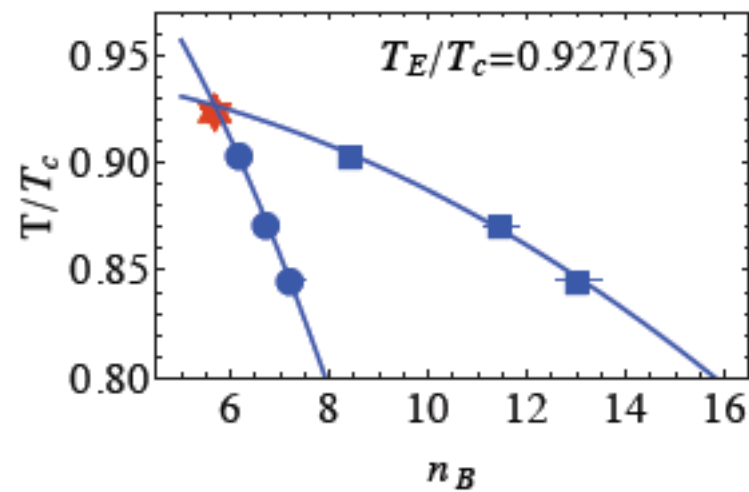
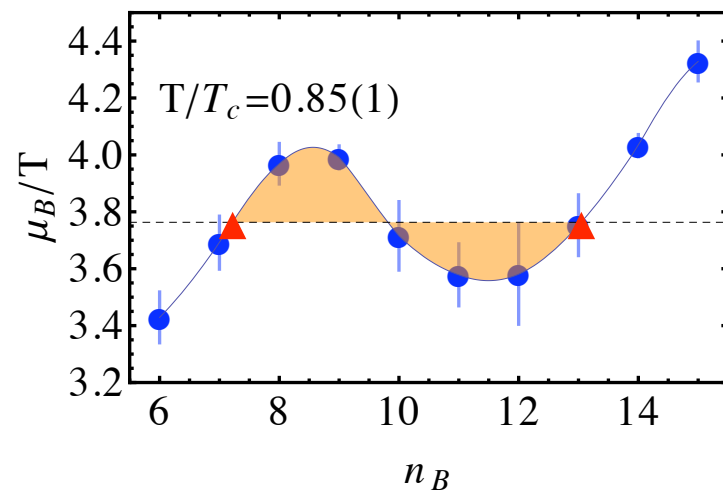
$$\langle \mu \rangle_{n_B} = \frac{F(n_B + 1) - F(n_B)}{(n_B + 1) - n_B}$$

Alexandru, Li, Liu II

Wilson clover fermions

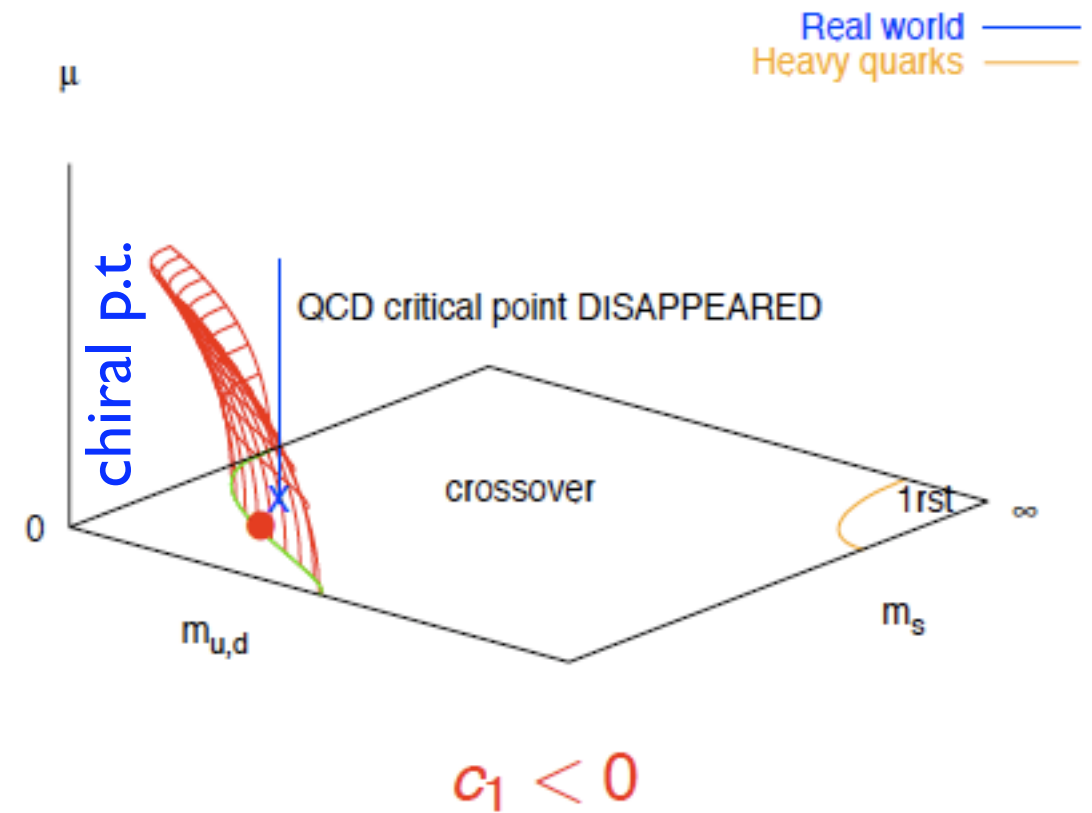
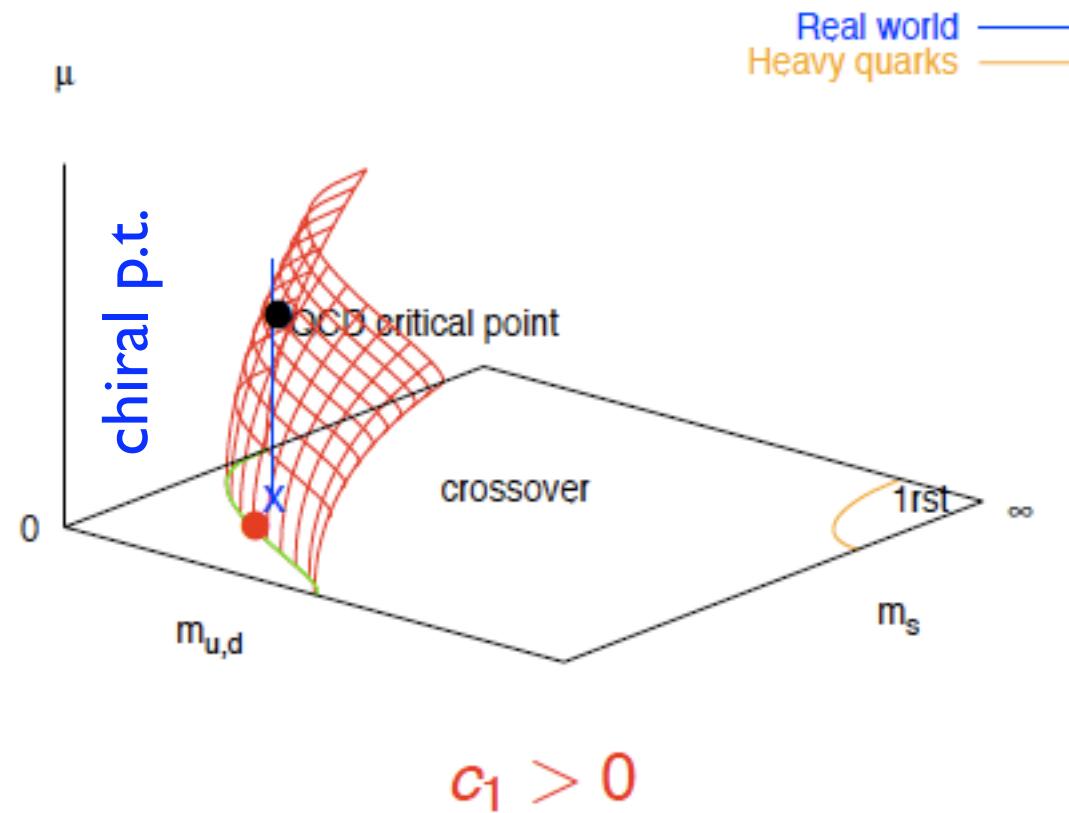
$6^3 \times 4, m_\pi \sim 700 - 800 \text{ MeV}$

