WIMP Dark Matter and the QCD Equation of State

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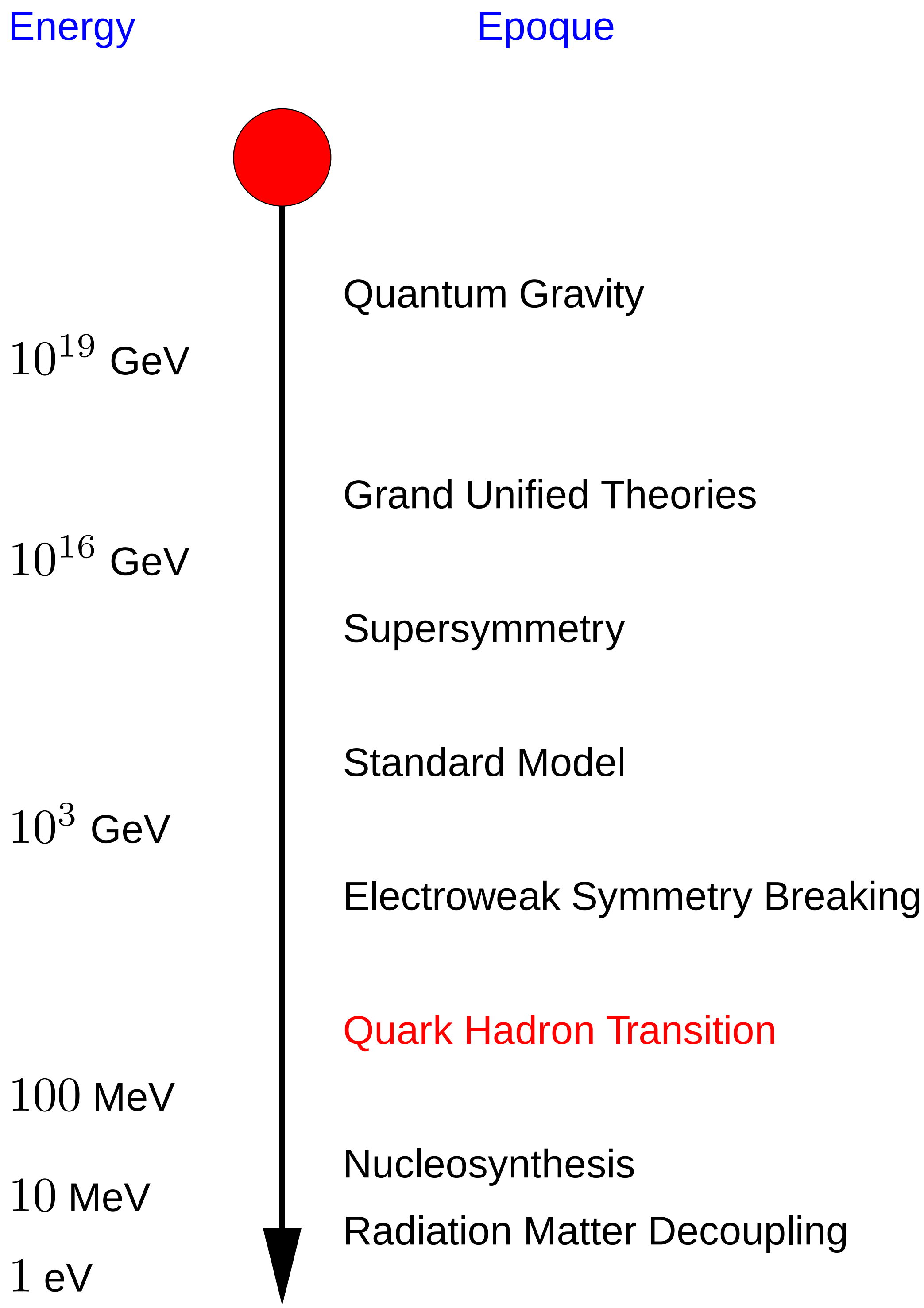
- Motivation + overview
- Cosmology I: expansion and contents of the universe
- Cosmology II: thermodynamics in the universe
- Calculating the QCD pressure
- Results for WIMP relic densities

M. Hindmarsh + O.P., Phys. Rev. D71 (2005) 087302

Overview

- 25% of the mass of the Universe is made of unknown “Dark Matter”
- Leading candidate: the WIMP (**W**eakly **I**nteracting **M**assive **P**article)
- Supersymmetric extensions of the Standard Model of particle physics predict WIMPs in mass range 10 – 1000 GeV.
- WIMP density today depends on masses, couplings, and the thermodynamics of the Universe at $T \sim$ 0.4 – 40 GeV . . .
- . . . when the major component of the Universe was a quark-gluon plasma .
- WIMP density known to 10% accuracy now, and 1% within 3–5 years.
- QCD important for precision cosmology

The Early Universe: Physics of Non-Abelian Plasmas



Cosmology I: Friedmann equation

Robertson-Walker metric: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2$

Spatial sections: $d\mathbf{x}^2 = (1 - Kr^2)^{-1} dr^2 + r^2 d\Omega^2$

Describes expanding space of constant curvature K.

