
Exercise 1: Green's functions and the free scalar field *(8 points)*

Consider the euclidean action of a real scalar field ϕ

$$S_E[\phi] = \int \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 \right) d^4x. \tag{1}$$

Use the generating functional familiar from the lecture to

i) Compute the 4-point Green's function $G_4(x_1, x_2, x_3, x_4)$ and draw all the Feynman diagrams.

ii) Show that $G_n(x_1, ..., x_n) = 0$ for odd n. Is this also true in an interacting theory?

Exercise 2: Classical perturbation theory *(6 points)*

The classical field equations in real, euclidean ϕ^4 theory in the presence of an additional source J are given as

$$\left(\partial_\mu\partial_\mu - m^2 - \frac{\lambda}{3!}\phi^2(x) \right) \phi(x) = -J(x). \tag{2}$$

Use the expansion

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \lambda^3\phi_3(x) + \mathcal{O}(\lambda^4), \tag{3}$$

to solve the equation order-by-order in λ around $\lambda = 0$. Give the expressions for $\phi_k(x)$ for $k \in \{0, ..., 3\}$.

Exercise 3: Next to leading order computation in ϕ^4 theory *(6 points)*

Consider the euclidean action of ϕ^4 theory

$$S_E[\phi] = \int \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right) d^4x = S_{E,0}[\phi] + S_{E,I}[\phi]. \tag{4}$$

In case of an interacting theory, the generating functional $Z[J]$ can be rewritten as

$$Z[J] = \frac{1}{Z[0]} \int e^{-S_E[\phi] + \int J(x)\phi(x)d^4x} D\phi = \frac{Z_0[0]}{Z[0]} e^{-S_{E,I}\left[\frac{\delta}{\delta J}\right]} e^{\frac{1}{2}\int J(x)\Delta_F(x,y)J(y)d^4x d^4y}, \tag{5}$$

where

$$Z_0[0] = \int e^{-S_{E,0}[\phi]} D\phi \tag{6}$$

and

$$e^{-S_{E,I}\left[\frac{\delta}{\delta J}\right]} e^{\frac{1}{2}\int J(x)\Delta_F(x,y)J(y)d^4x d^4y} = e^{\frac{1}{2}\int J(x)\Delta_F(x,y)J(y)d^4x d^4y} \left(1 + \lambda w_1[J] + \lambda^2 w_2[J] + \mathcal{O}(\lambda^3) \right). \tag{7}$$

Compute $w_2[J]$ using

$$w_2[J] = \frac{1}{2}\left(\frac{1}{4!}\right)^2 e^{-\frac{1}{2}\int J(x)\Delta_F(x,y)J(y)d^4x d^4y} \left[\int \left(\frac{\delta}{\delta J(z)} \right)^4 d^4z \right]^2 e^{\frac{1}{2}\int J(x)\Delta_F(x,y)J(y)d^4x d^4y} \tag{8}$$

and give the diagrammatic representation.
Hint: Use the result of $w_1[J]$.