

Exercise 1: Fermion propagator (4 + 2 points)

Consider free Dirac fermions,

$$S_0[\bar{\psi}, \psi] = \int \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi d^4x, \quad (1)$$

with generating functional

$$Z[\bar{\eta}, \eta] = \frac{1}{Z} \int e^{iS_0[\bar{\psi}, \psi] + i \int \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) d^4x} D\psi D\bar{\psi}. \quad (2)$$

- i) Explicitly solve the integral and show, that the generating functional can be given as

$$Z[\bar{\eta}, \eta] = e^{-\int \bar{\eta}(x) S_F(x, y) \eta(y) d^4x d^4y}, \quad (3)$$

with

$$S_F(x, y) = \int \frac{i}{\not{k} - m + i\epsilon} e^{-ik(x-y)} \frac{d^4k}{(2\pi)^4}. \quad (4)$$

- ii) Show, that S_F is indeed the Feynman propagator, by calculating

$$\langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle = \left(-i \frac{\delta}{\delta \bar{\eta}(x)} \right) \left(i \frac{\delta}{\delta \eta(y)} \right) Z[\bar{\eta}, \eta] \Big|_{\bar{\eta}=\eta=0}. \quad (5)$$

Exercise 2: Yukawa theory (2 + 4 points)

Consider the following Lagrangian for a theory with interacting real scalar fields ϕ and Dirac fermion fields ψ , referred to as a Yukawa theory

$$\mathcal{L}[\phi, \psi, \bar{\psi}] = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \bar{\psi} (i\not{\partial} - m_\psi) \psi - \frac{1}{2} m_\phi^2 \phi^2 + g \bar{\psi} \psi \phi. \quad (6)$$

- i) Give the generating functional of the theory $Z[J, \eta, \bar{\eta}]$ and express it as a function of the generating functional of the non-interacting theory $Z_0[J, \eta, \bar{\eta}]$, as you have done it in the lecture for scalar ϕ^4 theory.
- ii) Calculate the connected Green's function to order $\mathcal{O}(g^2)$

$$\langle \Omega | T(\psi(x)\bar{\psi}(y)) | \Omega \rangle, \quad (7)$$

using the generating functional.

Exercise 3: Fadeev-Popov method (2+4+2 points)

Let us solve the 2-dim. Gaussian integral

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy, \quad (8)$$

in a Fadeev-Popov inspired way.

i) Argue, that the integral is invariant under rotation

$$\begin{pmatrix} x^g \\ y^g \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (9)$$

where g is a $SO(2)$ matrix.

ii) Use the relation

$$1 = \Delta_{FP}(x, y) \int \delta(F(x^g, y^g)) dg, \quad (10)$$

to calculate $\Delta_{FP}^{-1}(x, y)$, using the following „gauge“

$$F(x^g, y^g) = y^g = x \sin \phi + y \cos \phi. \quad (11)$$

Hint: Use the following relation to simplify the δ -function

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad (12)$$

where x_i are the roots of $f(x_i) = 0$.

iii) Use your results to show that

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy = \int dg \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta_{FP}(x, y) \delta(F(x, y) = y) e^{-x^2-y^2} dx dy = \pi, \quad (13)$$

where $\int dg$ is the volume of the gauge orbit.