General relativity: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$

- $T^{\mu\nu} = [p(T) + \rho(T)] u^\mu u^\nu - p(T) g^{\mu\nu}$
- $\mathbf{r} = a(t)\mathbf{x}$, $\dot{\mathbf{r}} = \mathbf{v} = (\dot{a}/a)\mathbf{r} \equiv H\mathbf{r}$ defines Hubble parameter H.
- Reduced Planck mass m_P: $m_P^2 = 1/8\pi G$, Planck mass M_P: $M_P^2 = 1/G$

00-component: $$H^2 + \frac{K}{a^2} = \frac{1}{3m_P^2}\rho + \frac{1}{3}\Lambda$$

(Friedmann)/H^2: $\frac{K}{a^2 H^2} = \frac{\rho}{3m_P^2 H^2} - 1 = \frac{\rho}{\rho_c} - 1 = \Omega - 1$

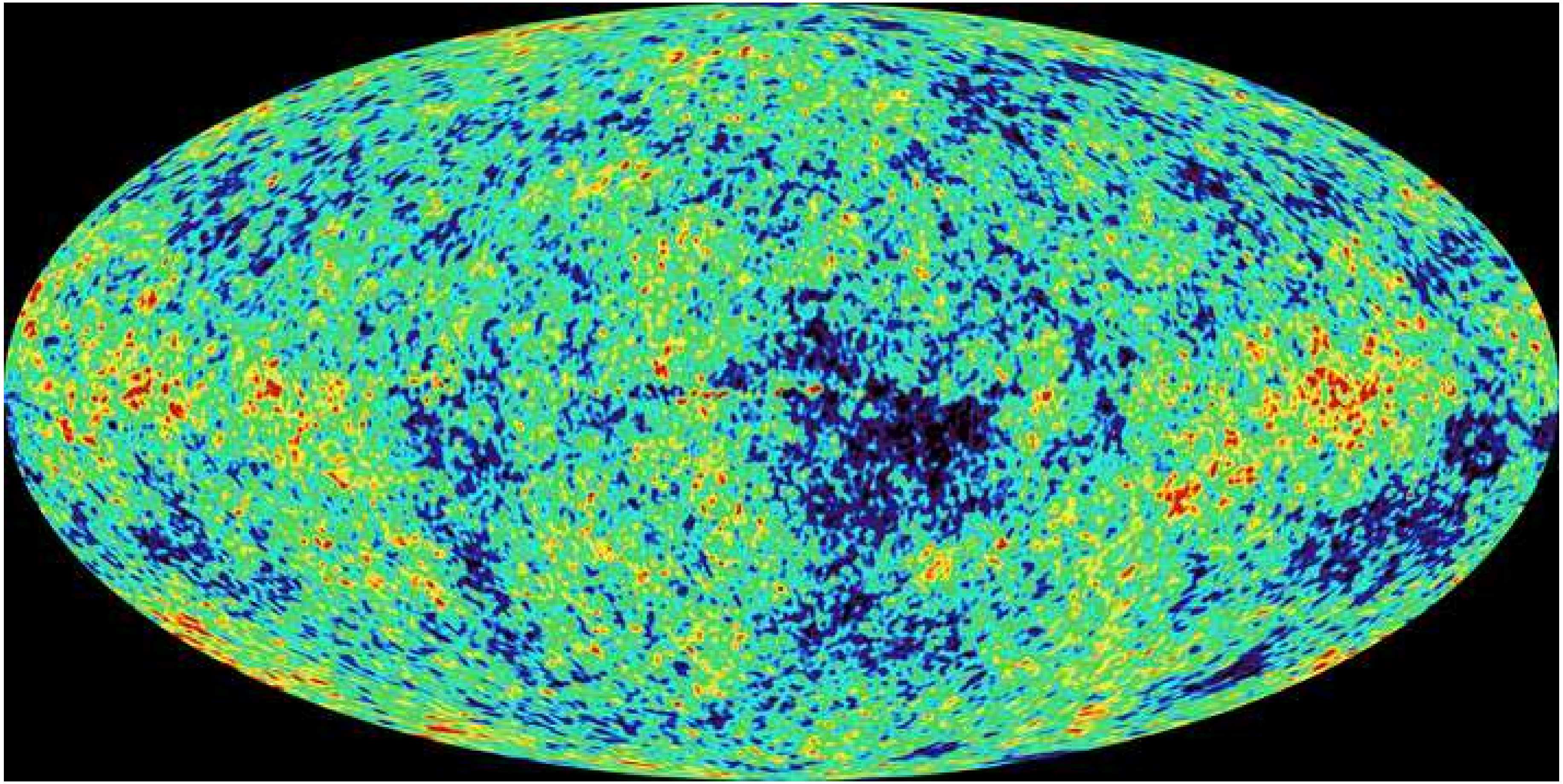
Define critical density $\rho_c = 3m_P^2 H^2$.

Define density parameters $\Omega_i = \rho_i / \rho_c$

Total density parameter $\Omega = \sum_i \Omega_i$ — $\begin{cases} \Omega > 1 & \text{closed} \\ \Omega = 1 & \text{flat} \\ \Omega < 1 & \text{open} \end{cases}$

Species i	Ω_i	ρ_i *scales as*
Photons γ	5×10^{-5}	a^{-4}
Neutrinos ν	< 0.015	a^{-4}, a^{-3}
Baryons b	0.044 ± 0.04	a^{-3}
Total matter m	0.27 ± 0.04	a^{-3}
Curvature K	< 0.02	a^{-2}
Dark energy Λ	0.73 ± 0.04	constant

WMAP: Cosmic Microwave Background temperature



-200 $< \Delta T <$ 200 μK

Measuring dark matter component $\Omega_c h^2$: fitting to a model

- Inflation motivates an 8-parameter model for the Universe ($\Omega = 1$)
- Fitting to the current data gives $h = 0.75$, $\Omega_c h^2$:

WMAP	WMAP+	Pub
0.127 ± 0.017	0.134 ± 0.006	Peiris et al 2003
$0.123^{+0.020}_{-0.018}$	$0.1438^{+0.0084}_{-0.0080}$	Tegmark et al 2003

- Cosmic Microwave background:
  $\Omega_b h^2 = 0.0224 \pm 0.0009$
- Big Bang Nucleosynthesis (abundances of He^4, D, He^3, Li^7):
  $\Omega_b h^2 = 0.0214 \pm 0.002$[a]

$$\Rightarrow \Omega_b \ll \Omega_c$$

[a] Kirkman et al astro-ph/0301006

Cosmology II: thermodynamics

Particle reaction rates large compared with expansion rate $H \propto 1/t$ ⇒equilibrium

$$n\langle\sigma v\rangle \gg H \qquad \begin{cases} \sigma & \text{Scattering cross-section} \\ n & \text{Number density of scatterers} \\ v & \text{Relative speed} \\ \langle\ldots\rangle & \text{Thermal average} \end{cases}$$

