Strong coupling expansions for QCD at finite temperature and density

Owe Philipsen

Westfälische Wilhelms-Universität Münster

- Motivation
- Introduction to strong coupling expansions
- SCE for finite temperature: free energy, screening masses
- The phase transition from SCE, finite chemical potential

in collaboration with J. Langelage (Bielefeld)

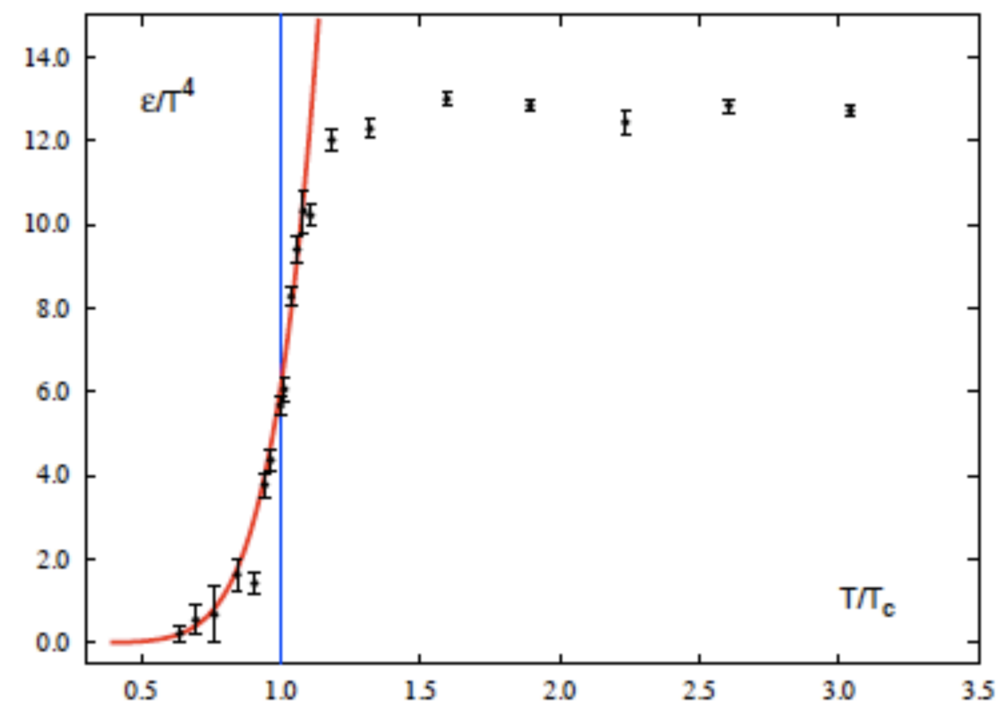
Why strong coupling expansions?

- SCE produce convergent series, finite radius of convergence
- Complementary to weak coupling approach, Monte Carlo
- Only analytical approach from first principles for confined phase
- Study onset of finite T-effects
- T=0: important qualitative insights (confinement, glueball spectrum...)
- finite T: establish connection between QCD and strong coupling limit

  Kawamoto et al. ;   Fromm, de Forcrand

Model for confined phase: hadron resonance gas

Karsch, Redlich, Tawfik 03



- Non-interacting gas of hadrons/resonances;
  implicitly includes binding effects, neglects weak decays
- Allows to factorise partition function into product over
  one-particle partition functions

SCE at T=0: 'free energy density'

Partition function; link variables as degrees of freedom

$$Z = \int \prod_{x,\mu} dU(x;\mu) \exp\left(-S_{YM}\right) \equiv \int DU \exp\left(-S_{YM}\right)$$

Wilson's gauge action

$$S_W = -\frac{\beta}{N} \sum_p \mathrm{Re}\mathrm{Tr}(U_p) = \sum_p S_p \qquad \beta = \frac{2N}{g^2}$$

Free energy density; thermodynamic limit

$$f = -\frac{1}{\Omega} \ln Z \qquad \Omega = VL_t \longrightarrow \infty$$

SCE: calculational technology

Expansion in irreducible characters $\chi_r(U) = \mathrm{Tr} D_r(U)$

$$\exp(-S_p) = c_0(\beta) \left\{ 1 + \sum_{r \neq 0} d_r c_r(\beta) \chi_r(U_p) \right\}$$

Expansion parameters $c_r(\beta)$ are combinations of modified Bessel functions (for $SU(N)$)

$$c_f \equiv u \sim \beta + \ldots$$
$$c_{ad} \sim \beta^2 + \ldots$$

Higher dimensional representations go with higher orders in β

Rewrite partition function as sum over sets of plaquettes

$$
\begin{aligned}
Z &= \prod_p \int DU \exp(-S_p) \\
&= \prod_p c_0(\beta) \int DU \left\{ 1 + \sum_{r \neq 0} d_r c_r(\beta) \chi_r(U_p) \right\} = c_0^{6\Omega} \sum_G \Phi(G)
\end{aligned}
$$

Remark: Each plaquette is now allowed to occur only once in a given graph!

Contribution of a graph: $\Phi(G) = \int DU \prod_{p \in G} d_r c_{r_p} \chi_{r_p}(U_p)$

Integration rule 1

$$\int dU \chi_r(U) = \delta_{r,0}$$

$\longrightarrow$ Graphs must have closed surfaces

Integration rule 2

$$\int dU \chi_r(UV)\chi_s(U^{-1}W) = \delta_{rs}\frac{1}{d_r}\chi_r(VW)$$

Used to perform the occurring group integrations

The free energy density

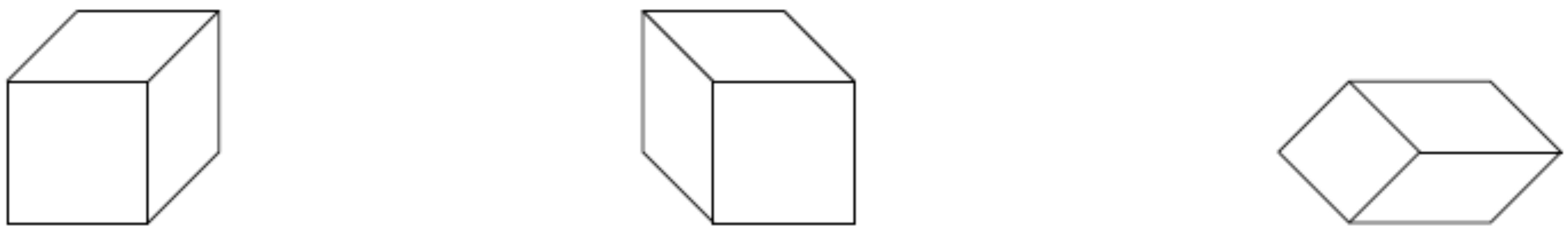
Taking logarithm

$$\begin{aligned} f &= -6 \ln c_0 - \frac{1}{\Omega} \sum_{C=\{X_i\}} a(C) \prod_i \Phi(X_i) \\ &\equiv -6 \ln c_0 - \frac{1}{\Omega} \sum_C \Phi(C) \end{aligned}$$

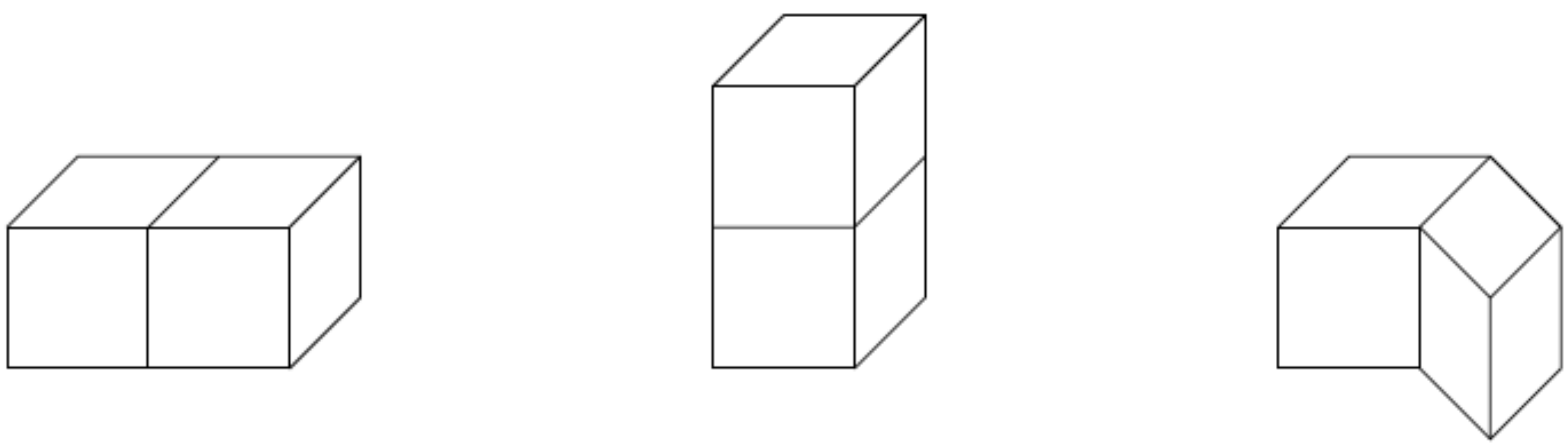
$a(C)$: combinatorial factor for cluster C of polymers, depends on how the polymers are connected ($a(C) = 1$ if only one polymer)

