Exercise 1: The four momentum operator of electrodynamics. (*10 points*)
In the lecture you have discussed the quantized four potential of electrodynamics

$$\hat{A}^\mu(x) = \int \sum_{\lambda=1}^{2} \epsilon^\mu_\lambda(\mathbf{p}) \left(\hat{a}_\lambda(\mathbf{p})e^{-\mathrm{i}px} + \hat{a}^\dagger_\lambda(\mathbf{p})e^{\mathrm{i}px}\right) \frac{d^3p}{(2\pi)^3 2E(\mathbf{p})}. \tag{1}$$

With the polarization vectors $\epsilon^\mu_\lambda(\mathbf{p})$ and dispersion relation $\omega = E(\mathbf{p}) = |\mathbf{p}|$. Furthermore let us consider radiation gauge

$$\hat{A}^0 = 0, \qquad \nabla \cdot \hat{\mathbf{A}} = 0, \tag{2}$$

making it possible to choose two real linearly independent polarization vectors that are normalizable

$$\epsilon^\mu_\lambda \epsilon_{\mu,\lambda'} = \epsilon^i_\lambda \epsilon_{i,\lambda'} = -\delta_{\lambda\lambda'}. \tag{3}$$

i) Calculate the normal ordered zero component of the four momentum operator

$$\hat{P}^0 = \int \hat{\mathcal{H}} d^3x = \frac{1}{2}\int (|\hat{\mathbf{E}}|^2 + |\hat{\mathbf{B}}|^2) d^3x \tag{4}$$

Hint: Use $|\hat{\mathbf{E}}|^2 = \hat{F}^{i0}\hat{F}_{0i}$ *and* $|\hat{\mathbf{B}}|^2 = \frac{1}{2}\hat{F}^{ij}\hat{F}_{ij}$.

ii) An analogous calculation can be done for the spatial components of the four momentum operator, leading to

$$\hat{P}^i = \int \left(\hat{\mathbf{E}} \times \hat{\mathbf{B}}\right)^i d^3x = \frac{1}{2}\int p^i \sum_{\lambda=1}^{2} \hat{a}^\dagger_\lambda(\mathbf{p})\hat{a}_\lambda(\mathbf{p}) \frac{d^3p}{(2\pi)^3 2E(\mathbf{p})}, \tag{5}$$

making it possible to give a normal ordered expression of the complete four momentum operator

$$\hat{P}^\mu = \int \frac{p^\mu}{E(\mathbf{p})} \sum_{\lambda=1}^{2} \hat{a}^\dagger_\lambda(\mathbf{p})\hat{a}_\lambda(\mathbf{p}) \frac{d^3p}{2(2\pi)^3}. \tag{6}$$

Prove that the first excited Fock state is an eigenstate of the four momentum operator

$$\hat{P}^\mu \left|a_\lambda(\mathbf{p})\right\rangle = \hat{P}^\mu \hat{a}^\dagger_\lambda(\mathbf{p}) \left|0\right\rangle = p^\mu \left|a_\lambda(\mathbf{p})\right\rangle. \tag{7}$$

Info: Showing Eq.(5) will be rewarded with 5 bonus points.

Exercise 2: Differential cross section. (*10 points*)

In the lecture you derived the differential cross section of 2-n scattering. The two incoming particles with masses m_1 and m_2 carry the four momenta p_1 and p_2 and the differential cross section is given as

$$d\sigma = \frac{1}{2\omega(s, m_1^2, m_2^2)}(2\pi)^4\delta^4(k_1 + ... + k_n - p_1 - p_2)|M_{fi}|^2 \prod_{i=1}^{n} \frac{d^3k_i}{(2\pi)^3 2k_i^0}, \tag{8}$$

with $s = (p_1 + p_2)^2$ and $w(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}$.

i) Show that the differential cross section reduces to

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}(2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2)|M_{fi}|^2 \frac{d^3k_1}{(2\pi)^3 2k_1^0} \frac{d^3k_2}{(2\pi)^3 2k_2^0}, \tag{9}$$

in the case of 2-2 scattering.

ii) Let us consider two incoming particles of the same mass $m_1 = m_2 = m$ and two outgoing ones of mass M. Show that the cross section of 2-2 scattering is given as

$$\frac{d\sigma}{d\Omega} = \frac{\sqrt{1 - 4\frac{M^2}{s}}}{64\pi^2 s\sqrt{1 - 4\frac{m^2}{s}}}|M_{f_i}|^2, \tag{10}$$

in the center-of-mass system.

Hint: We define the total energy of the collision in the center-of-mass system $E_{tot} = \sqrt{s}$.