Exercise 1: Renormalizability *(12 points)*

Calculate the divergence index of the following interaction terms and determine the space-time dimensions for which the interaction becomes renormalizable

$$a)\ \bar{\psi}\psi\phi, \quad b)\ \phi^6, \quad c)\ \bar{N}N, \quad d)\ \pi^T \cdot \pi, \quad e)\ \epsilon^{abc}\pi^b\partial^\mu\pi^c, \quad f)\ \bar{N}\gamma^5\gamma^\mu T^a N,$$

where ϕ, ψ, π and N represent scalar bosons, fermions, pions (mesons) and nucleons (baryons).

Exercise 2: QED β-function *(3+3+2 points)*

The QED beta-function to two loop order is given as

$$\beta(e) = \frac{e^3}{12\pi^2} + \frac{e^5}{64\pi^4} + \mathcal{O}(e^7). \tag{1}$$

We want to study the behavior of the coupling for $\mu \to 0$.

i) Use separation of variables to solve the differential equation

$$\mu \frac{de}{d\mu} = \beta(e), \tag{2}$$

and obtain a solution of the form

$$e^2 = f(e^2 \ln e^2). \tag{3}$$

Hint: When performing the separation of variables, expand the inverse beta-function $1/\beta(e)$ at $e = 0$, neglecting all terms $\mathcal{O}(e)$.

ii) To obtain an equation of the form $e^2 = f(\mu)$, where no coupling is appearing on the right hand side anymore, we need to iteratively solve the equation from i). Iterate the equation, using the approximation

$$e^2 \approx -\frac{1}{\frac{1}{12\pi^2}\ln\left(\frac{\mu^2}{\mu_0^2}\right)} \tag{4}$$

in the second iteration.

iii) Perform the limit $\mu \to 0$ and comment on the behavior of the coupling.