The graphs to be calculated

Examples of leading order graphs



Some corrections



Introducing a physical temperature

Münster, Langelage, Philipsen 08

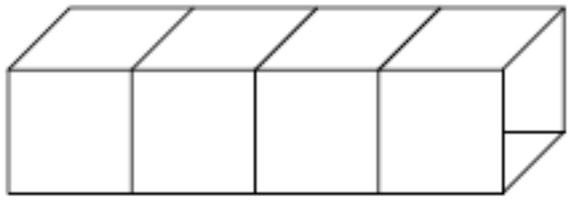
- Consider two lattices, one with finite temporal extent, periodic b.c.
- Subtract vacuum contribution (renormalisation, cf. continuum)
- Physical free energy density:

$$f(T) = f(N_t) - f(\infty)$$

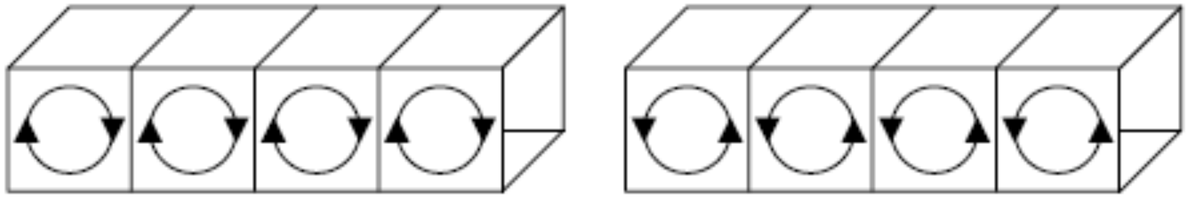
- New class of diagrams contributing to difference

Leading order contributions

Leading order graph for $SU(2)$



Leading order graphs for $SU(N \geq 3)$



Contribution of such a graph

$$\Phi = \int DU \prod_p d_f \, u \, \mathrm{Tr}(U_p) = u^{4N_t}$$

The leading order result

Free energy density

$$f(N_t, u) = -\frac{1}{VN_t}\sum_C \Phi(C)$$

Summing all leading order terms gives

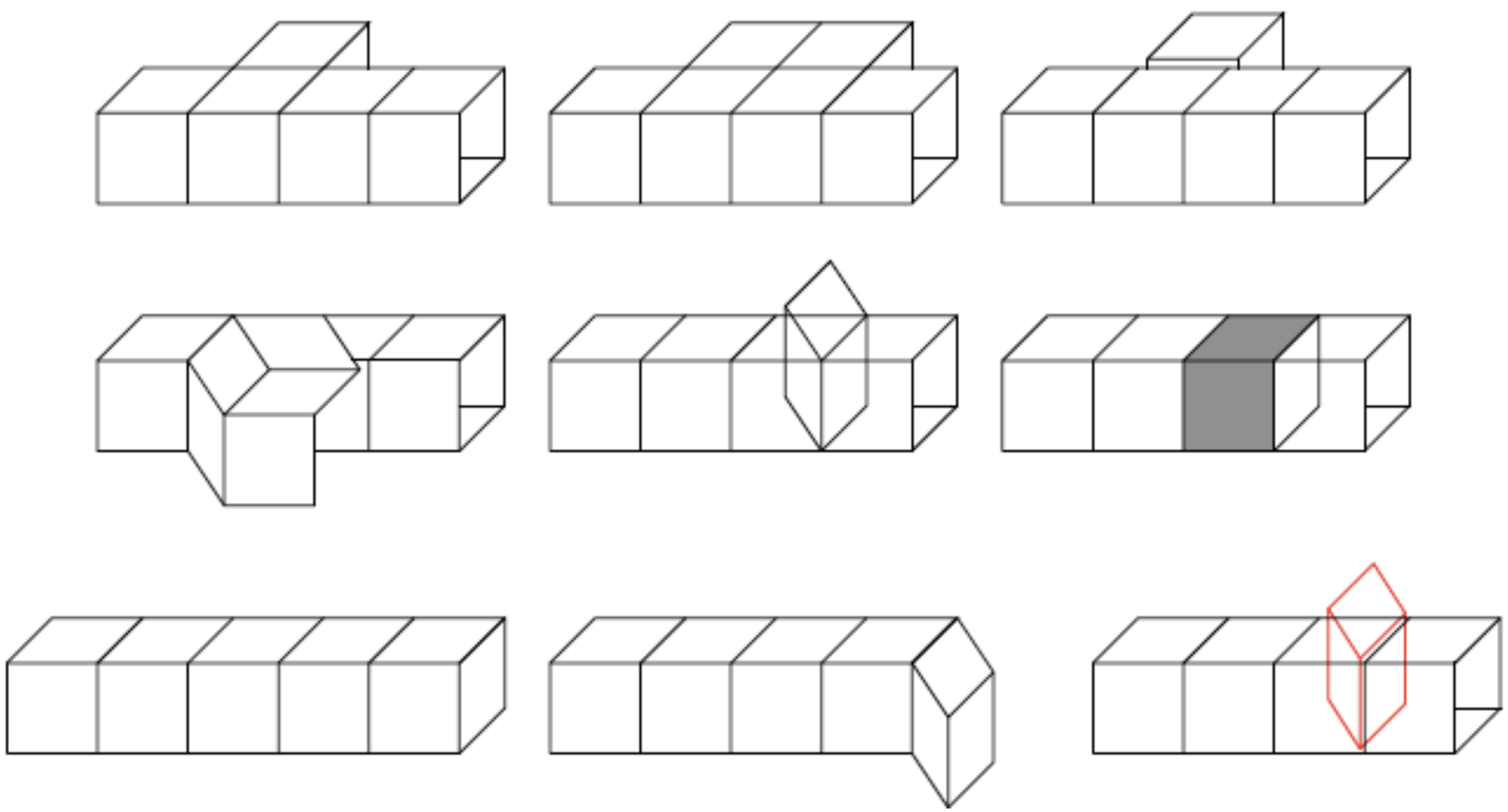
$$f_{LO}(N_t, u) = -\frac{3}{N_t}u^{4N_t} \qquad \text{for } SU(2)$$

$$f_{LO}(N_t, u) = -\frac{6}{N_t}u^{4N_t} \qquad \text{for } SU(N \geq 3)$$

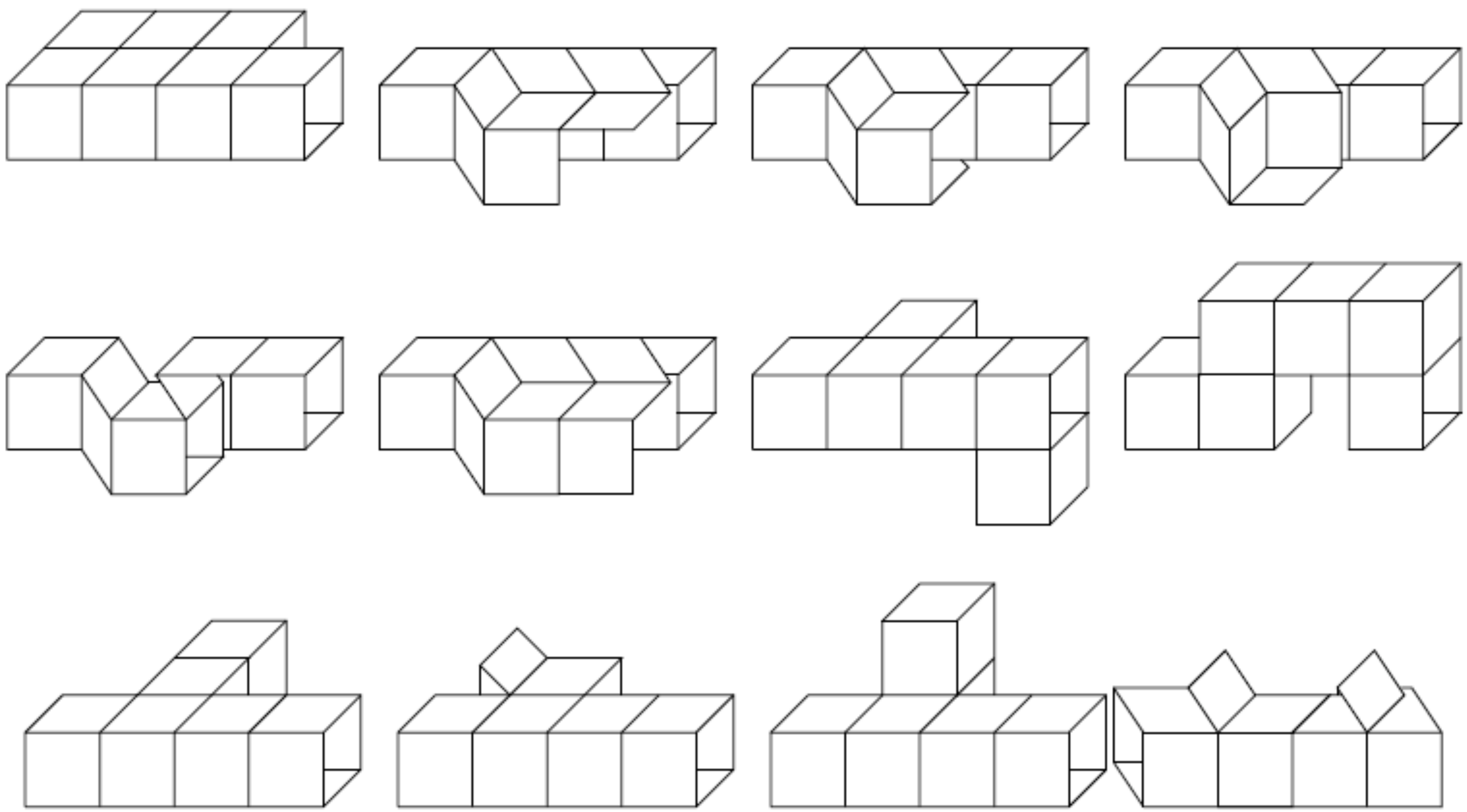
Strong coupling (T=0) limit has zero free energy, pressure!

