Exercise 1: Partition function of the harmonic oscillator *(10 points)*
Recall the Hamilton function of the 1 dimensional harmonic oscillator

$$H(p,x) = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

i) Use the Hamilton function to compute the partition function of the system

$$Z_1 = \int e^{-\beta H} \frac{dpdx}{2\pi}, \tag{2}$$

with $\beta = 1/T$.

ii) We can obtain a similar result from the quantum mechanical case. Recall the energies of the discrete spectrum of the quanum mechanical harmonic oscillator E_n and calculate

$$Z_2 = \sum_n e^{-\beta E_n}. \tag{3}$$

iii) Finally another possibility to obtain the partition function is using the Mehler formula given by

$$\langle x| e^{-\beta H} |x\rangle = \sqrt{\frac{m\omega}{2\pi \sinh(\beta\omega)}} \exp\left(-\frac{x^2 m\omega}{\sinh(\beta\omega)}(\cosh(\omega\beta) - 1)\right). \tag{4}$$

Calculate the partition function using

$$Z_3 = \int \langle x| e^{-\beta H} |x\rangle \, dx. \tag{5}$$

iv) Calculate the free energy

$$F = -\frac{1}{\beta}\log(Z), \tag{6}$$

for all three cases Z_1, Z_2 and Z_3 and discuss the limit $\beta << 1$ for F_2 and F_3.

Exercise 2: Gaussian integrals *(4 points)*
Calculate the following Gaussian integrals

i) Calculate

$$\langle x^{2n}\rangle = \sqrt{\frac{a}{2\pi}} \int\limits_{-\infty}^{\infty} x^{2n} e^{-\frac{1}{2}ax^2} dx \tag{7}$$

ii) Calculate

$$\langle x^{2n}\rangle = \left(\frac{d}{dJ}\right)^{2n} \left[\sqrt{\frac{a}{2\pi}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + Jx} dx\right]\Bigg|_{J=0} \tag{8}$$

Exercise 3: Euclidean path integral *(6 points)*

Following the same steps as in the lecture, derive the one dimensional, euclidean path integral

$$\langle x_2 | U(\tau_2 - \tau_1) | x_1 \rangle = \int_{y(\tau_1)=x_1}^{y(\tau_2)=x_2} e^{-S_E[y]} Dy, \tag{9}$$

with evolution operator (for imaginary time $\tau_2 > \tau_1$)

$$U(\tau_2 - \tau_1) = e^{-(\tau_2-\tau_1)H}, \tag{10}$$

with a time independent Hamilton operator $H = H_0 + V = \frac{p^2}{2m} + V$. The quantity $S_E[y]$ is the euclidean action

$$S_E[y] = \int \left(\frac{m\dot{y}^2}{2} + V \right) d\tau, \tag{11}$$

that needs to be identified in the derivation.