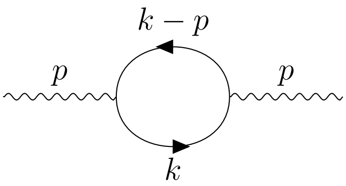
Exercise 1: Lamb shift *(4+6+6+4=20 points)*

In the following we want to calculate the first loop correction to the photon propagator in QED.

i) Use the QED Feynman rules in momentum space to calculate the quantity $-\mathrm{i}\Pi_2^{\mu\nu}$ of the fermion loop correction to the photon propagator.



$= \mathrm{i}\Pi_2^{\mu\nu}(p)$

Hint: Write out the Dirac indices. You should obtain a trace over the product of γ-matrices.

ii) Calculate the trace, introduce Feynman parameters and change the integration variables the following way

$$k^\mu \to k^\mu + p^\mu(1-x). \tag{1}$$

To simplify the calculation, neglect all contributions $\sim p^\mu, p^\nu, p^\mu p^\nu$.
Comment: Neglecting terms $\sim p^\mu, p^\nu, p^\mu p^\nu$, leads to $\Pi_2^{\mu\nu} = g^{\mu\nu}\Pi_T(p^2)$ in the next step, instead of $\Pi_2^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi_T(p^2) = t^{\mu\nu}(p^2)\Pi_T(p^2)$, familiar from QCD. Nevertheless for the determination of Z_3, it is sufficient to consider the term $\sim g^{\mu\nu}$ only.

iii) The remaining integral is divergent and needs to be regularized. Introduce a Wick rotation and use the procedure of dimensional regularization to calculate the integral.
Show that the result (after analytical continuation to Minkowski space) is given as

$$\Pi_2^{\mu\nu} = -8\mu^{4-d} g^{\mu\nu} p^2 \frac{e^2}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 \frac{x(1-x)}{[m^2+p^2(1-x)x]^{2-\frac{d}{2}}}dx = g^{\mu\nu}\Pi_T(p^2). \tag{2}$$

Hint: The following identity is very useful

$$\int \frac{k^\mu k^\nu}{(k^2-\Delta)^n}\frac{d^dk}{(2\pi)^d} = \frac{1}{d}g^{\mu\nu}\int \frac{k^2}{(k^2-\Delta)^n}\frac{d^dk}{(2\pi)^d}, \tag{3}$$

as well as the following integral

$$\int_0^\infty \frac{k^{d+1}}{[k^2+\Delta]^2}dk = \frac{d}{4}\frac{1}{\Delta^{1-\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right). \tag{4}$$

iv) Use this result to calculate the Photon wavefunction renormalization $A_{R,\mu} = Z_3^{-1/2}A_\mu$ in the $\overline{MS}$ scheme via

$$Z_3 = 1 - \Pi_T'(p^2=0) + \mathcal{O}(e^4), \tag{5}$$

as done in case of the gluon propagator in the lecture.