
Exercise 1: Wick's theorem (*8 points*)

In the lecture you showed Wick's theorem for the time ordered product of two field operators

$$T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\right) =: \hat{\phi}_I(x_1)\hat{\phi}_I(x_2) : + \langle 0| T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\right)|0\rangle \,. \tag{1}$$

Use this result to prove that

$$\begin{aligned} T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\hat{\phi}_I(x_3)\right) =&: \hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\hat{\phi}_I(x_3) : + : \hat{\phi}_I(x_1) : \langle 0| T\left(\hat{\phi}_I(x_2)\hat{\phi}_I(x_3)\right)|0\rangle \\ &+ : \hat{\phi}_I(x_2) : \langle 0| T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_3)\right)|0\rangle + : \hat{\phi}_I(x_3) : \langle 0| T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\right)|0\rangle \end{aligned} \tag{2}$$

Hint: You will need to decompose the fields at some point, as you have done it in the lecture, writing $\hat{\phi}_I = \hat{\phi}_I^+ + \hat{\phi}_I^-$. Also show it for $x_1^0 \geq x_2^0 \geq x_3^0$ at first and discuss afterwards that it holds in all cases.

Exercise 2: Feynman propagator of real scalar field theory (*4+2 points*)

In the lecture the Feynman propagator of real scalar field theory was given as

$$\Delta_F(x-y) = \mathrm{i}\int \frac{d^4p}{(2\pi)^4}\, \frac{e^{-\mathrm{i}p(x-y)}}{p^2 - m_0^2 + \mathrm{i}\epsilon}\bigg|_{\epsilon=0} \,. \tag{3}$$

i) Prove that the propagator is equivalent to

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3 2E(\mathbf{p})}\left(e^{-\mathrm{i}p(x-y)}\theta(t_x - t_y) + e^{\mathrm{i}p(x-y)}\theta(t_y - t_x)\right), \tag{4}$$

when performing the p_0-integration.
Hint: Find the poles of the denominator and use the residue theorem. You will have to rescale $\epsilon' = \frac{\epsilon}{2E(\mathbf{p})}$.

ii) Show that the Feynman propagator $\Delta_F(x-y)$ is the Green function of the Klein-Gordon operator

$$(\partial_\mu \partial^\mu + m_0^2)\Delta_F(x-y) = -\mathrm{i}\delta^{(4)}(x-y). \tag{5}$$

Exercise 3: Mandelstam variables (*6 points*)

The Mandelstam variables are very useful when calculating $2 \to 2$ scattering processes. They are defined as

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \tag{6}$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \tag{7}$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2, \tag{8}$$

where p_1 and p_2 are the four-momenta of the incoming and p_3 and p_4 the four-momenta of the outgoing particles.

i) Show that the following relation holds

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{9}$$

ii) Show that in the case of equal masses $m_1 = m_2 = m_3 = m_4 = m$ the following conditions always hold

$$s \geq 4m^2, \quad t \leq 0, \quad u \leq 0. \tag{10}$$