Maxwell construction on finite volumes:



No thermodynamic limit yet, heavy pions

Approach 2: follow chiral critical line $\rightarrow$ surface



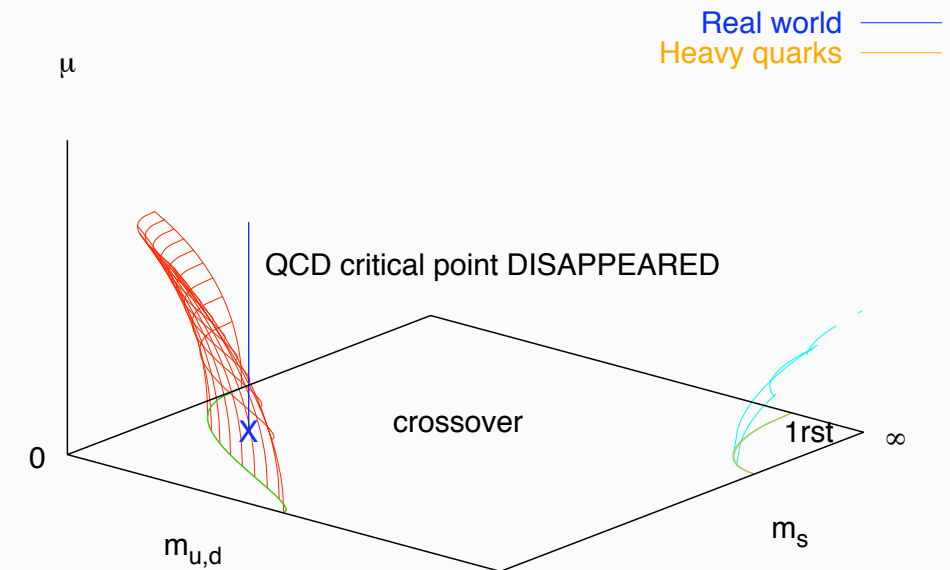
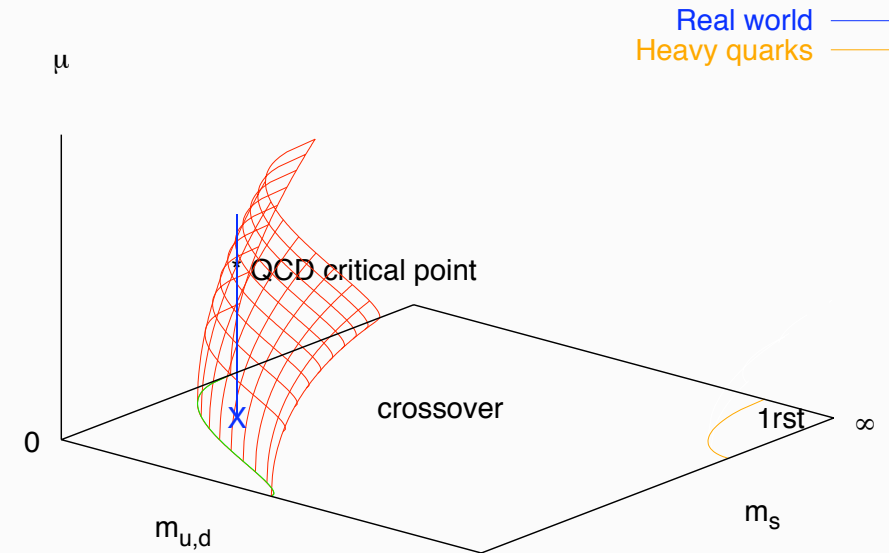
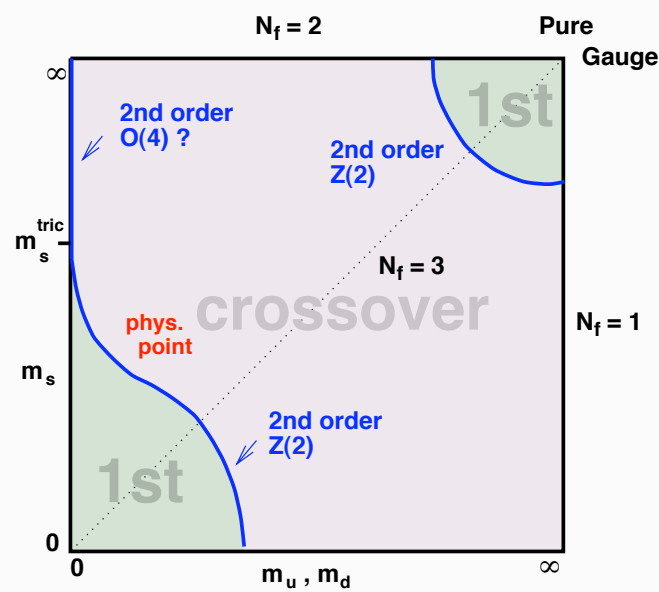
$$\frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1} c_k \left(\frac{\mu}{\pi T} \right)^{2k}$$

1. Tune quark mass(es) to $m_c(0)$: 2nd order transition at $\mu = 0, T = T_c$
known universality class: 3d Ising
2. Measure derivatives $\left. \frac{d^k m_c}{d\mu^{2k}} \right|_{\mu=0}$:

Turn on imaginary μ and measure $\frac{m_c(\mu)}{m_c(0)}$

de Forcrand, O.P. 08,09

Finite density: chiral critical line $\longrightarrow$ critical surface

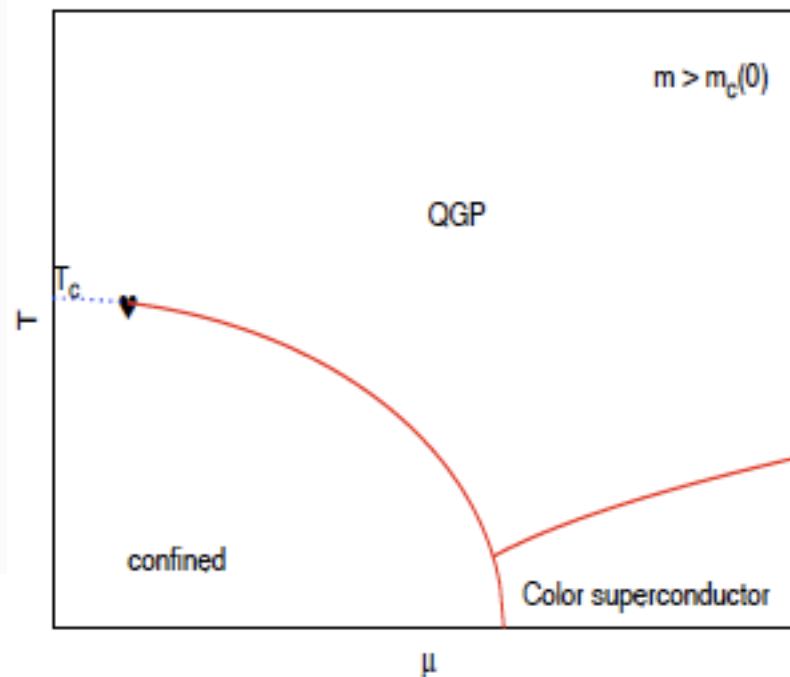


$$\frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1} \mathbf{c}_k \left(\frac{\mu}{\pi T} \right)^{2k}$$