The hard part: corrections

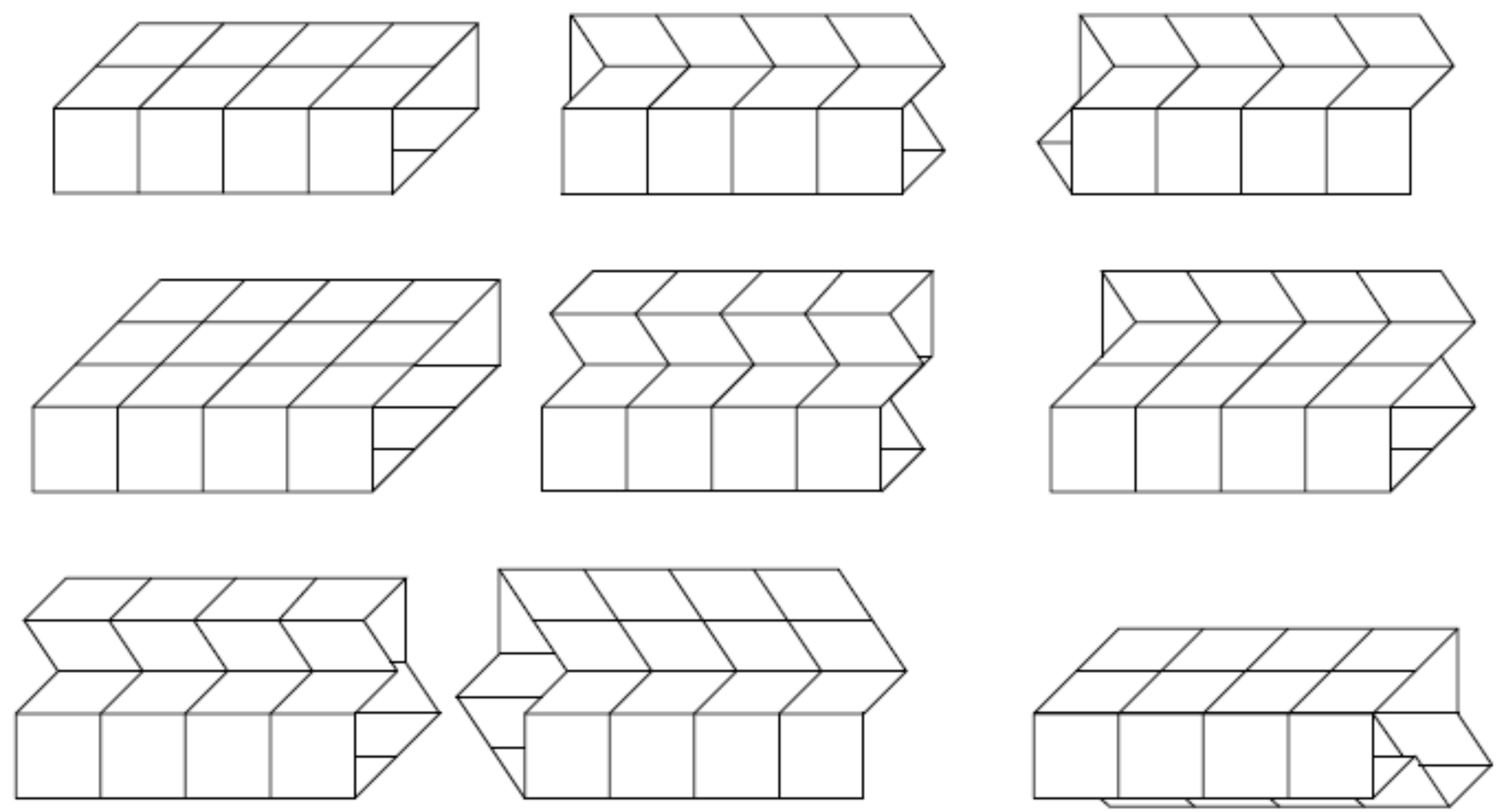
Some examples up to $\mathcal{O}(u^6)$ in the correction



Some examples of $\mathcal{O}(u^8)$ in the correction



Some examples of polymers with a bigger cross-section



The series for the free energy density

SU(2)

$$f(1,u) = -3\,u^4\left(1 + 5.33u^2 + 67.36u^4 + 797.24u^6\right)$$

$$f(2,u) = -\frac{3}{2}u^8\left(1 - 4.00u^2 + 36.67u^4 - 144.38u^6 + 1195.14u^8\right)$$

$$f(3,u) = -u^{12}\left(1 - 6.00u^2 + 50.00u^4 - 281.23u^6 + 2113.70u^8\right)$$

SU(3)

$$\begin{aligned} f(1,u) &= -6u^4(1 + 7.33u^2 - 5.00u^3 + 91.62u^4 - \\ &\qquad -104.45u^5 + 1013.46u^6) \end{aligned}$$

$$\begin{aligned} f(2,u) &= -3u^8(1 + 9.00u^2 - 45.00u^3 + 92.08u^4 - \\ &\qquad -29.25u^5 + 587.85u^6 - 2404.83u^7 + 2577.14u^8) \end{aligned}$$

Free energy from free glueball gas

Leading orders correspond to exponentiated glueball masses [Münster 1981, Seo 1982]

$$f(N_t, u) = -\frac{1}{N_t}\left[e^{-m(A)N_t} + 2e^{-m(E)N_t} + \mathcal{O}(u^4)\right] \qquad \text{for } SU(2)$$

$$f(N_t, u) = -\frac{1}{N_t}\left[e^{-m(A)N_t} + 2e^{-m(E)N_t} + 3e^{-m(T)N_t} + \mathcal{O}(u^4)\right]$$

$$\text{for } SU(N \geq 3)$$

Confirms expected hadron resonances gas picture in the pure gauge theory[Karsch, Redlich, Tawfik 2003]

$$f(T) \sim -T \sum_n e^{-\frac{E_n}{T}}$$

Getting more information from the series

Improved convergence, critical coupling

Padé approximants

$$[L, M](u) = \frac{\sum_{l=0}^{L} b_l u^l}{1 + \sum_{m=1}^{M} c_m u^m}$$

Coefficients constrained by

$$f(u)\Big|_{N_t \text{ fixed}} = [L, M](u) + \mathcal{O}(u^{L+M+1})$$

Singularities = Zeroes of the denominator

$$1 + \sum_{m=1}^{M} c_m u^m = 0$$

Phase transition limits radius of convergence

Define specific heat by

$$C(u) = u^2 \frac{d^2}{du^2} f(u)$$

Near the phase transition

$$C(u) \sim \frac{1}{|u - u_c|^\alpha} \qquad 0 < \alpha < 1$$

Padé singularities usually have exponent 1

Calculate Padé of

$$D(u) \equiv \frac{d}{du} \ln C(u) \sim \frac{1}{|u - u_c|}$$

Estimates for the critical couplings for SU(2)

	α	β_c	β_c (Monte Carlo)
$N_t = 1$	0.061(19)	0.92(15)	0.860(10)
$N_t = 2$	0.052(19)	1.65(35)	1.880(3)
$N_t = 3$	0.078(50)	2.26(63)	2.177(3)
$N_t = 4$	0.102(37)	2.66(54)	2.299(6)

Table: Estimates for the critical coupling β_c and the critical exponent of the deconfinement phase transition. The exponent for 3d Ising universality is $\alpha = 0.12$. The Monte-Carlo results are taken from [Fingberg et al., 1992] for $N_t = 2, 3, 4$ and from [Velytsky, 2007] for $N_t = 1$