Early Universe very close to thermal equilibrium: expansion isentropic.

$$S = sa^3 = \text{const.} \qquad \text{Entropy density } s.$$

Thermodynamic relations:

$$s = \frac{dp}{dT}, \quad sT = \rho + p \qquad \left(\rightarrow \rho = T^2 \frac{d}{dT}\left(\frac{p}{T}\right)\right)$$

⇒Need to calculate pressure only

- **Pressure from non-interacting particles with g degrees of freedom:**

Relativistic Boson, $m \ll T$	$\times$ *(Fermion)*	*Non-relativistic,* $m \gg T$
$p_r = g\frac{\pi^2}{90}T^4$	$\left(\frac{7}{8}\right)$	$p_{nr} = gT\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}\exp(-m/T)$
$\rho_r = g\frac{\pi^2}{30}T^4$	$\left(\frac{7}{8}\right)$	$\rho_{nr} = \frac{m}{T}p_{nr} \gg p_{nr}$

$\Rightarrow$Equations of state: $p_r = \rho_r/3, \qquad p_{nr} \simeq 0$

- **Pressure from interacting particles:**

Define effective numbers of d.o.f. for energy & entropy densities:

$$\rho = \frac{\pi^2}{30}g_{\text{eff}}(T)T^4, \quad s = \frac{2\pi^2}{45}h_{\text{eff}}(T)T^3,$$

$\Rightarrow$Calculate from **Quantum Field Theory**

Decoupling or "freeze-out"

...when interaction rates become less than expansion rate

Weakly interacting, cross-section $\sigma \sim G_F^2 E^2$.

Estimate decoupling temperature: $n\langle\sigma v\rangle \simeq H = (\sqrt{g_{\rm eff}/3})T^2/m_P$

- **Neutrinos:** $\nu e \leftrightarrow \nu e$

  Cross-section: $\sigma \sim G_F^2 T^2$

  Gives $T_{f,\nu} \sim 1{\rm MeV}$

  Hot dark matter ($T_{f,\nu} \gg m_\nu$)

- **WIMPs:** $XX \leftrightarrow q\bar{q}$

  Cross-section: $\langle\sigma v\rangle \sim G_F^2 m_q^2$

  Gives $T_{f,X} \simeq m_X/25 \sim 0.4 - 40$ GeV

  Cold dark matter ($T_{f,X} \ll m_X$)

WIMP relic density calculation

Mass m, number density n, annihilations $XX \rightarrow \ldots$ with total cross-section σ.

Boltzmann equation (non-rel. particles): [a]

$$\dot{n} + 3\frac{\dot{a}}{a}n = -\langle \sigma v_{\rm M\o l} \rangle \left(n^2 - n_{eq}^2\right)$$

Let $x = m/T$, $Y = n/s$:

$$\frac{dY}{dx} = \left(\frac{\pi}{45}\right)^{\frac{1}{2}} g_*^{\frac{1}{2}}(T)(mM_{\rm P}) \langle \sigma v_{\rm M\o l} \rangle \left(Y^2 - Y_{\rm eq}^2\right) \frac{1}{x^2}$$

where:

$$g_*^{1/2}(T) = \frac{h_{\rm eff}}{g_{\rm eff}^{1/2}} \left(1 + \frac{T}{3} \frac{d \ln h_{\rm eff}}{dT}\right).$$

Approximate solution: $\langle \sigma v_{\rm M\o l} \rangle = a_{(0)} + a_{(1)}/x + \cdots$

$$\Omega_X h^2 \simeq \frac{10^{-10} \ \ {\rm GeV}^{-2}}{a_{(0)}} \frac{x_f}{g_*^{\frac{1}{2}}(T)}$$

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[a]Møller velocity: $v_{\rm M\o l} = \left((\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2\right)^{\frac{1}{2}}$ (Gondolo & Gelmini 1991)

Degrees of freedom for $T = 0.4 - 40$ GeV: mostly coloured

	species	m	g	species	m	g	
	γ	0	2	g	0	16	
	ν_e	$\lesssim 1$ eV	2	u	3 MeV	12	
	ν_μ	$\lesssim 1$ eV	2	d	7MeV	12	
	ν_τ	$\lesssim 1$ eV	2	s	76 MeV	12	
	e	0.5 MeV	4	c	1.2 GeV	12	
	μ	106 MeV	4	b	4.2 GeV	12	
	τ	1.7 GeV	4	t	174 GeV	12	
	W	80 GeV	6				
	Z	91 GeV	3				
T=40 GeV:			$\frac{7}{8}18 + 2$			$\frac{7}{8}60 + 16$	68.5/84.25
T=0.4 GeV:			$\frac{7}{8}14 + 2$			$\frac{7}{8}36 + 16$	47.5/61.75

Cannot ignore strong QCD interactions!

QCD, theory of strong interactions

$$\mathcal{L}_{\text{QCD}} = \frac{1}{4g^2} \sum_{a=1}^{N_c^2-1} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i [\gamma_\mu D_\mu + m_i] \psi_i \ .$$

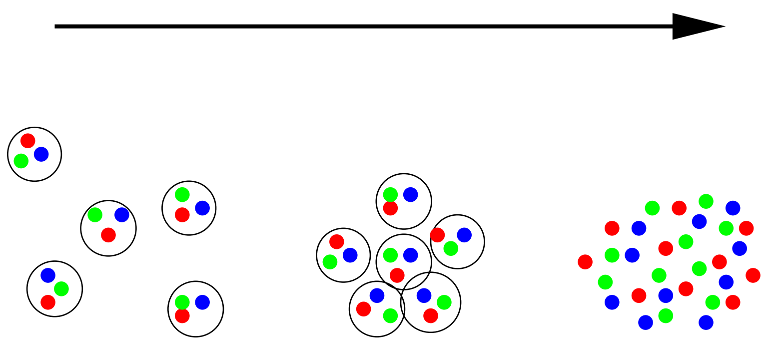
$N_c = 3; \quad N_f = 2, ..., 6; \quad m_u, m_d, m_s, m_c, m_b, m_t; \quad g \sim 1.$