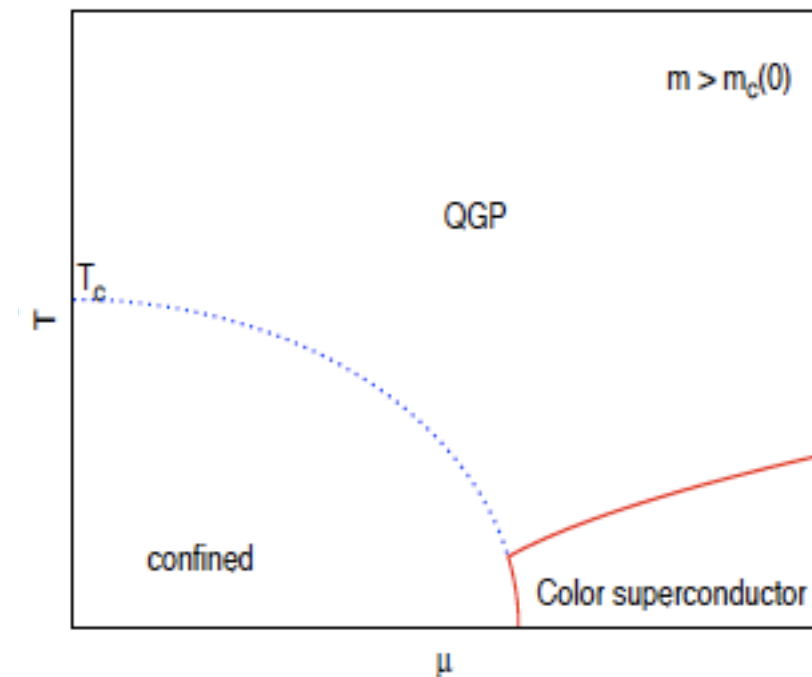
$$c_1 > 0$$

$$c_1 < 0$$

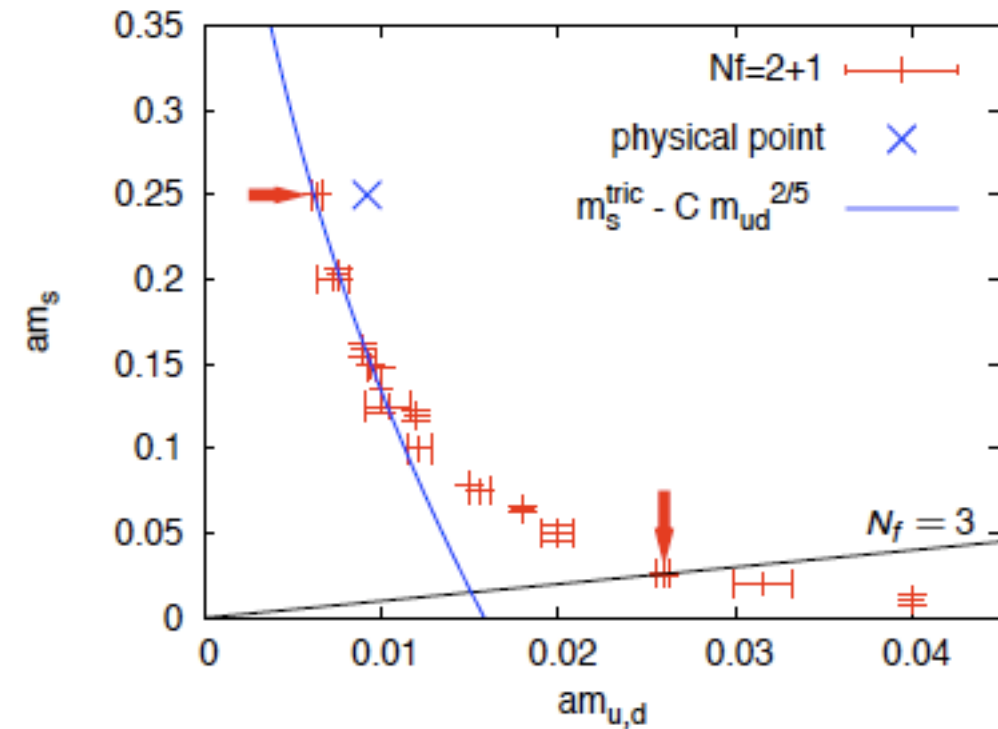
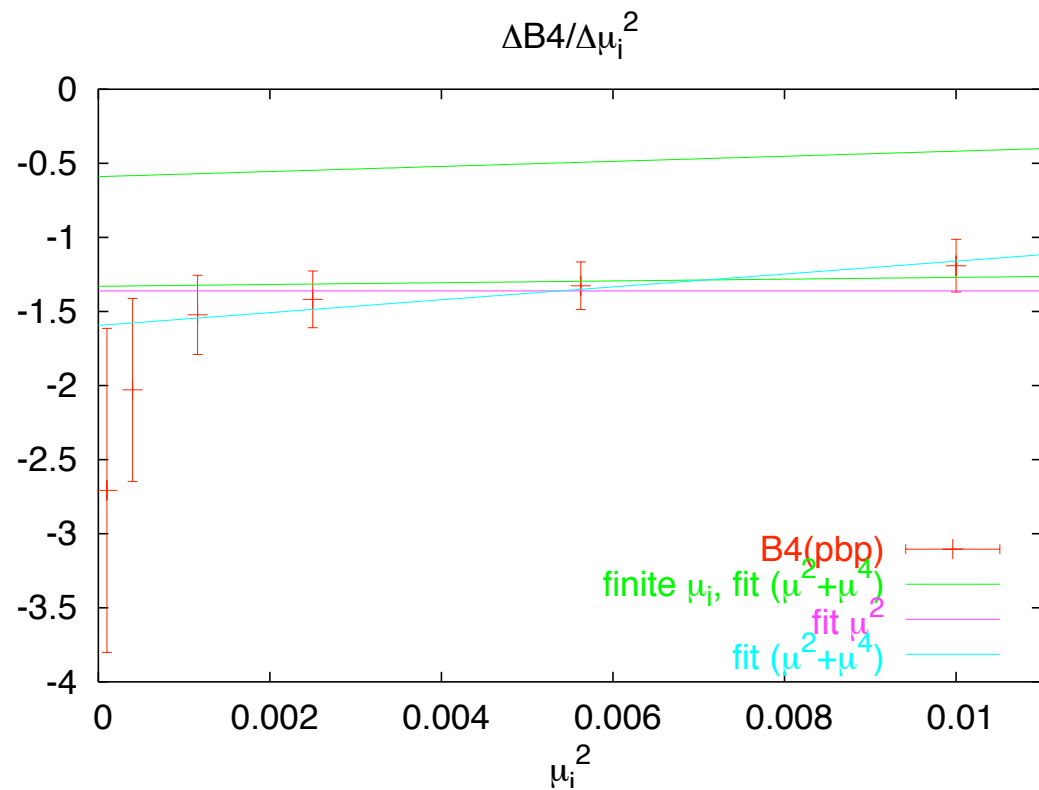
Standard scenario
transition strengthens



Exotic scenario
transition weakens



Curvature of the chiral critical surface



$N_f=3$: a) fit to imaginary chemical potential
b) calculation of coefficient by finite differences

consistent $8^3 \times 4$ and $12^3 \times 4$, $\sim 5 \times 10^6$ traj.

$$\frac{m_c(\mu)}{m_c(0)} = 1 - \underbrace{3.3(3) \left(\frac{\mu}{\pi T}\right)^2 - 47(20) \left(\frac{\mu}{\pi T}\right)^4}_{\text{8th derivative of P}} - \dots$$