Energy density in comparison with MC

Energy density per plaquette

$$\varepsilon(\beta) = -\frac{1}{6}\frac{d}{d\beta}f(\beta)$$

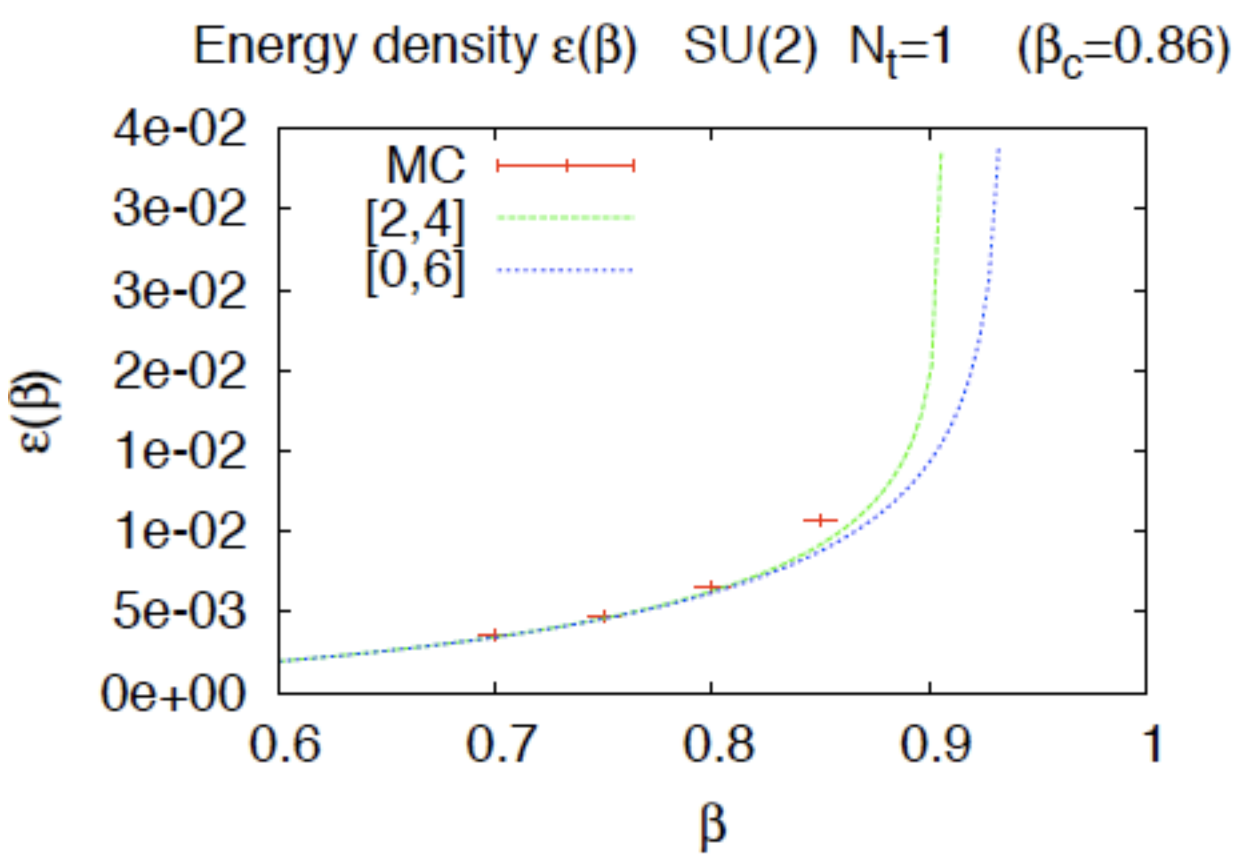
From Monte-Carlo simulations

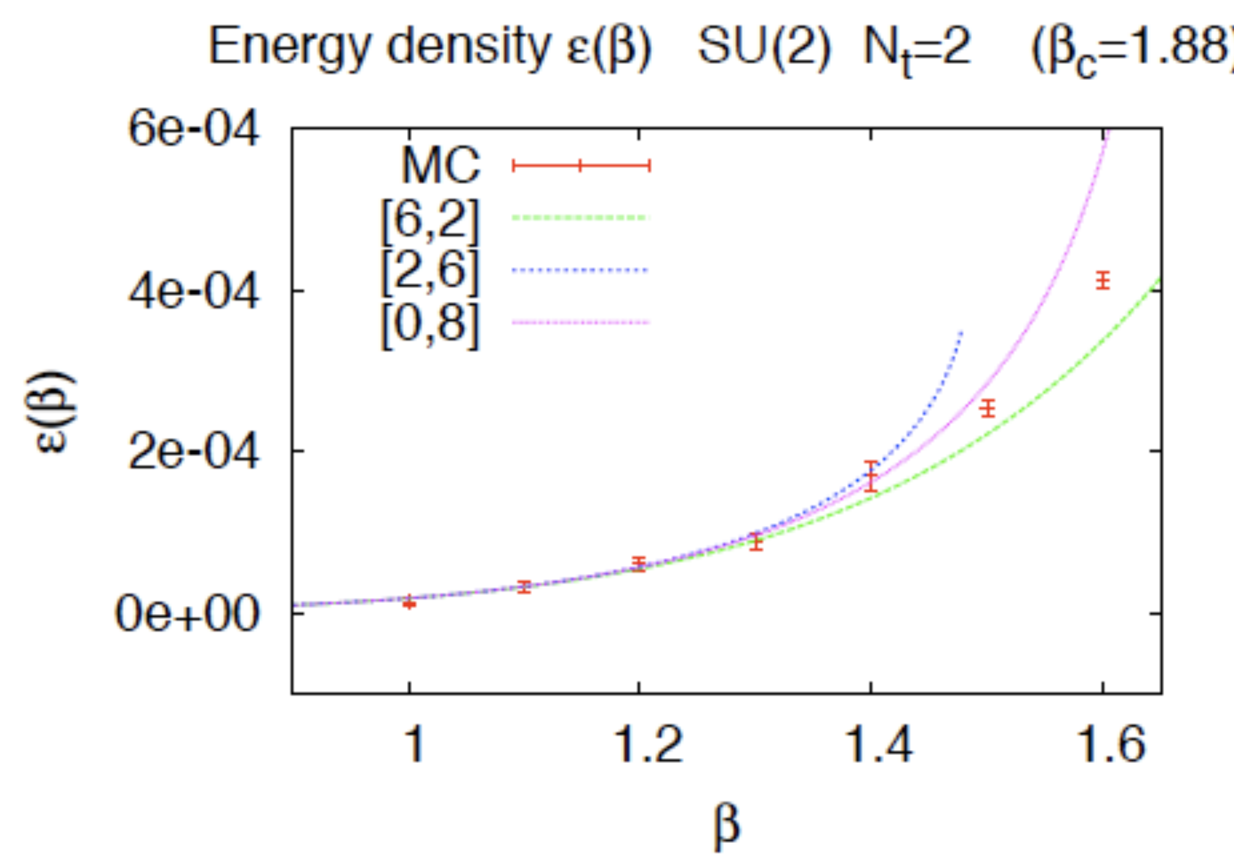
$$\varepsilon(\beta) = \langle \mathrm{Tr}\, U_p \rangle_{N_t} - \langle \mathrm{Tr}\, U_p \rangle_{N_t < \infty}$$