⇒Confinement, non-perturbative physics

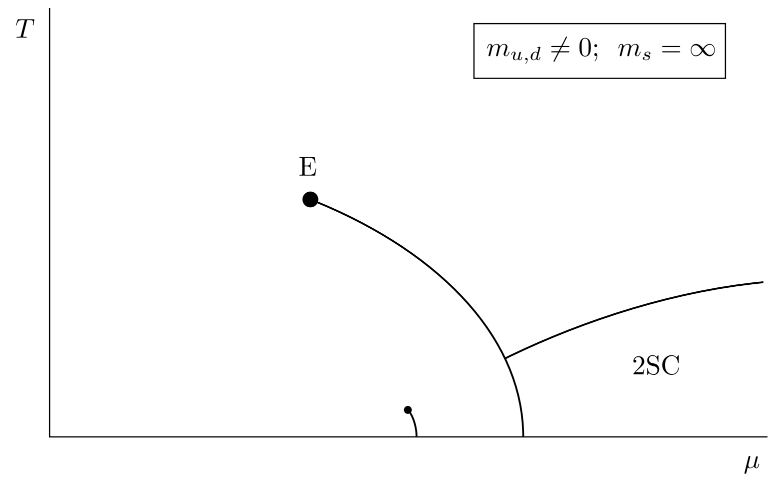
High temperature/density: phase transition (schematically)

asymptotic freedom: $\alpha_s(q \to \infty) \to 0$

T, μ_B



Hadrongas Quark-Gluon-Plasma



The QCD pressure

....is minus the grand canonical free energy density

$$p(T,\mu) \equiv -f(T,\mu) = -\lim_{V\to\infty} \frac{T}{V} \ln Z = \lim_{V\to\infty} \frac{T}{V} \ln \left\{ \mathrm{Tr} \left[\exp\left(-\frac{\hat{H}_{\text{QCD}} - \mu \hat{B}}{T} \right) \right] \right\}$$

For early universe: $\mu = 0$

QFT at finite T: two problems of perturbation theory

a) strong coupling $g(T)$

b) infrared problem at $T \neq 0$: $\frac{g^2}{e^{E/T}-1} \overset{E,p\ll T}{\sim} \frac{g^2 T}{m}$ $\Rightarrow$divergent for $m = 0$

- current dark matter calculations use simple potential models of QCD interactions:[a]
- here: lattice around T_c , hadronic gas model $T < T_c$, effective theory $T \gg T_c$
- in principle this can be done better!

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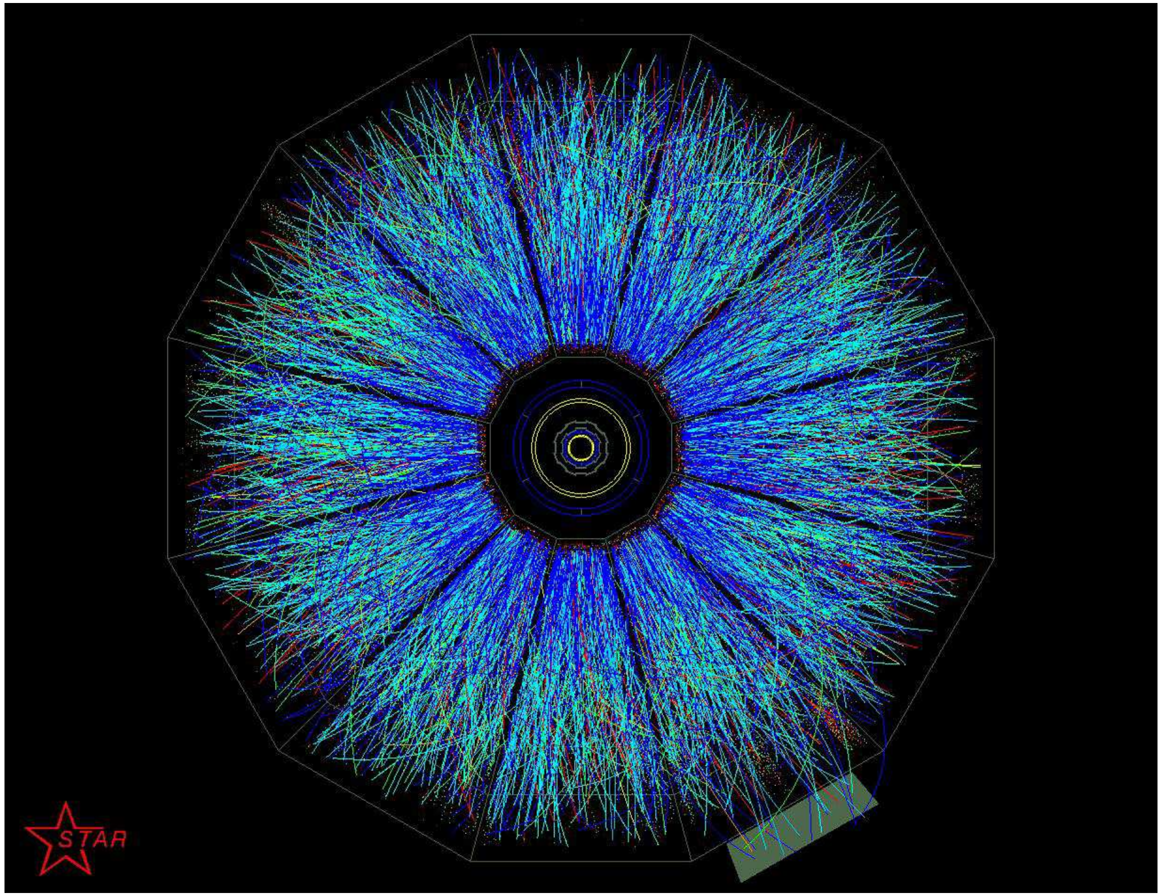
[a]Olive 1981

How experiment probes the phase transition & QGP....



