Exercise 1: Regularization and Renormalization in ϕ^4-theory *(14+6=20 points)*

In the lecture you discussed that the leading order self-energy correction to the real scalar field propagator in ϕ^4-theory is given by

$$-\mathrm{i}\Pi(p^2) = (-\mathrm{i}\lambda_0)\frac{1}{2}\int \frac{\mathrm{i}}{k^2 - m_0^2 + \mathrm{i}\epsilon}\frac{d^4k}{(2\pi)^4}, \tag{1}$$

which is a divergent integral that needs to be regularized.

i) Use the procedure of Pauli-Villars regularization to solve the integal. Since the integral is $\sim \frac{k^3}{k^2}$ divergent, we have to introduce two parameters M_1 and M_2

$$\frac{1}{k^2 - m_0^2} \to \frac{1}{k^2 - m_0^2} - \frac{a_1}{k^2 - M_1^2} - \frac{a_2}{k^2 - M_2^2}. \tag{2}$$

First calculate a_1 and a_2 such, that all terms $\sim \frac{k^4}{k^6}$ and $\sim \frac{k^2}{k^6}$ are vanishing. Now one can set $M_1 = M_2 = M$ to isolate the divergent terms in the limit $M \to \infty$. Finally solve the integral.
Hint: Introduce a Wick rotation and use Feynman parameters. Since the limit $M \to \infty$ is taken at the end, you can simplify $M^2 - m_0^2 \approx M^2$.

ii) As done in the lecture expand the self energy around the pole $p^2 = m^2$, with m the one-loop renormalized mass $m^2 = m_0^2 + \delta m^2$ and give the one loop corrected two point function $G_2(x, y)$. Compare to the lecture.