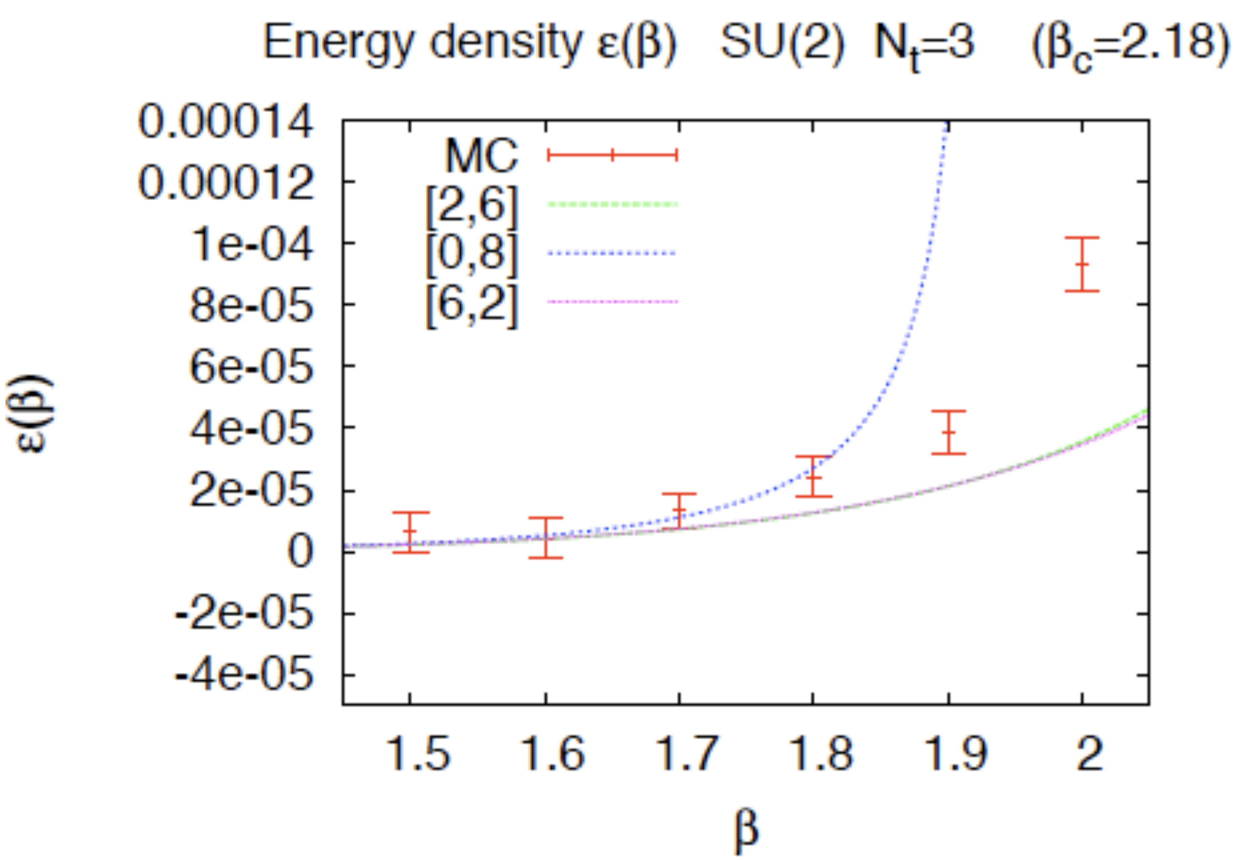
Infinite volume lattice: 12^4

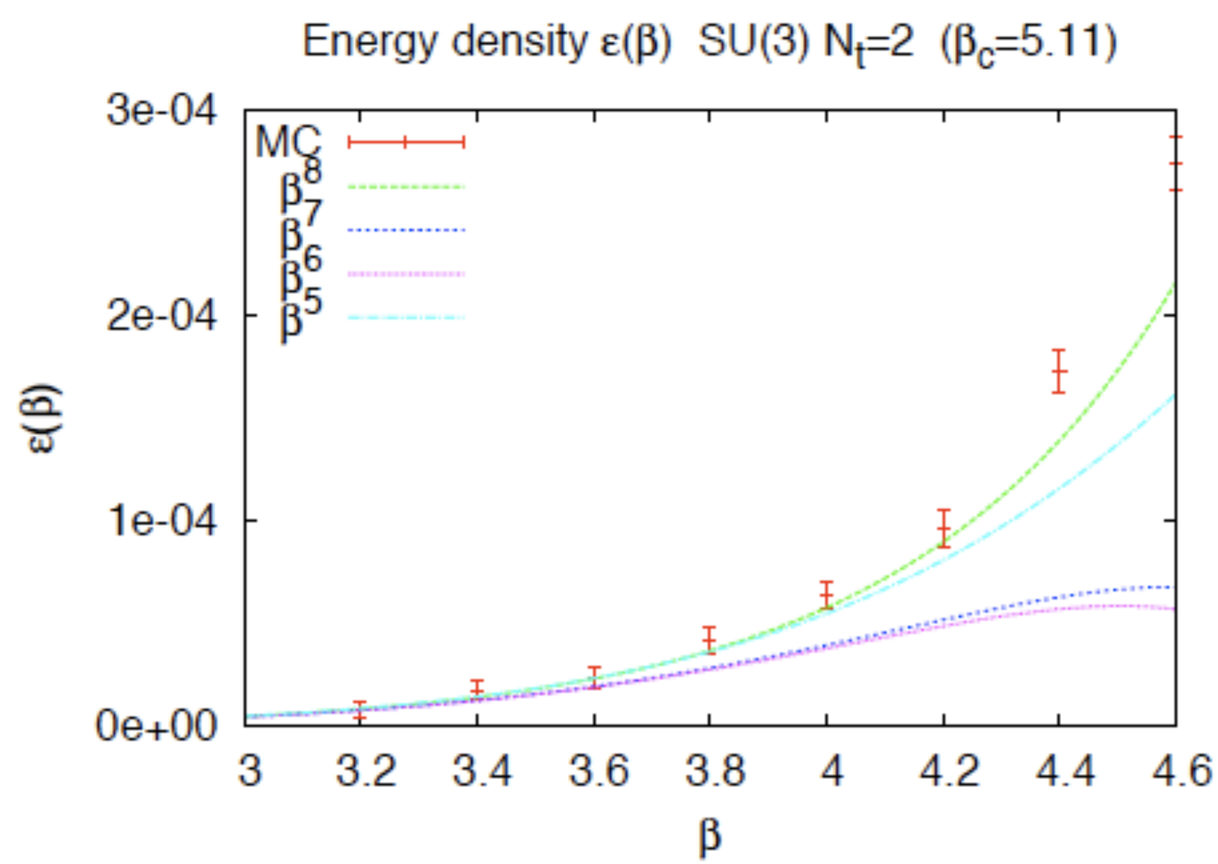
Finite T lattice: $12^3 \times N_t$

Quality check: comparison with Monte Carlo






Energy density ε(β) SU(2) N_t=3 (β_c=2.18)
MC
[2,6]
[0,8]
[6,2]
0.00014
0.00012
1e-04
8e-05
6e-05
4e-05
2e-05
0
-2e-05
-4e-05
1.5
1.6
1.7
1.8
1.9
2
ε(β)
β


Energy density ε(β) SU(3) N_t=2 (β_c=5.11)
MC
β8
β7
β6
β5
3e-04
2e-04
1e-04
0e+00
3
3.2
3.4
3.6
3.8
4
4.2
4.4
4.6
ε(β)
β

Temperature dependence of screening masses

Spatial glueball correlators

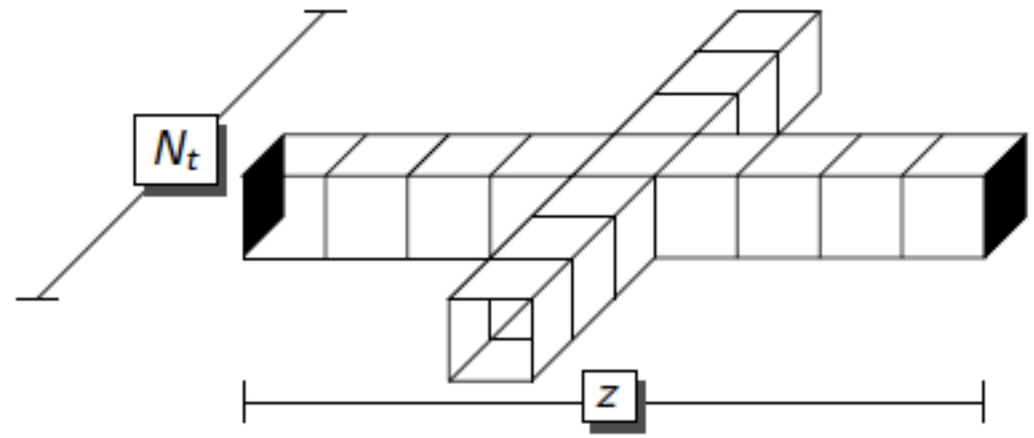
$$C(z) = \langle \mathrm{Tr}\, U_{p_1}(0)\ \mathrm{Tr}\, U_{p_2}(z) \rangle = N^2 \frac{\partial^2}{\partial\beta_1 \partial\beta_2} \ln\ Z(\beta, \beta_1 \beta_2) \Big|_{\beta_{1,2}=\beta}$$

$$m = -\lim_{z\to\infty} \frac{1}{z} \ln\ C(z)$$

Mass difference at finite T

$$\Delta m(T) = m(T) - m(0) \quad = \quad -\lim_{z\to\infty} \frac{1}{z} [\ln C(T; z) - \ln C(0; z)]$$

Leading order graph at finite T



NLO result

$$\Delta m(T) = -\frac{2}{3} N_t\, u^{4N_t-6}\,(1+2u^2)$$

To be compared to Monte-Carlo result [Datta, Gupta, 2003]

$$\frac{\Delta m(T)}{m(0)} \lesssim 0.15 \qquad \text{at} \qquad T \simeq 0.97\, T_c$$

Including fermions

Wilson's action (without indices)

$$S_q = \sum_x \left\{ \overline{\psi}(x)\psi(x) - K_f \sum_\mu \overline{\psi}(x+\mu)(1+\gamma_\mu)U(x;\mu)\psi(x) \right\}$$

Grassmann integration leads to

$$\int D\psi D\overline{\psi} e^{-S_q} = \det Q[U] \equiv \exp\left(-S_{eff}\right)$$

$$Q[U]_{yx} \equiv \delta_{yx} - K_f M[U]_{yx}$$

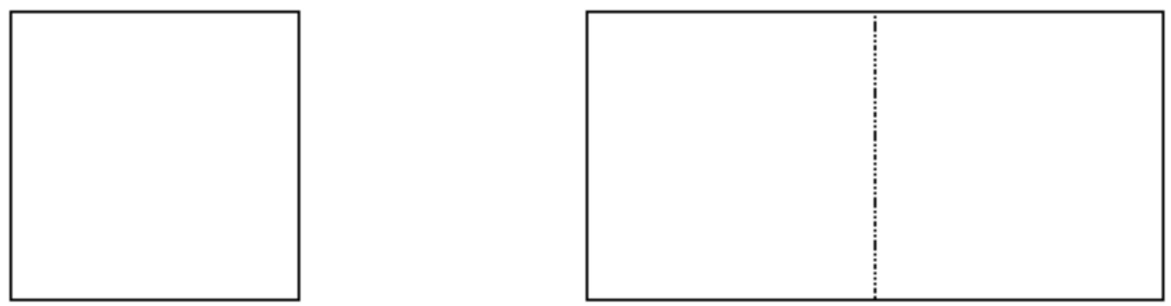
Hopping parameter expansion

$$S_{eff} = -\mathrm{Tr}\ln(1 - K_f M[U]) = \sum_{l=1}^{\infty} \frac{K_f^l}{l} \mathrm{Tr}\, M[U]^l$$

Leading orders of the hopping expansion

Due to the δ-functions, contributing terms in the effective action have to correspond to closed loops on the lattice

Some lowest order graphs to $\mathcal{O}(K_f^4)$ and $\mathcal{O}(K_f^6)$

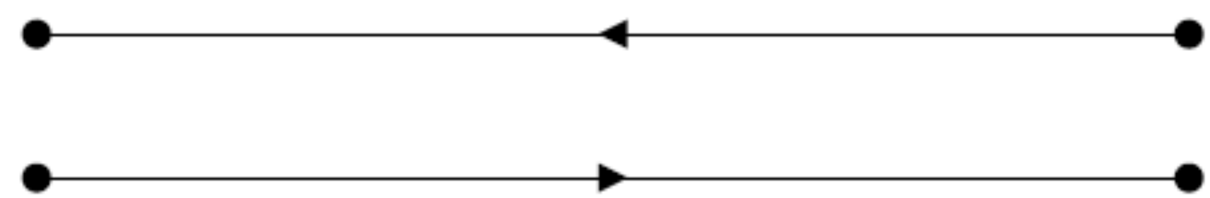


Meson and baryon masses from exponential decay of suitable operators, e.g.