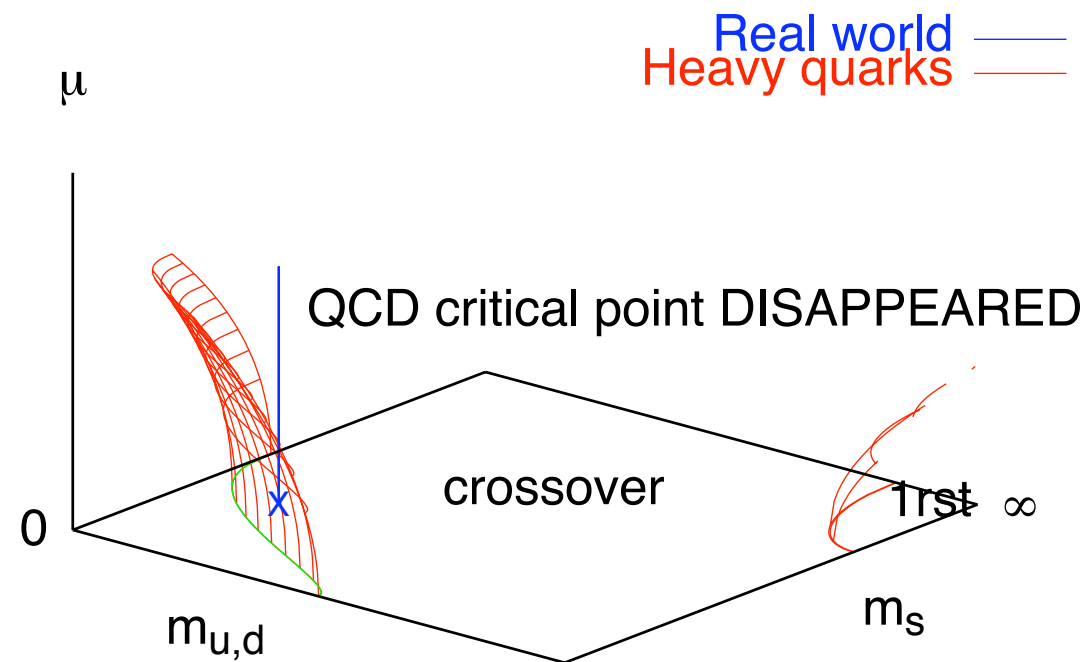
$16^3 \times 4$, Grid computing, $\sim 10^6$ traj.

$$\frac{m_c^{u,d}(\mu)}{m_c^{u,d}(0)} = 1 - 39(8) \left(\frac{\mu}{\pi T}\right)^2 - \dots$$

Importance of higher order terms ?

de Forcrand, O.P. 08,09

➔ On coarse lattice exotic scenario:
no chiral critical point at small density



Weakening of p.t. with chemical potential also for:

-Heavy quarks

de Forcrand, Kim, Takaishi 05

-Light quarks with finite isospin density

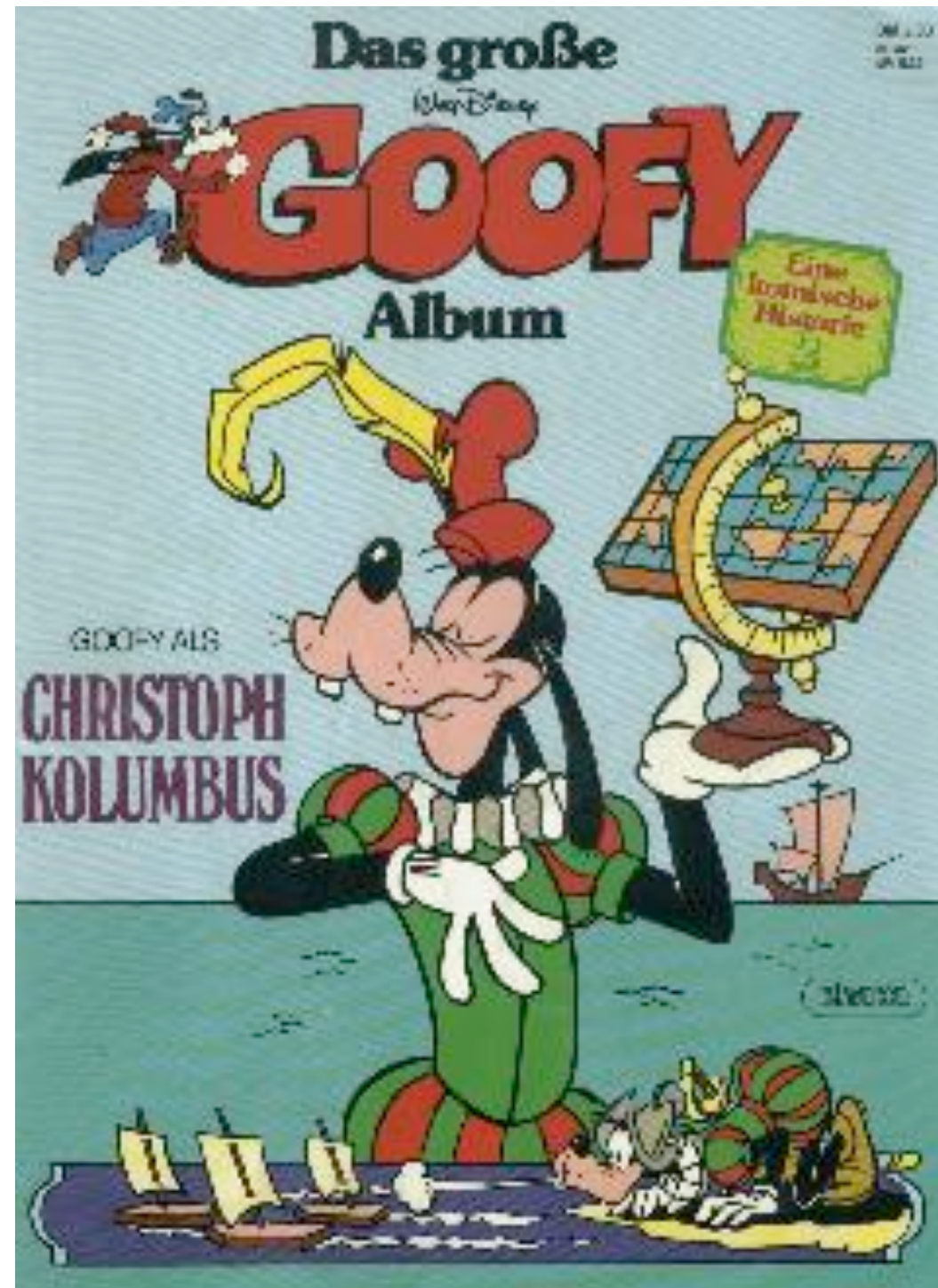
Kogut, Sinclair 07

-Electroweak phase transition with finite lepton density

Gynther 03

Un-discovering a critical point feels like...

Un-discovering a critical point feels like...

