
Exercise 1: Gaussian integrals II *(10 points)*

Prove the following Gaussian integrals

i) Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ be a n-dimensional vector and $A \in \mathbb{R}^{n\times n}$ be a real, symmetric matrix, with positive eigenvalues $\lambda_i > 0$, that can be diagonalized. Prove that

$$\int e^{-\frac{1}{2}x^T A x} d^n x = \sqrt{\frac{(2\pi)^n}{det(A)}}. \tag{1}$$

Hint: Diagonalize the matrix A (recall the spectral theorem).

ii) In the same manner prove that

$$\int e^{-\frac{1}{2}x^T A x + J^T x} d^n x = \sqrt{\frac{(2\pi)^n}{det(A)}} e^{\frac{1}{2}J^T A^{-1} J}. \tag{2}$$

iii) Consider a real constant $a > 0$. Prove the following one-dimensional integral of Gaussian type

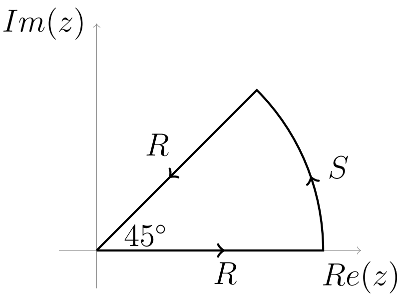
$$\int_{-\infty}^{\infty} e^{\mathrm{i}ax^2} dx = \sqrt{\frac{\mathrm{i}\pi}{a}}. \tag{3}$$

Hint: Use the theorem of Cauchy for the given contour

$$\oint_C e^{iaz^2} dz = 0.$$

Split it up into 3 parts and identify the integral (3) in the limit $R \to \infty$.
Show that the contribution of the S part vanishes, using Jordan's identity

$$\frac{2\alpha}{\pi} \leq \sin\alpha \leq \alpha, \quad 0 \leq \alpha \leq \frac{\pi}{2}$$



Exercise 2: Vacuum expectation value for the harmonic oscillator *(4 points)*

In the lecture you found the following euclidean generating functional for the one-dimensional harmonic oscillator

$$Z_E[j] = \exp\left(\frac{1}{4m\omega}\int\int j(\tau_1)e^{-\omega|\tau_1-\tau_2|}j(\tau_2)d\tau_1 d\tau_2\right). \tag{4}$$

Calculate the vacuum expectation value of

$$\langle 0| x(\tau)x(\tau') |0\rangle, \tag{5}$$

using $Z_E[j]$.

Exercise 3: The Schrödinger equation and path integrals *(6 points)*
Recall the time dependent, one-dimensional Schrödinger equation

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} = H(x)\psi(x,t), \tag{6}$$

with time independent Hamilton operator $H(x)$.
The Schrödinger equation can be recovered from the path integral representation of the quantum mechanical transition amplitude. To show this, first recall the infinitesimal form of the transition amplitude

$$\langle x| W(\epsilon) |x'\rangle = \sqrt{\frac{m}{2\pi\mathrm{i}\epsilon}} \exp\left(\mathrm{i}\epsilon\left[\frac{m}{2}\left(\frac{x-x'}{\epsilon}\right)^2 - V\left(\frac{x+x'}{2}\right)\right]\right). \tag{7}$$

Use it in the propagation of the wave function

$$\psi(t+\epsilon,x) = \int_{-\infty}^{\infty} \langle x| W(\epsilon) |x'\rangle \psi(t,x') dx' \tag{8}$$

and introduce $\eta = x' - x$.
Argue that the dominant contribution to the integral is given for

$$0 \leq |\eta| \leq \sqrt{\frac{2\pi\epsilon}{m}}, \tag{9}$$

and use this result to Taylor expand the integrand up to order ϵ. Solve the remaining integral and take the limit $\epsilon \to 0$.
Hint: The integral of Exercise 1 iii) and the derivative of it with respect to a will be of great use.