$$m(\pi^+) = -\ln 2K_u - \ln 2K_d$$
$$m(p) = -2\ln 2K_u - \ln 2K_d$$

Temperature effects

Due to the compactification, possible new graphs are those which wind around the temporal dimension



With the defintion of the Polyakov loop

$$L(\vec{x})) = \mathrm{Tr} \prod_{t=1}^{N_t} U(t, \vec{x}; 0)$$

the contribution of the leading orders to the action can be written as

$$S_{Pl} = -(2K_f)^{N_t} \sum_{\vec{x}} \left[L(\vec{x}) + L^\dagger(\vec{x}) \right]$$

Now two character expansions, compute again $f(T) = f(N_t) - f(\infty)$

The free energy density

Leading order result for two flavours

$$
\begin{aligned}
f(N_t;K_f) &= -\frac{4}{N_t}\Big[(2K_u)^{2N_t}+(2K_d)^{2N_t}+2(2K_u)^{N_t}(2K_d)^{N_t}\Big] \\
&- \frac{1}{N_t}\Big[8(2K_u)^{3N_t}+8(2K_d)^{3N_t}+12(2K_u)^{N_t}(2K_d)^{2N_t} \\
&\qquad +12(2K_u)^{2N_t}(2K_d)^{N_t}\Big]
\end{aligned}
$$

This can be written as

$$
\begin{aligned}
f(N_t;K_f) &= -\frac{1}{N_t}\Big[\sum_{0^-} e^{-m(0^-)N_t}+3\sum_{1^-} e^{-m(1^-)N_t}\Big] \\
&\quad -\frac{1}{N_t}\Big[2\sum_{\frac{1}{2}^+} e^{-m(\frac{1}{2}^+)N_t}+4\sum_{\frac{3}{2}^+} e^{-m(\frac{3}{2}^+)N_t}\Big]
\end{aligned}
$$

HRG arises as strong coupling effective theory!

Refinements: investigation of the phase transition

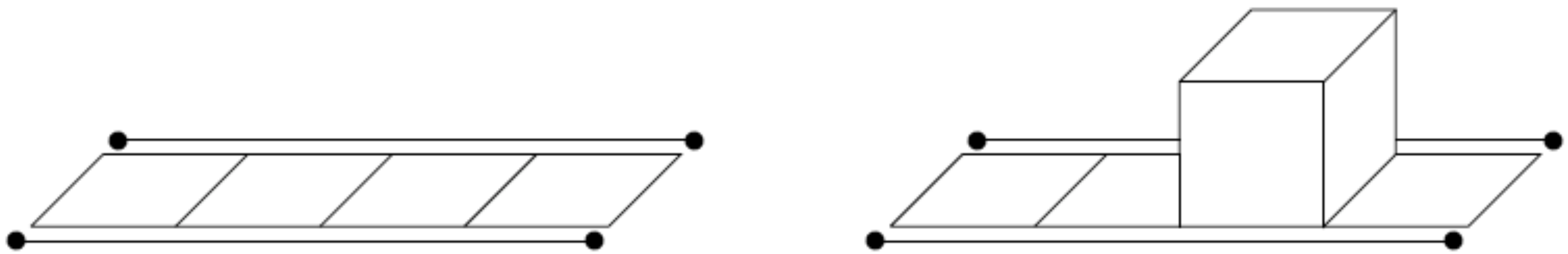
Langelage, Philipsen 09

Easier to calculate and better convergence: Polyakov loop susceptibility

$$-S(J) = \frac{\beta}{2N} \sum_p \left(\mathrm{tr}\, U_p + \mathrm{tr}\, U_p^\dagger\right) + J \sum_{\vec{x}} \left(L_{\vec{x}} + L_{\vec{x}}^\dagger\right)$$

$$\chi_L = \frac{1}{V} \frac{\partial^2}{\partial J^2} \ln Z(J)\Big|_{J=0}$$

Again double character expansion, leading graphs:



Examples for corrections

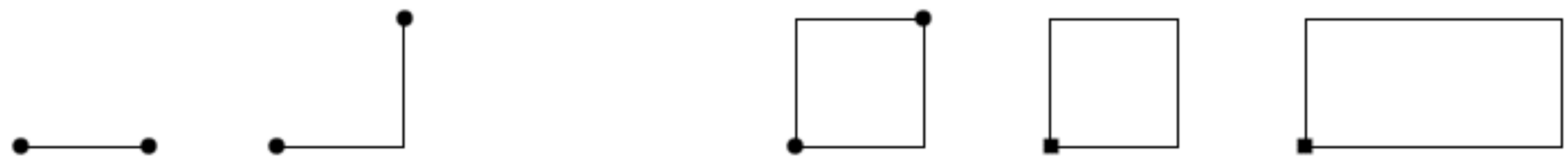


Figure 3: Left: Self avoiding walks with two fundamental Polyakov loops. Right: Self avoiding polygons with one adjoint or two fundamental Polyakov loop.

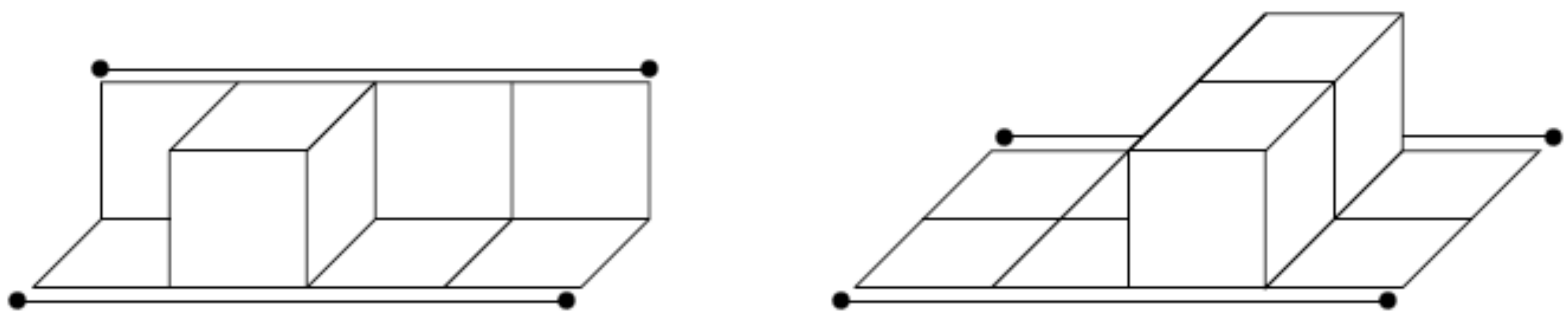


Figure 4: Examples of corrections to self avoiding walks of length $L = 2$ and $N_\tau = 4$.

Results for SU(2)

$$\begin{aligned}\chi_L(1,u) &= 1 + 6\,u + 30\,u^2 + 150\,u^3 + 738\,u^4 + 3622\,u^5 + \frac{52982}{3}u^6 + \\ &+ \frac{773434}{9}u^7 + \frac{11239612}{27}u^8 + \mathcal{O}\left(u^9\right), \\ \chi_L(2,u) &= 1 + 6\,u^2 + 30\,u^4 + 222\,u^6 + 1218\,u^8 + \frac{24602}{3}u^{10} + \mathcal{O}\left(u^{12}\right),\end{aligned}$$

etc.

$$\chi_L \sim \frac{1}{(u_c - u)^\gamma}\,, \qquad D_\chi(u) \equiv \frac{d}{du}\ln\left(\chi_L\right) \sim \frac{\gamma}{(u_c - u)}$$

Model DLog by Pade's as before

Results for Nt=1

Padé	u_c	β_c	γ
[5, 2]	0.21055	0.86825	1.167
[3, 3]	0.20967	0.86439	1.138
[4, 2]	0.20957	0.86396	1.146
[2, 4]	0.20987	0.86527	1.135
[3, 2]	0.20927	0.86264	1.126
[2, 2]	0.20820	0.85796	1.102
Mean	0.2095(12)	0.864(5)	1.14(3)

unbiased

MC: $\beta_c = 0.8730(2)$

Padé	u_c	β_c
[6, 2]	0.21221	0.87553
[4, 3]	0.21159	0.87281
[2, 5]	0.21138	0.87189
[3, 3]	0.21229	0.87588
[2, 4]	0.21238	0.87628
[4, 2]	0.21279	0.87808
[3, 2]	0.20986	0.86523
[2, 3]	0.21464	0.88621
[2, 2]	0.21495	0.88757

Padé	γ_1	γ_2
[3, 4]	1.1250	1.2378
[4, 3]	1.1244	1.2331
[2, 5]	1.1246	1.2157
[5, 2]	1.1244	1.2208
[3, 3]	1.1225	1.2579
[2, 4]	1.1236	1.2661
[4, 2]	1.1244	1.2950
[3, 2]	1.1240	1.2308
[2, 3]	1.1238	1.2215

biased

$$\bar{\beta}_c = 0.877(11)$$

QCD: SU(3) to leading order hopping expansion

$$-S_q^{eff} = \sum_{\mathbf{x}} \left[h(\kappa) e^{\mu} \,\mathrm{tr}\, W_{\mathbf{x}} + h(\kappa) e^{-\mu} \,\mathrm{tr}\, W_{\mathbf{x}}^{\dagger} \right]$$

