Exercise 1: Dyson-Schwinger equation *(4+4 points)*

Consider an euclidean, real scalar field theory with a $\mathcal{L}_{\text{int}}[\phi] = \frac{\lambda}{4!}\phi^4$ interaction. In the lecture you derived the Dyson-Schwinger equation for a scalar field theory with generating functional $Z[J]$,

$$\left(-\frac{\delta S[\phi]}{\delta\phi(x)}\left[\frac{\delta}{\delta J}\right] + J(x)\right) Z[J] = 0. \tag{1}$$

Use this expression to derive exact equations that are satisfied

i) by the two- and four-point (with three points identified) functions.

ii) by the four- and six-point (with three points identified) functions.

Exercise 2: Connected Green's functions and the effective action *(4 points)*

In the lecture you have found the following relations between amputated Green's functions G_n^a and the proper vertices $\Gamma^{(n)}$, for a general action $S[\phi]$

$$\begin{aligned} \Gamma^{(1)}(x) =& 0 \\ \Gamma^{(2)}(x_1,x_2) =& S(x_1,x_2) = [G_2^c(x_1,x_2)]^{-1} \\ \Gamma^{(3)}(x_1,x_2,x_3) =& -G_3^a(x_1,x_2,x_3) \\ \Gamma^{(4)}(x_1,x_2,x_3,x_4) =& -G_4^a(x_1,x_2,x_3,x_4) + \int G_3^a(x_1,x_2,y)G_2^c(y,z)G_3^a(x_3,x_4,z)d^4y d^4z \\ &+ 2\text{ permutations.} \end{aligned} \tag{2}$$

Identify the permutations and use the relations to give the connected Green's functions

$$G_3^c(x_1,x_2,x_3) \quad \text{and} \quad G_4^c(x_1,x_2,x_3,x_4), \tag{3}$$

as functions of $\Gamma^{(i)}$.

What happens in the special case of ϕ^4-theory?

Exercise 3: Integrals of Grassmann variables *(4+4 points)*

Calculate the following integrals for complex Grassmann numbers θ_i

i) An integral of two-point gaussian type

$$\int \left(\prod_i d\theta_i^\star d\theta_i\right) \theta_k \theta_l^\star e^{-\theta_i^\star A_{ij}\theta_j} \tag{4}$$

ii) Two four-point gaussian integrals

$$\int \left(\prod_i d\theta_i^\star d\theta_i\right) \theta_k \theta_l \theta_m \theta_n e^{-\theta_i^\star A_{ij}\theta_j}, \tag{5}$$

and

$$\int \left(\prod_i d\theta_i^\star d\theta_i\right) \theta_k \theta_l \theta_m^\star \theta_n^\star e^{-\theta_i^\star A_{ij}\theta_j}. \tag{6}$$