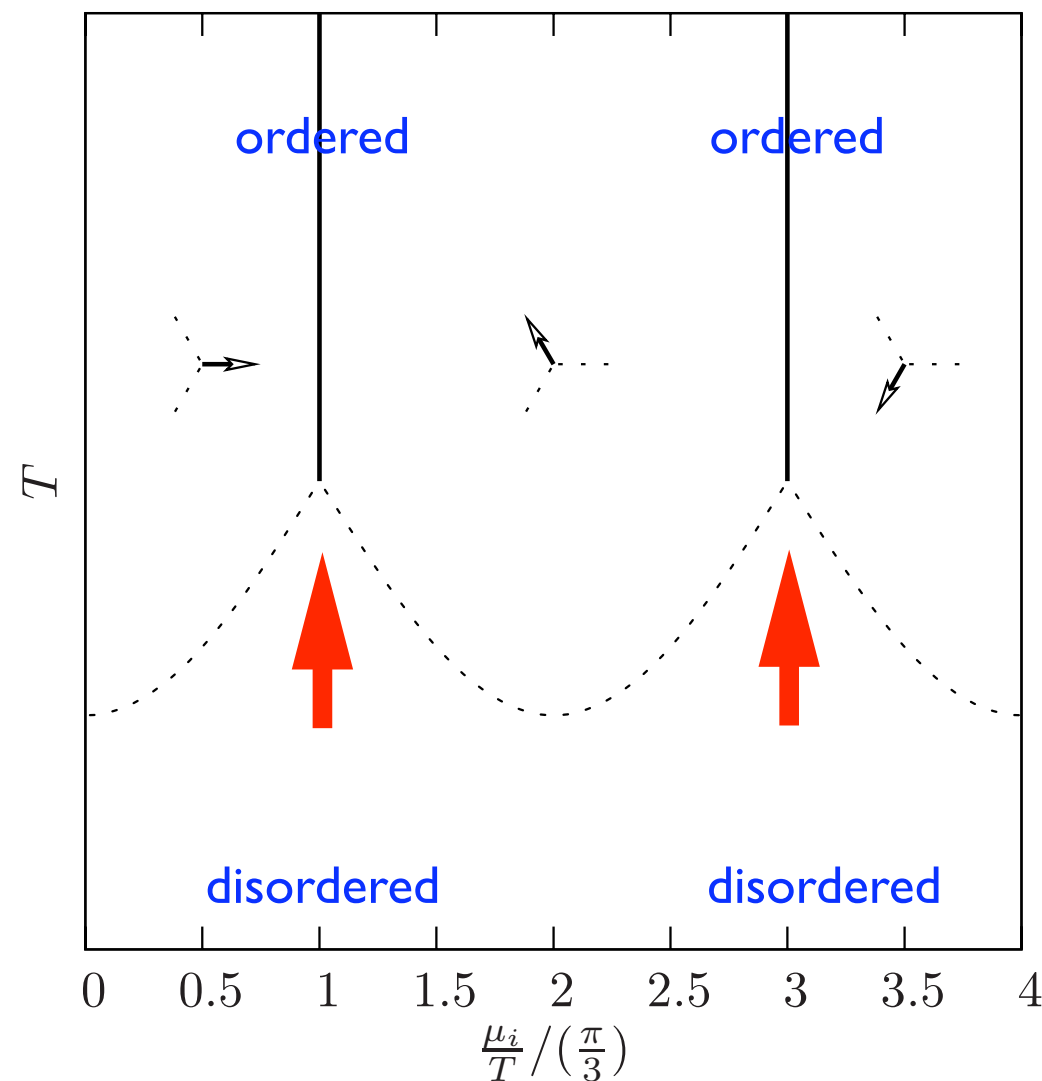


Understanding the curvature from imaginary μ

Nf=4: D'Elia, Di Renzo, Lombardo 07 Nf=2: D'Elia, Sanfilippo 09 Nf=3: de Forcrand, O.P. 10

Strategy: fix $\frac{\mu_i}{T} = \frac{\pi}{3}, \pi$, measure $\text{Im}(L)$, order parameter at $\frac{\mu_i}{T} = \pi$

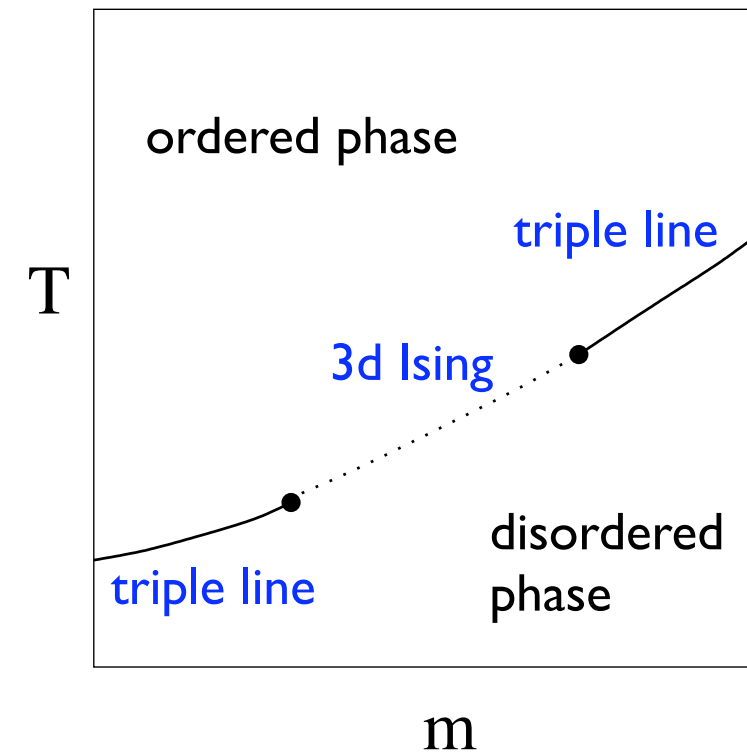
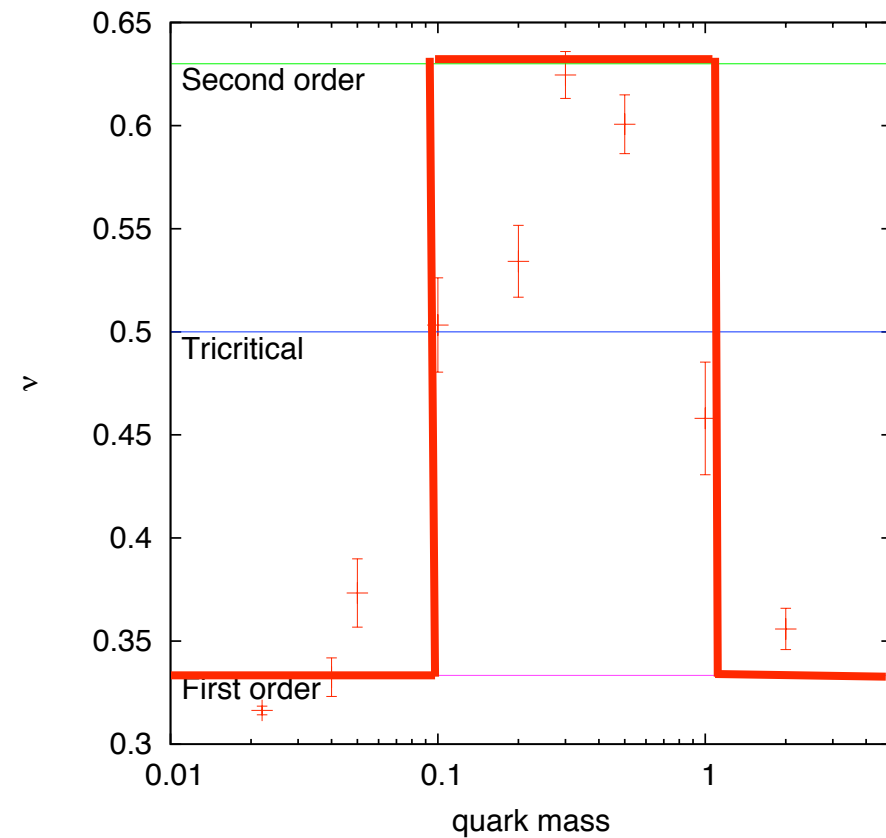
determine order of $Z(3)$ branch/end point as function of m