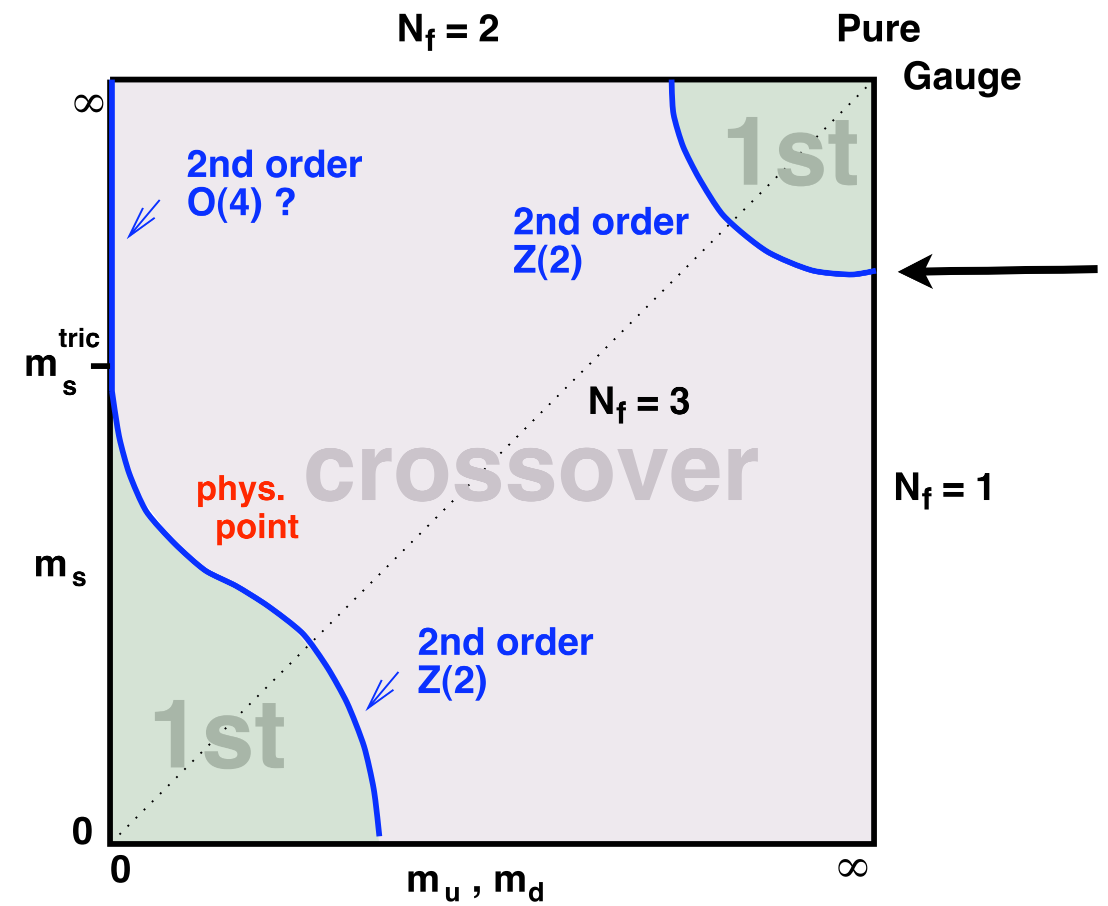
$$h(\kappa) = 2N_f (2\kappa)^{N_\tau}$$

Double character expansion as before

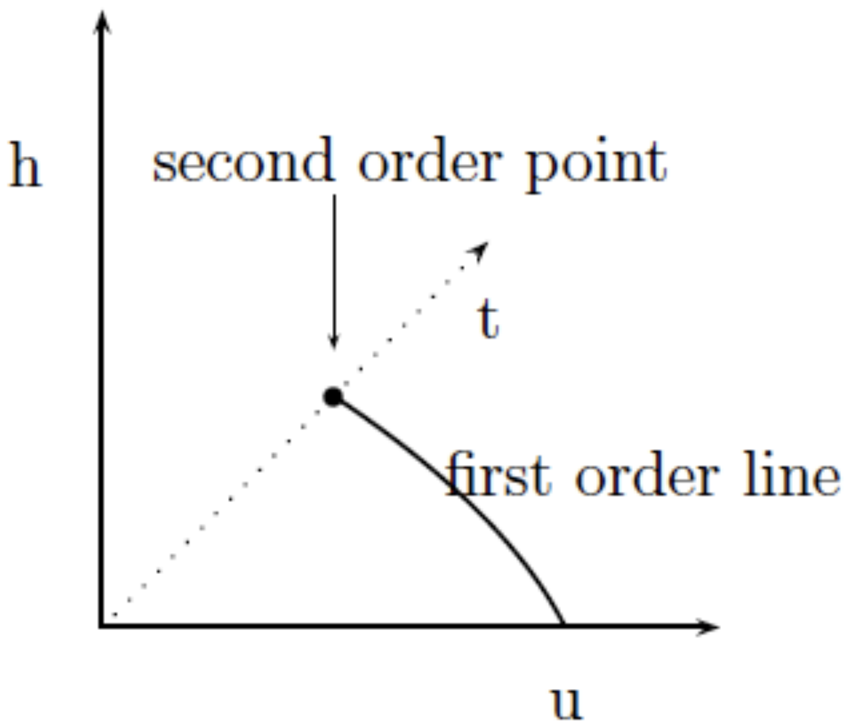
$$
\begin{aligned}
\chi_L(u,h) \;=\;& \Bigg[1 + ch + \left(-\frac{4}{3}c^3 + \frac{1}{2}c\right)h^3 + \left(-\frac{5}{3}c^4 + \frac{4}{3}c^2 - \frac{7}{24}\right)h^4 \\
&+ \left(\frac{2}{15}c^5 + \frac{1}{3}c^3 - \frac{1}{8}c\right)h^5 + \left(\frac{28}{15}c^6 - \frac{7}{5}c^4 - \frac{7}{120}c^2 + \frac{119}{720}\right)h^6\Bigg] \\
&+ \Bigg[6 + 18ch + \left(6c^2 + 3\right)h^2 + \left(-40c^3 + 15c\right)h^3 \\
&+ \left(-90c^4 + 66c^2 - \frac{69}{4}\right)h^4 + \left(-\frac{32}{5}c^5 - 8c^3 + 6c\right)h^5\Bigg]u \\
&+ \Bigg[30 + 180ch + \left(144c^2 + 72\right)h^2 + \left(-760c^3 + 285c\right)h^3 \\
&+ \left(-\frac{5985}{2}c^4 + \frac{8985}{4}c^2 - \frac{4485}{8}\right)h^4\Bigg]u^2 \\
&+ \Bigg[150 + 1470ch + \left(\frac{4113}{2}c^2 + \frac{4113}{4}\right)h^2 \\
&+ \left(-6856c^3 + 2571c\right)h^3\Bigg]u^3 \\
&+ \Bigg[786 + 10752ch + \left(\frac{1131747}{32}c^2 + \frac{1088547}{64}\right)\Bigg]u^4 \\
&+ \Bigg[4011 + 73521ch\Bigg]u^5 + \frac{152247}{8}u^6,
\end{aligned}
$$

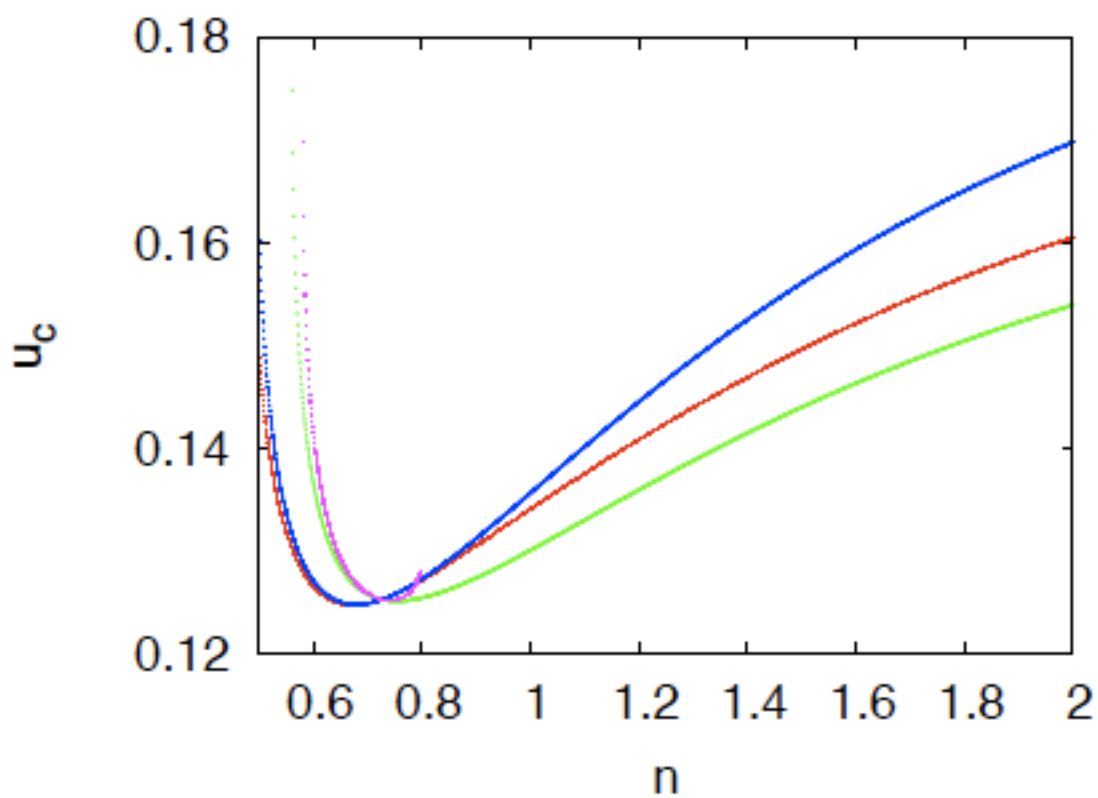
Now search for critical phase transition

