Exercise 1: Zassenhaus Formula *(10 points)*

Consider two operators A and B, with $[A,[A,B]] = [B,[A,B]] = 0$.

i) Derive the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}. \tag{1}$$

For this purpose show

$$[A^n, B] = nA^{n-1}[A,B] \tag{2}$$

first and then derive the following differential equation by definining $f(t) = e^{tA}e^{tB}$

$$\frac{df}{dt} = (A + B + t[A,B])f(t). \tag{3}$$

Find another solution of this differential equation.
Hint: Calculating the commutator $[e^{tA}, B]$ will be of great use.

ii) If the relations $[A,[A,B]] = [B,[A,B]] = 0$ do not hold, the formula can be extended to general cases. This extension is known as Zassenhaus formula and given by

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} e^{-\frac{1}{6}(2[B,[B,A]]+[A,[B,A]])} e^{Z_3} e^{Z_4} ..., \tag{4}$$

where $Z_3, Z_4, ...$ contain higher powers of the operators A and B. Use this formula to show the following relation for the Hamilton operator $H = H_0 + V$

$$e^{-\mathrm{i}\epsilon H} = e^{-\mathrm{i}\epsilon(H_0+V)} = e^{-\mathrm{i}\epsilon V/2} e^{-\mathrm{i}\epsilon H_0} e^{-\mathrm{i}\epsilon V/2} + \mathcal{O}(\epsilon^3), \tag{5}$$

with small parameter $\epsilon > 0$.

Exercise 2: Lie-Trotter Formula *(5 points)*

A useful formula in the derivation of Feynman's path integral is the Lie-Trotter formula

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}} \right)^m. \tag{6}$$

In case of the path integral, X and Y represent operators acting on the Hilbert space, but nevertheless the formula also holds if X and Y are $n \times n$ complex matrices.
Prove the formula by making use of the Taylor expansion of the matrix logarithm

$$\log\left(\mathbb{1} + A\right) = A + \mathcal{O}\left(||A||^2\right). \tag{7}$$

Exercise 3: Functional derivative - chain rule (*5 points*)

Recall the definition of the functional derivative

$$\delta F[x] = \int \frac{\delta F[x]}{\delta x(s)} \delta x(s) ds. \tag{8}$$

Consider the following functional

$$F[g(f)] = \int g(f(x))dx, \quad \text{with} \quad g(f) = \left(\frac{d}{dx} f(x)\right)^n. \tag{9}$$

i) Calculate the derivative using

$$\frac{\delta F[g(f)]}{\delta f(y)} = \int \frac{\delta F[g(f)]}{\delta g(f(s))} \frac{\delta g(f(s))}{\delta f(y)} ds. \tag{10}$$

ii) Calculate the derivative using

$$\frac{\delta F[g(f)]}{\delta f(y)} = \frac{d}{ds} F[g(f(x) + s\delta(x-y))]\Big|_{s=0}. \tag{11}$$