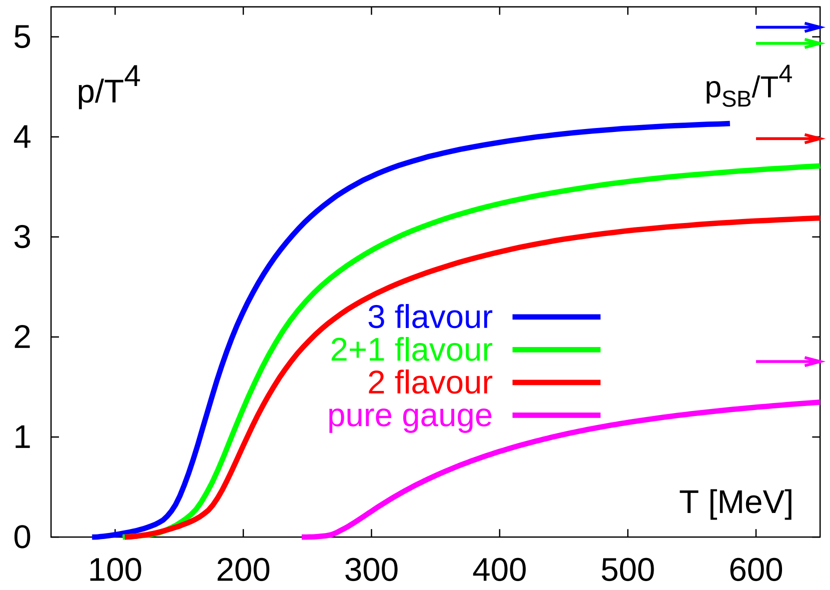
T_c and equation of state from the lattice

Karsch, Laermann, Peikert 2000

pure gauge :	$T_c = (271 \pm 2)$ MeV	continuum
chiral limit, $N_f = 2$:	$T_c = (173 \pm 8)$ MeV	on coarse lattice
$N_f = 3$:	$T_c = (154 \pm 8)$ MeV	"

Ideal gas: Stefan-Boltzmann

$$\frac{p_{SB}}{T^4} = \begin{cases} 3\frac{\pi^2}{90} & , T < T_c \\ (16 + \frac{21}{2} N_f)\frac{\pi^2}{90} & , T > T_c \end{cases}$$



$\Rightarrow T > T_c$: more degrees of freedom, but significant interaction! $\Rightarrow$sQGP?

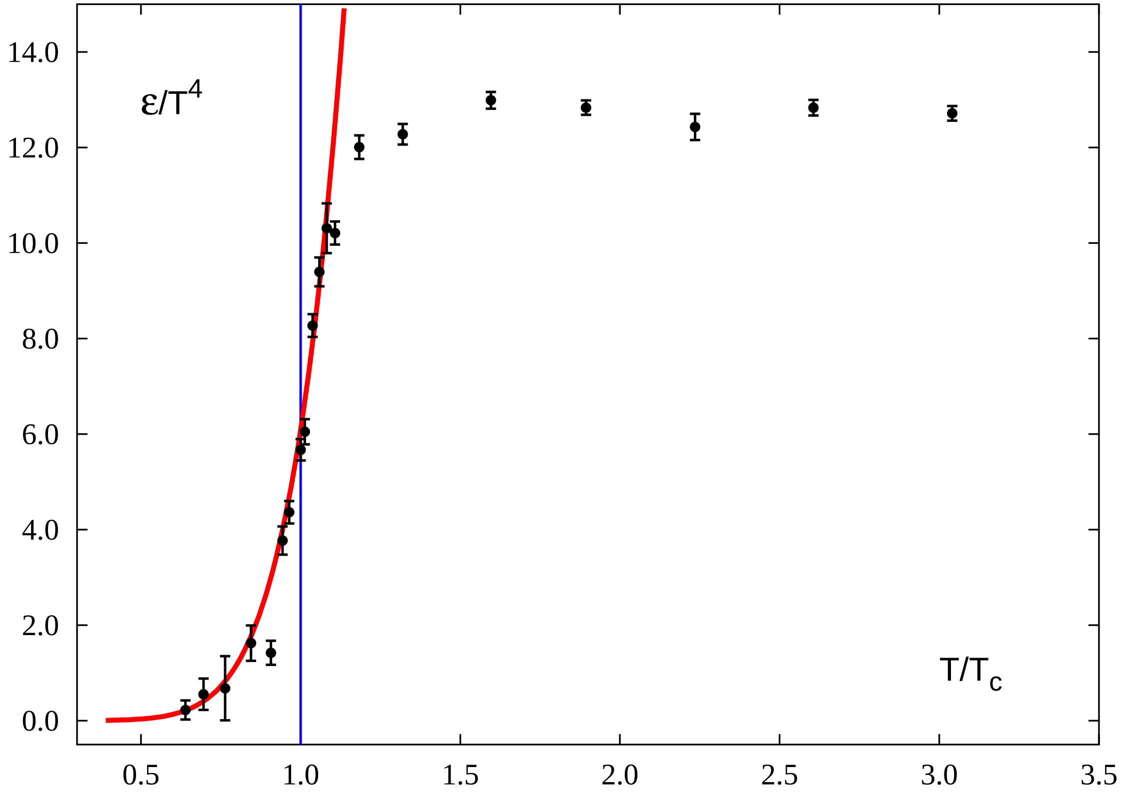
(not yet right, $m_\pi \sim 300$ MeV, coarse lattices etc.) limited to $T \lesssim 5T_c$

Pressure for $T < T_c$, hadronic resonance gas model

Below T_c, $q, \bar{q}, g$ confined into hadrons. Treat as ideal gas, including resonances

$$Z = \sum_n \exp(-E_n/T)$$

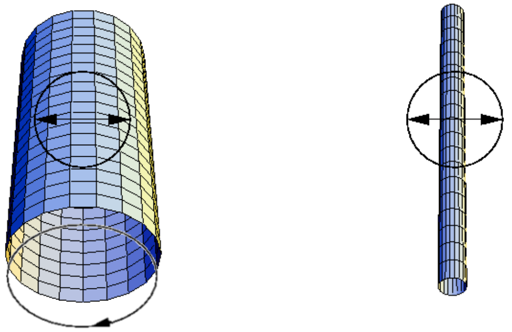
π^0	135 MeV
$\pi^\pm$	139 MeV
$K^\pm$	494 MeV
K^0	498 MeV
η	547 MeV
ρ	771 MeV
ω	782 MeV
$K^{*\pm}$	892 MeV
K^{*0}	896 MeV
p	938 MeV
n	940 MeV
η'	958 MeV
ϕ	1019 MeV
$\vdots$	$\vdots$



Karsch, Redlich, Tawfik 2003

Effective high temperature theory: dimensional reduction

Ginsparg;
Appelquist, Pisarski. . .