Scaling of Binder cumulant: $\nu = 0.33, 0.5, 0.63$

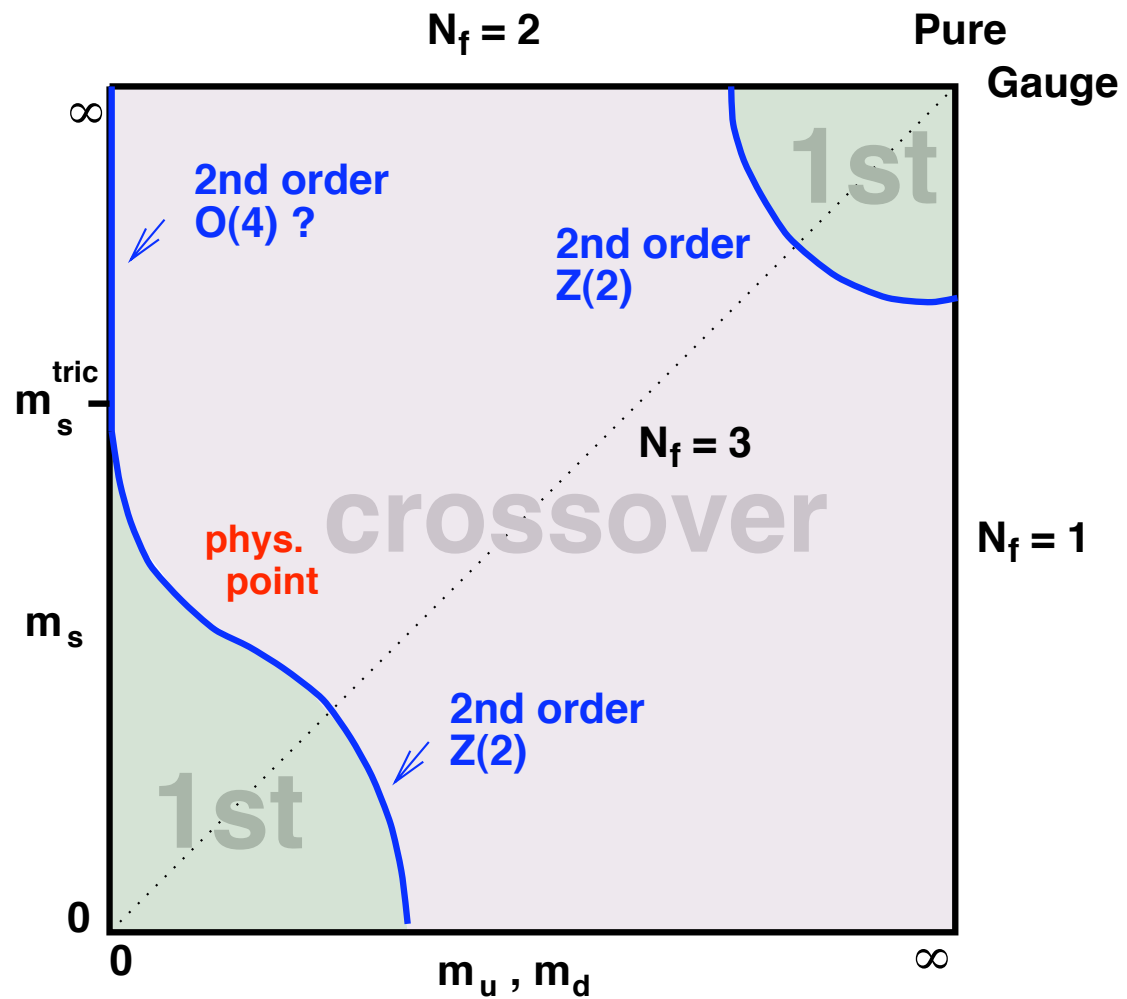
for 1st order, tri-critical, 3d Ising

Phase diagram at fixed $\frac{\mu_i}{T} = \frac{\pi}{3}, \pi$

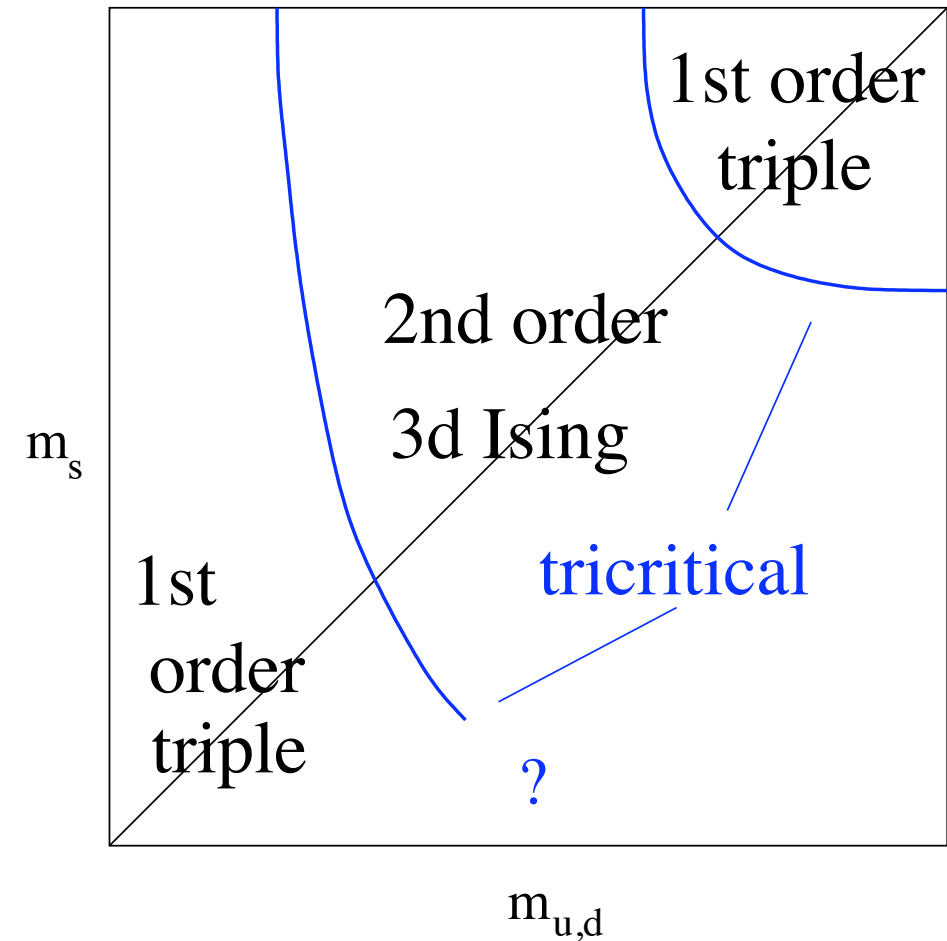


On infinite volume, this becomes a step function,
smoothness due to finite L

Critical lines at imaginary μ



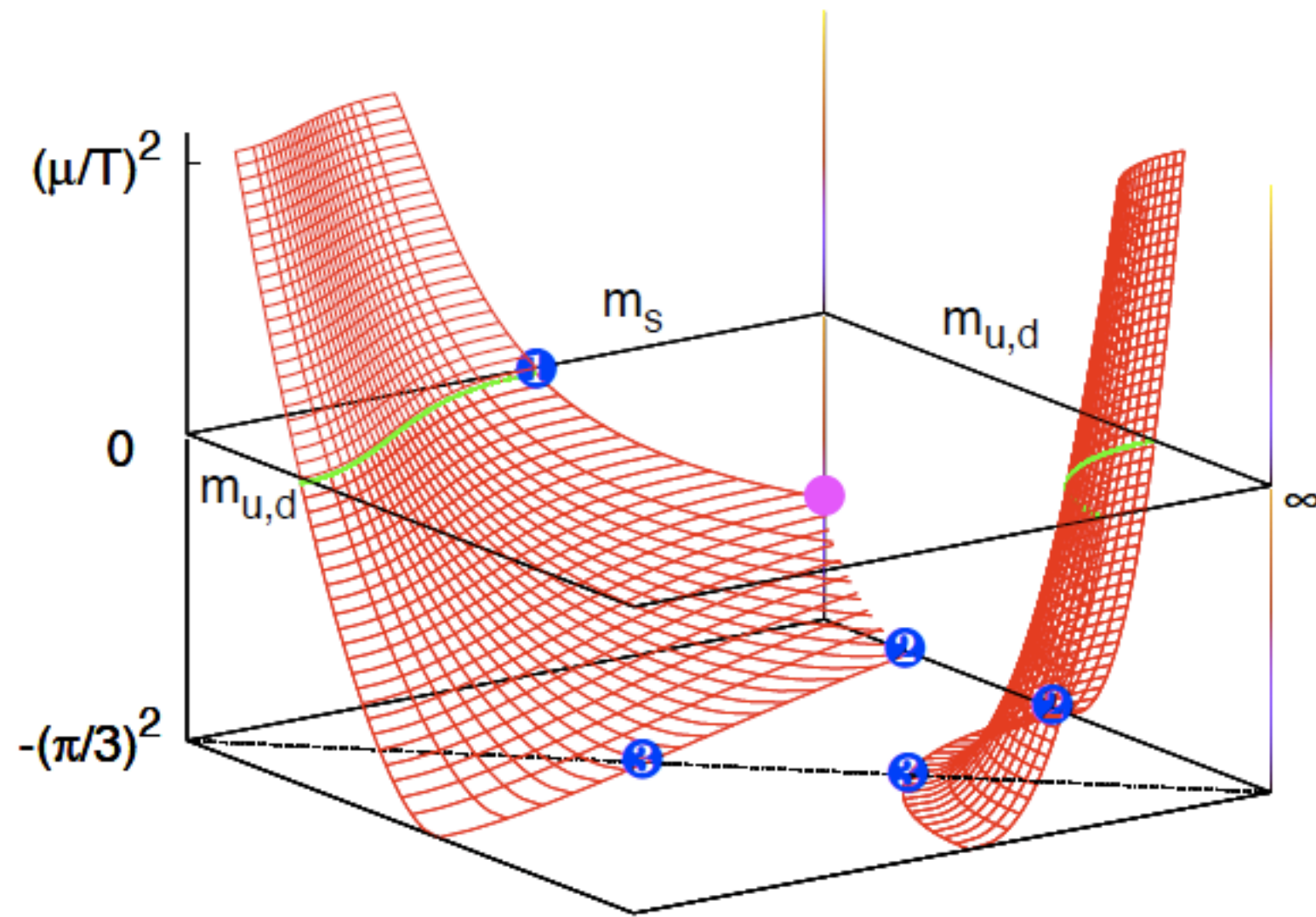