analysis for Nt=1



$$\chi_L(t) \sim \frac{1}{(t_c - t)^\lambda}$$



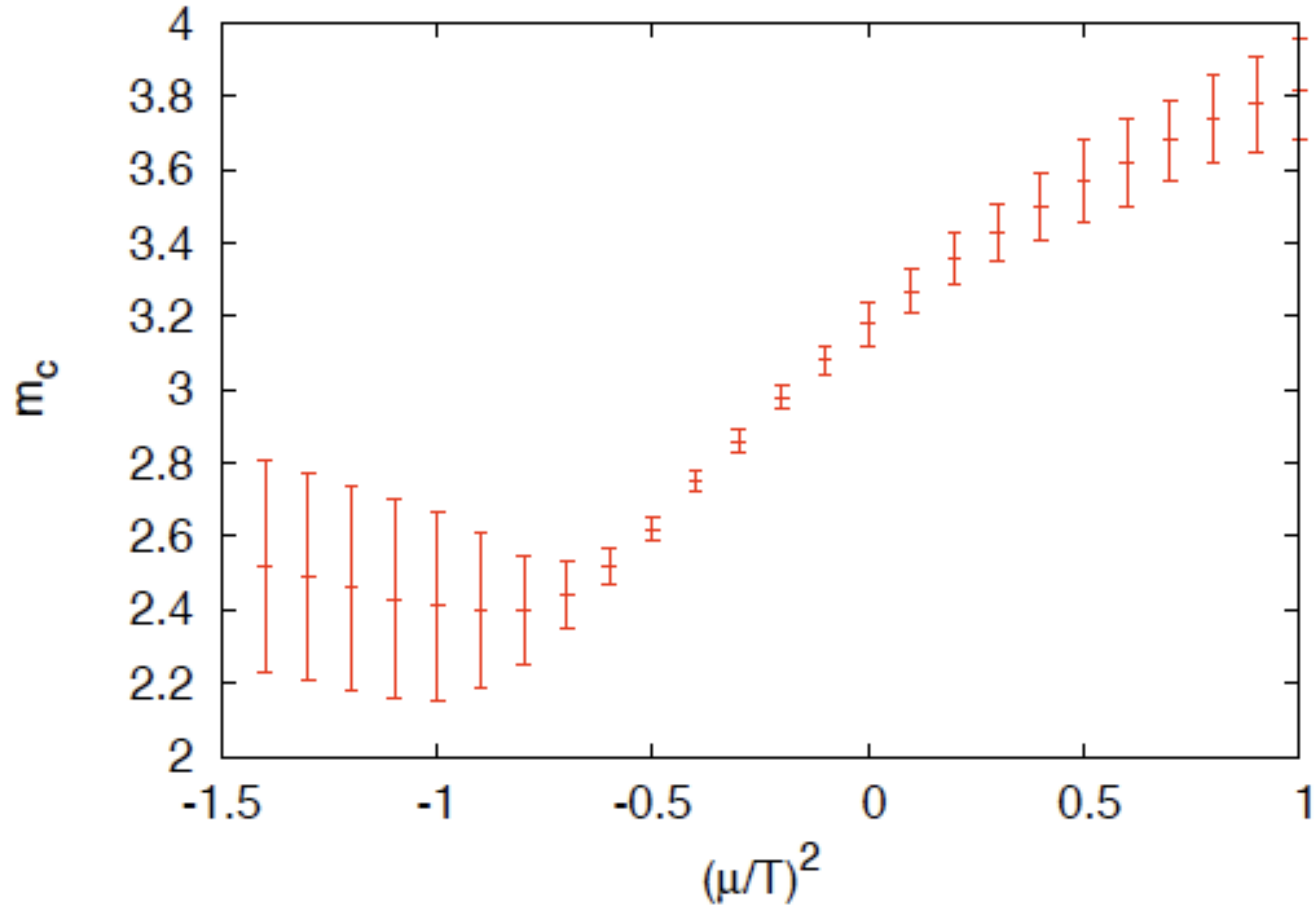


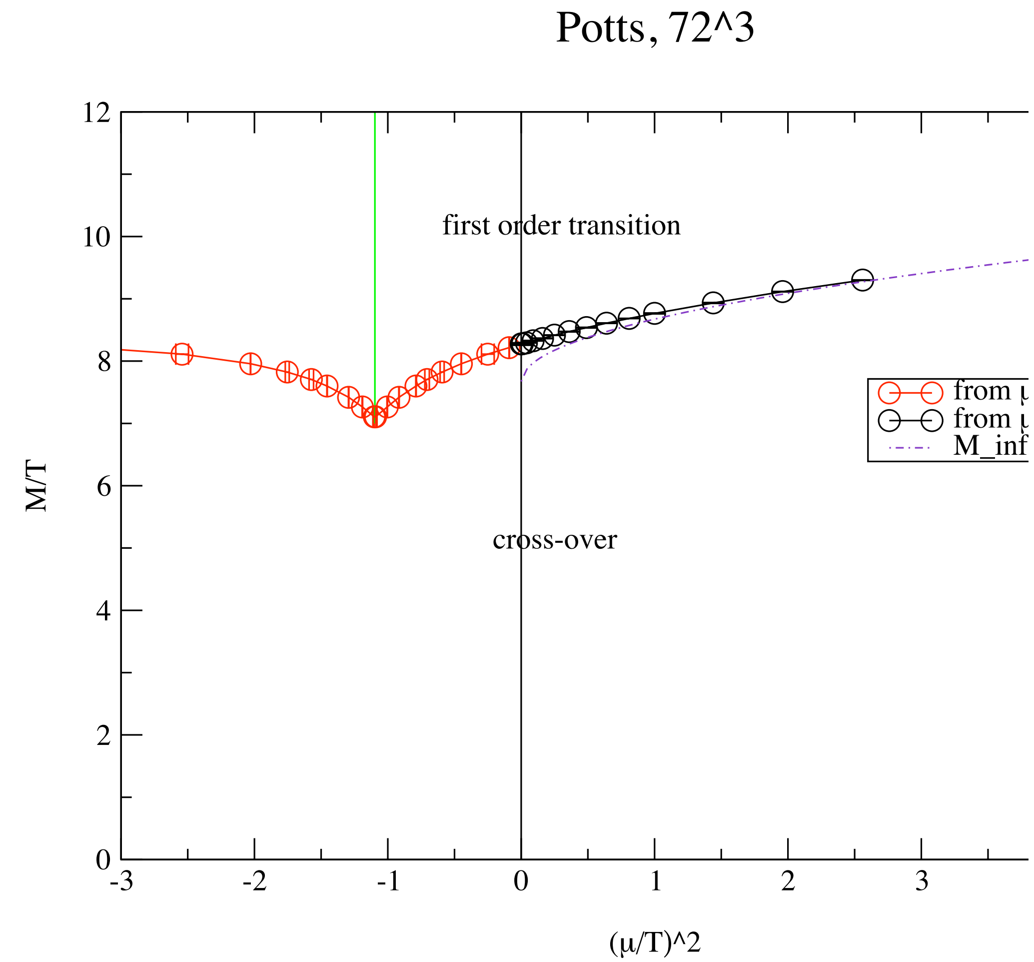
unbiased $n = 0.730(16), \quad \overline{t_c} = 0.172(4), \quad \overline{\lambda} = 1.03(3)$

bias with 3d Ising exponent for better accuracy

$$
\begin{aligned}
N_f = 1: \quad \kappa_c &= 0.062(4), & m_c/T &= 2.08(7), \\
N_f = 2: \quad \kappa_c &= 0.031(2), & m_c/T &= 2.78(7), \\
N_f = 3: \quad \kappa_c &= 0.021(1). & m_c/T &= 3.17(10).
\end{aligned}
$$

Switching on chemical potential





Potts model de Forcrand et al. 06

$$m_c(\mu^2) = 3.18 + 0.94(1)\mu^2 - 0.34(1)\mu^4 + 0.037(18)\mu^6 + \ldots,$$

first order region shrinking with real chemical potential, transition weakens!

Z(N) transition in the imaginary direction visible!

Conclusions

- EoS and screening masses in confined phase from SCE
- Exponential smallness of pressure and near T-independence of screening masses are genuine strong coupling effects
- Inclusion of dynamical (heavy) quarks possible
- HRG emerges from the fully interacting theory in the strong coupling limit
- Deconfinement critical line clearly identifiable
- Generalisation to finite chemical potential straightforward