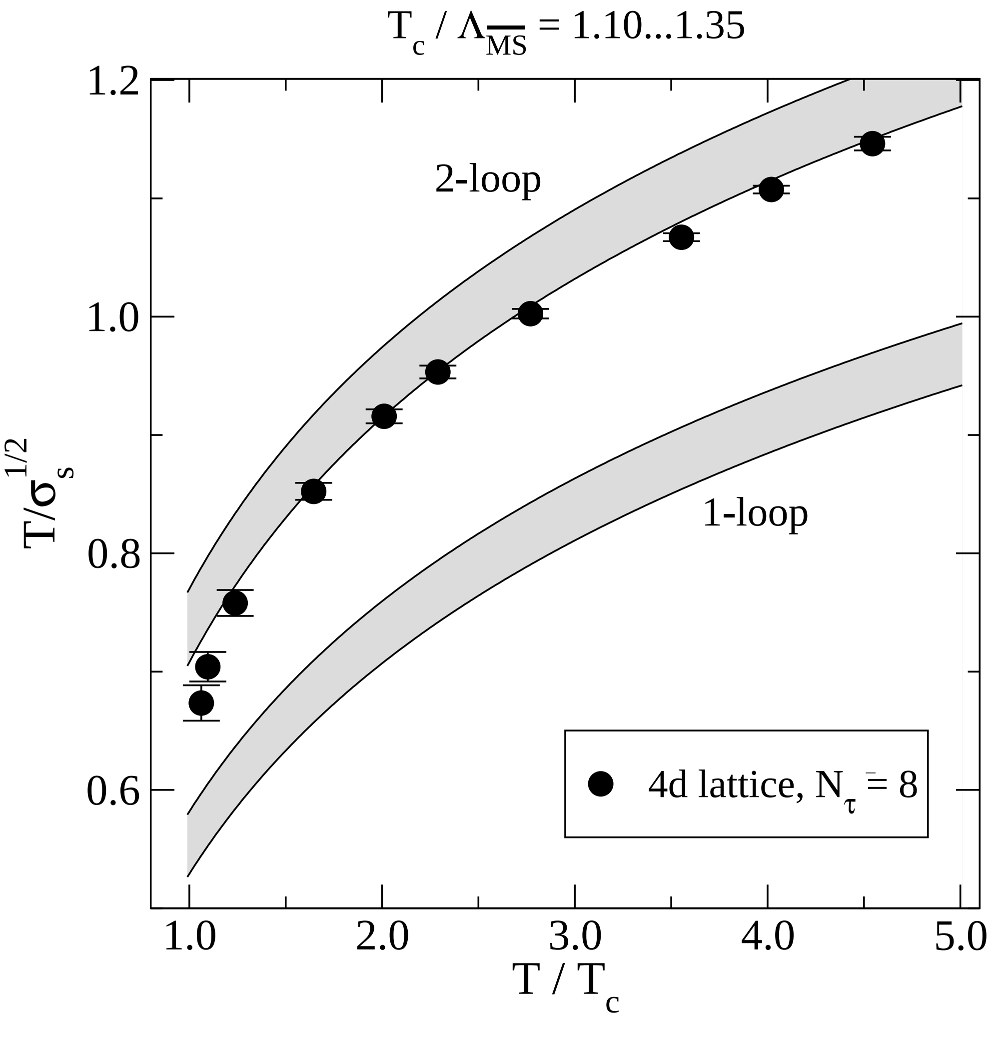
- Scale hierarchy $2\pi T, gT, g^2T$
- Perturbative integration of $p \gtrsim T$
  (Matsubara modes $\neq 0$, fermions!),
  expansion in $g(T)/(4\pi)$
- simulation of 3d effective theory
  IR modes $p \sim gT, g^2T$, bosonic;
  good $V \to \infty, a \to 0$ limits

dim.red.for pressure:[a] write $A^a_\mu(\tau, \mathbf{x}) = \sum_n A^a_{n\mu}(\mathbf{x})e^{i2\pi n\tau/\beta}$

1. Integrate out $A^a_{n\mu}(\mathbf{x})$ ($n \neq 0$, masses $2n\pi T$), $\psi, \bar{\psi}$: $O(g^0)$
2. Integrate out $A^a_0(\mathbf{x})$ (mass gT): $O(g^3)$
3. Integrate $A^a_i(\mathbf{x})$ (scale g^2T, non-perturbative): $O(g^6 \ln g, g^6)$