$$\mu = 0$$



$$\mu = i\frac{\pi T}{3}$$

- Connection computable with standard Monte Carlo!
- Here: heavy quarks in eff. theory

3d, imaginary chemical potential included:



Heavy quarks: 3d 3-state Potts and strong coupling

small μ/T : sign problem mild, doable for real μ !

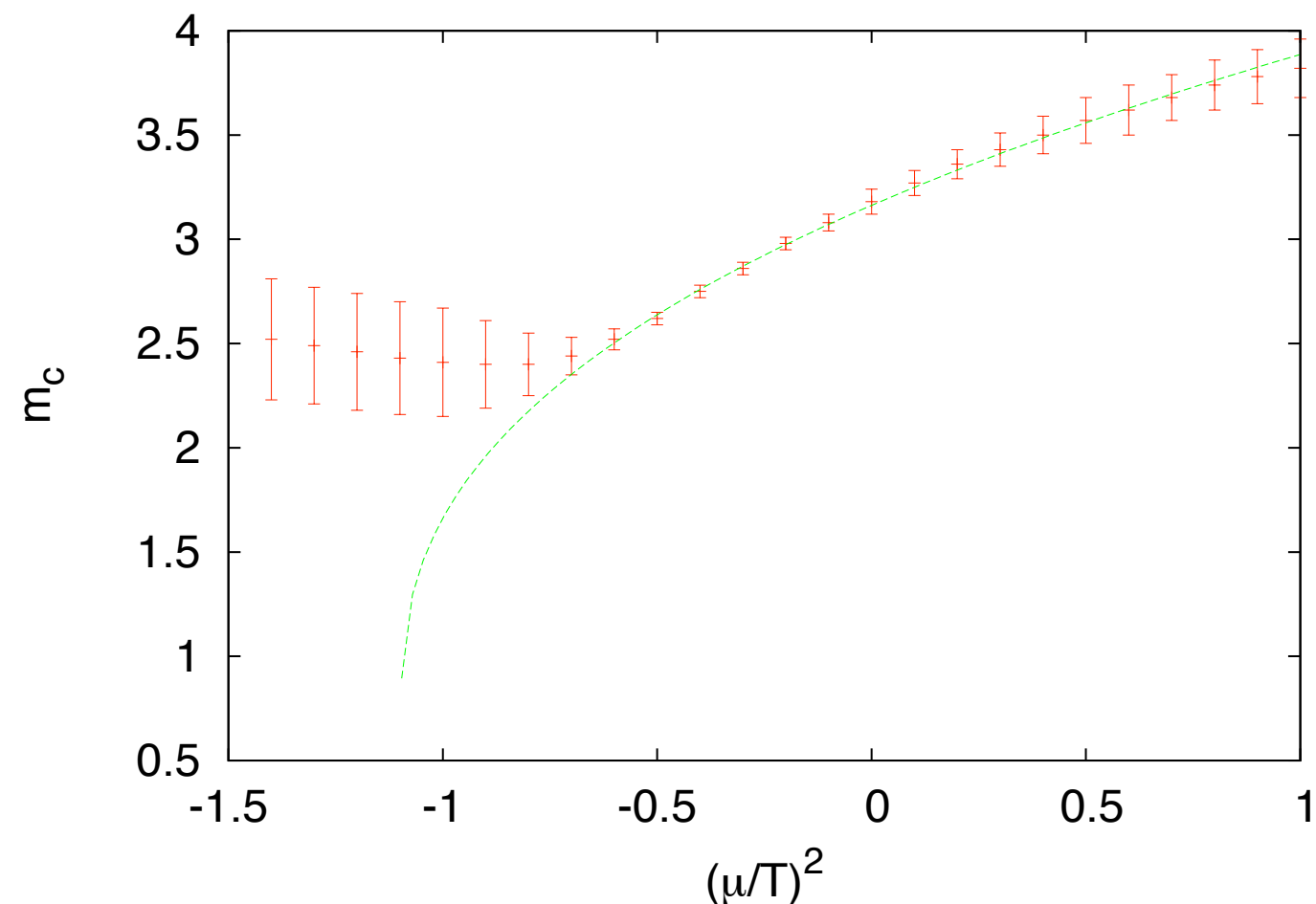
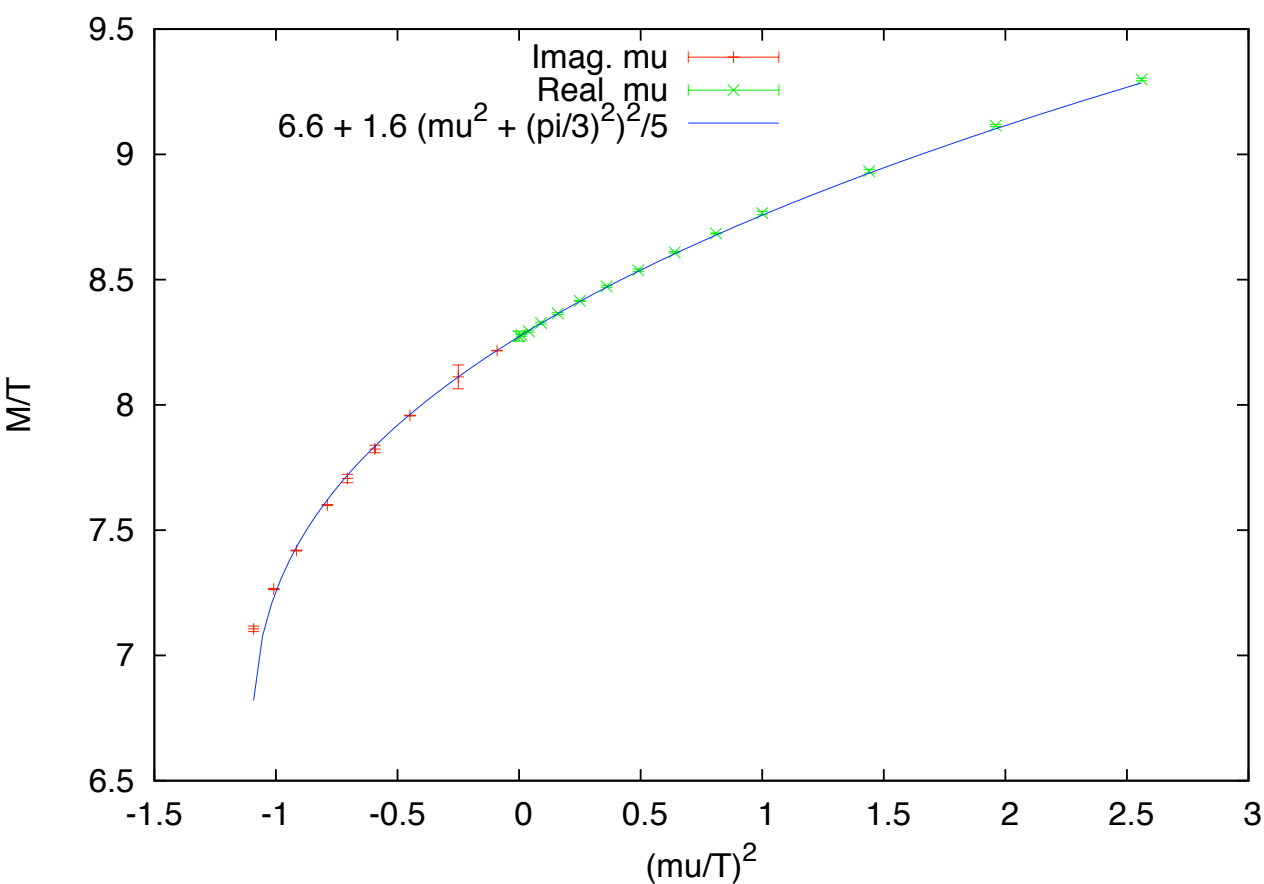
de Forcrand, Kim, Kratochvila, Takaishi