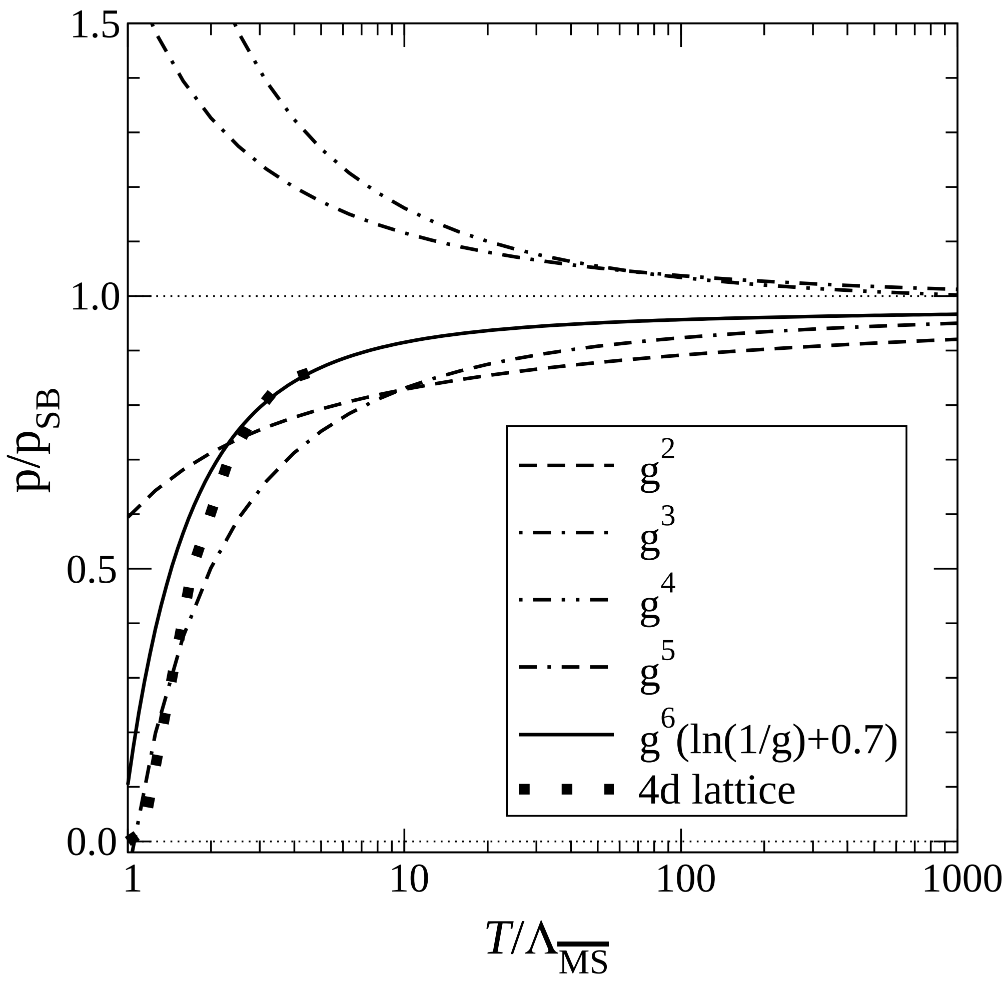
[a] Kajantie, Laine, Rummukainen & Schröder 2001

Example: spatial string tension computed in 3d and 4d ($N_f = 0$):



Pressure for $T > T_c$

Kajantie et al 2002, Karsch et al 2000



- Pure glue ($N_f = 0$)
- $\Lambda_{\overline{\mathrm{MS}}} = 237$ MeV
- $p_{\mathrm{SB}}(T) = 16\frac{\pi^2}{90}T^4$
- Fit order g^6 term

  (not yet computed)

  also fits slope!

N.B.: In principle no fit required, but four-loop computation ... will it match?

Lattice QCD pressure: not quite right yet

- Lattice data for $N_f \neq 0$ not at continuum limit or physical quark masses.
- Dimensional Reduction method matched only for pure glue $N_f = 0$
- Define QCD correction factor f from pure glue and scale to correct T_c:

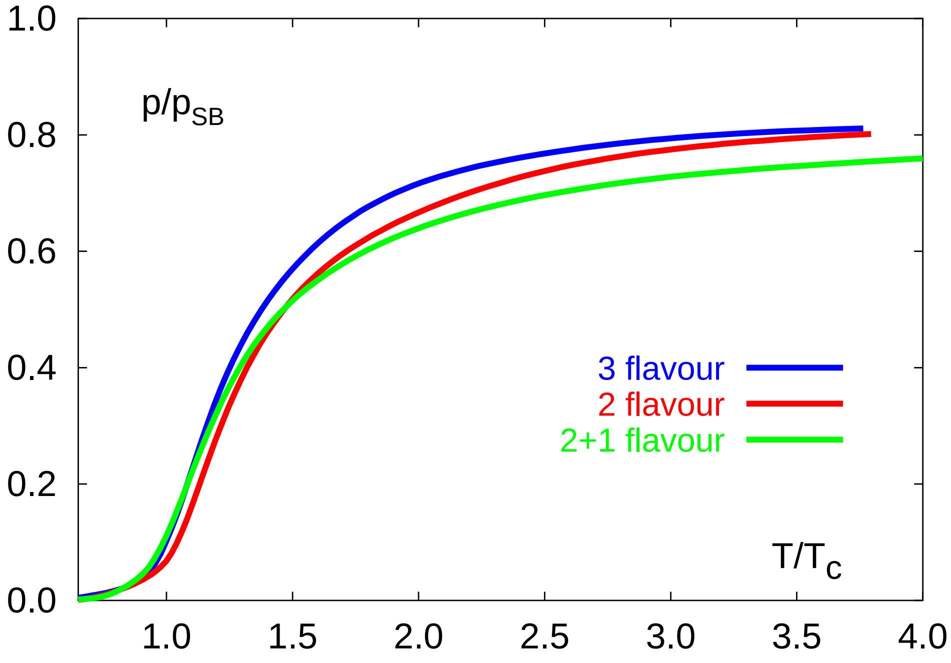
Definition:

$$f(T, N_f) = p(T, N_f)/p_{\rm SB}(T, N_f)$$

Observe approximate scaling:[a]

$$f(T, N_f) \simeq \bar{f}\,(T/T_c(N_f))$$

[a] Karsch, Laermann, Peikert 2000



Modelling the QCD pressure correction factor f

- Scale to $N_f = 0$, $T_c(0) = 269$ MeV using T_c listed below.

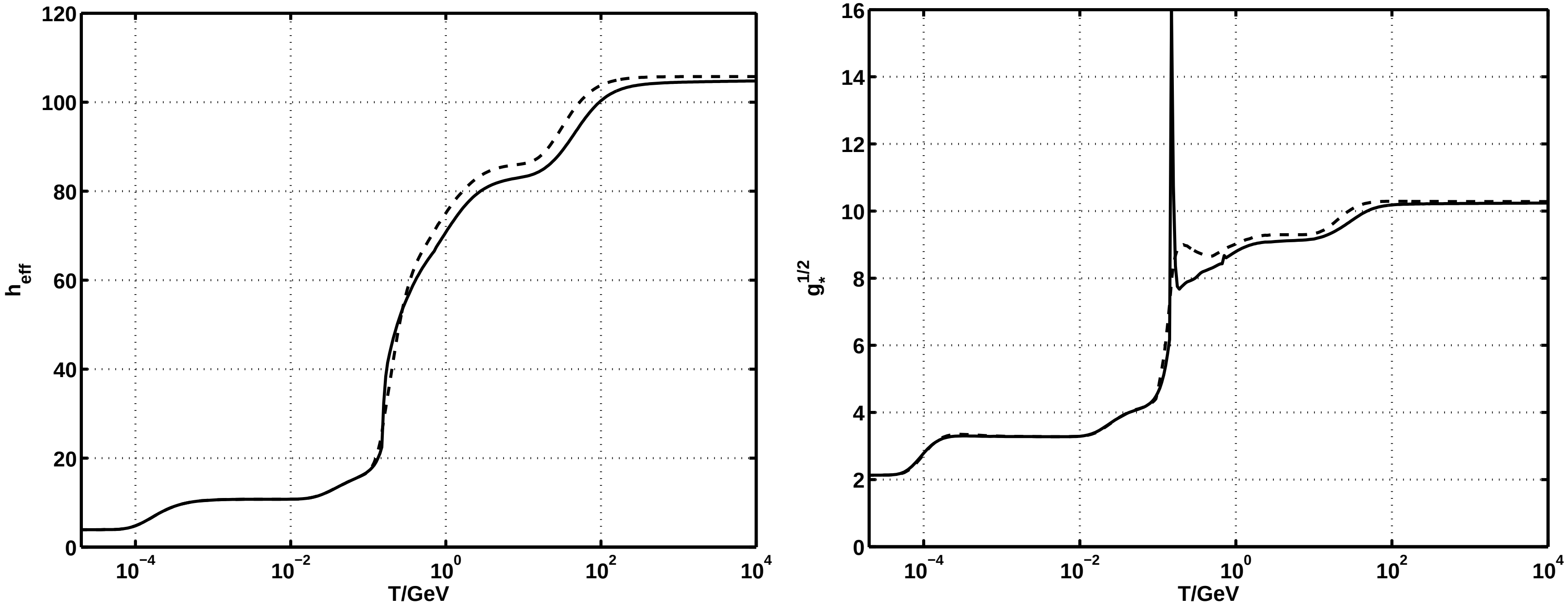
$T > 4.43T_c$	Dim. Red. $O(g^6 \ln(g))$, matched to lattice
$4.43T_c > T > T_{\rm HG}$	Interpolated lattice
$T_{\rm HG} > T$	Hadronic gas

EOS model	T_c/MeV	$T_{\rm HG}$/MeV	Notes
A	154	0	i.e. no hadronic gas
B	154	154	
C	185.5	200	chosen to keep $f(T)$ smooth.

- Multiply coloured degrees of freedom by f

Results: degrees of freedom

standard[a] (dashed) and modified (solid)



$$h_{\rm eff}(T) \qquad g_*^{1/2}(T) = \frac{h_{\rm eff}}{g_{\rm eff}^{1/2}}\left(1 + \frac{T}{3}\frac{d\ln h_{\rm eff}}{dT}\right)$$

Weak effect because p, ρ each shrink, but $p \approx \rho/3$ still

[a] Olive 1981

Calculating relic densities: packages

DarkSUSY[a] **MicrOMEGAs**[b]

- Both use same equation of state[c], incorporated as a look-up table.
- New look-up tables for DarkSUSY from www.pact.cpes.sussex.ac.uk/arXiv/0501232

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[a] Gondolo et al 2004

[b] Bélanger et al 2001, 2004

[c] Olive 1981

Relic densities in benchmark SUSY models

Battaglia et al 2003

QCD uncertainty estimated by scaling DR f by $^{(0.9)}_{(1.1)}$ and re-matching.

Model	m_χ/GeV	T_f/GeV	$\Omega_c h^2$ (DS)	$\Omega_c h^2{}^{(0.9f)}_{(1.1f)}$	Δ(%)
A'	242.83	9.8	0.0929	$0.0948^{(54)}_{(42)}$	$2.0^{2.6}_{1.4}$
B'	94.88	4.1	0.1213	$0.1242^{(56)}_{(31)}$	$2.4^{3.6}_{1.5}$
C'	158.09	6.5	0.1149	$0.1174^{(83)}_{(65)}$	$2.2^{2.9}_{1.5}$
G'	147.98	6.2	0.1294	$0.1323^{(33)}_{(13)}$	$2.2^{3.0}_{1.4}$
H'	388.38	16.0	0.1629	$0.1662^{(71)}_{(53)}$	$2.0^{2.6}_{1.5}$
I'	138.08	5.8	0.1319	$0.1351^{(62)}_{(40)}$	$2.4^{3.2}_{1.6}$
J'	309.17	12.6	0.0966	$0.0984^{(90)}_{(79)}$	$2.0^{2.5}_{1.4}$
L'	180.99	7.5	0.0988	$0.1011^{(18)}_{(03)}$	$2.3^{3.0}_{1.5}$

Conclusions

- The Universe is filled with Dark Matter: leading candidate is a (SUSY) WIMP
- QCD plasma effects increase WIMP density by few %
- Not insignificant: Planck may be able to reach $\Delta(\Omega_c h^2) = 0.0011$
- Precision QCD is necessary for precision cosmology
- Needed: lattice & dim. red. calculations for $N_f \neq 0$.
- Eqn. of State information from Heavy Ion Collisions?