Same universality class!

Potts:

QCD, $N_t=1$, strong coupling series:

Langelage, O.P. 09



tri-critical scaling:
$$\frac{m_c}{T}(\mu^2) = \frac{m_{tric}}{T} + K \left[\left(\frac{\pi}{3} \right)^2 + \left(\frac{\mu}{T} \right)^2 \right]^{2/5}$$
 ← exponent universal

Summary Lecture IV

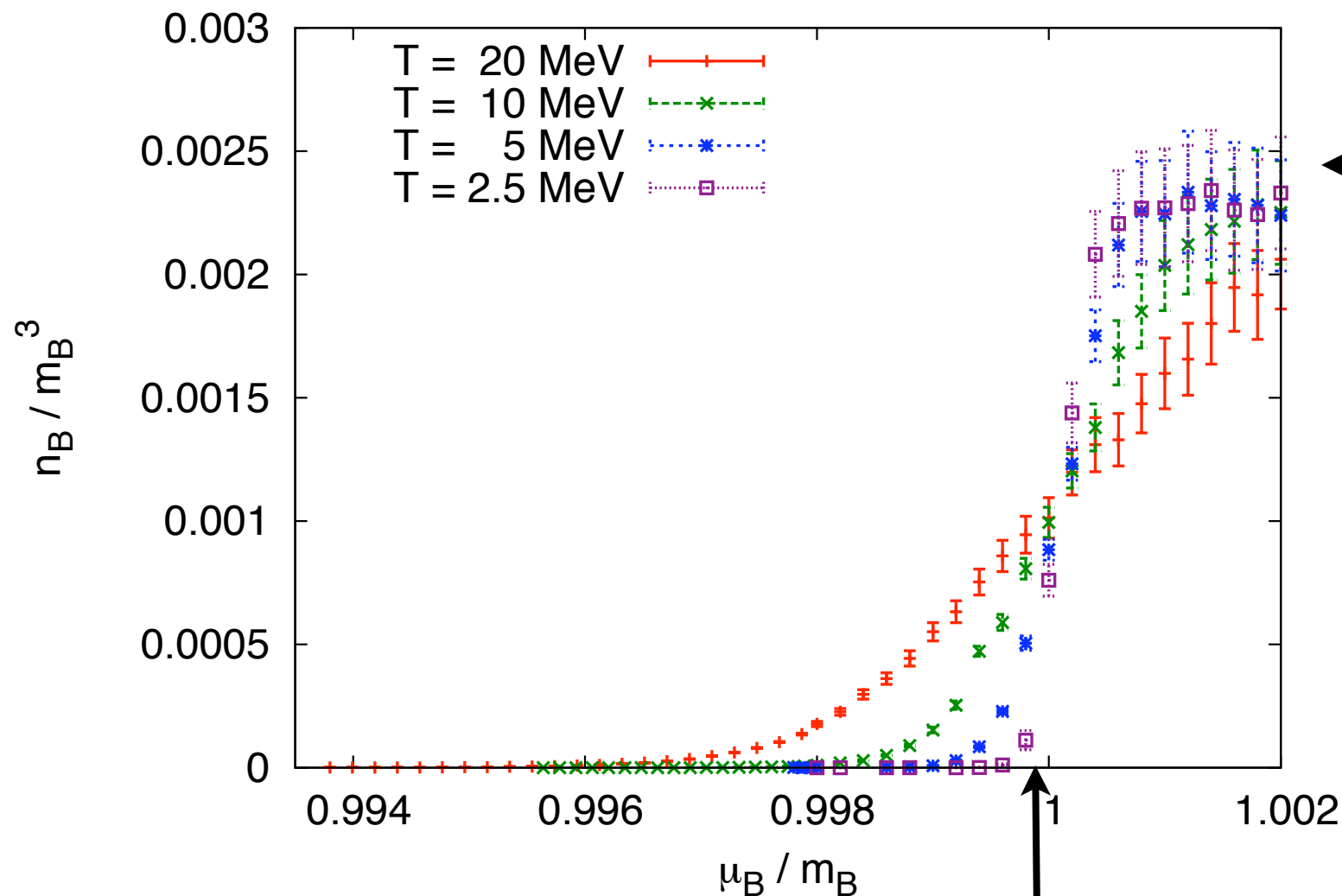
- Thermal transition at zero density is a crossover
- The sign problem is related to C-symmetry
- Direct MC methods to circumvent it related to small chemical potential
- In the controlled region there is no evidence for a chiral critical point!

New horizon: onset of cold nuclear matter

Based on 3d effective action by strong coupling and hopping exp.

... with very heavy quarks $m_\pi = 20 \text{ GeV}$

continuum limit with 5-7 lattice spacings per point



consistent with physical
nuclear density

$$\sim 0.16 \text{ fm}^{-3} \approx 0.15 \cdot 10^{-2} m_{\text{proton}}^3$$

Frankfurt group, PRL 13

Complex Langevin: no sign problem
convergence criteria satisfied
cf. Seiler, Stamatescu; Aarts, James

$$\frac{\mu}{T} \sim 4000$$