Exercise 1: Parity and charge conjugation

Recall the following bilinear covariants from the lecture

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^5\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{1}$$

i) In the lecture you discussed the parity transformation

$$P : (t, \mathbf{x}) \to (t, -\mathbf{x}), \tag{2}$$

and showed that the parity transformation of a Dirac spinor ψ is given as

$$\psi \to \psi^P = \gamma^0\psi. \tag{3}$$

Discuss the transformation properties of the bilinear covariants under parity transformation.
Hint: Split the vectorial bilinear covariants into $\bar{\psi}\gamma^0\psi$ and $\bar{\psi}\gamma^i\psi$ and discuss the parity transformation separately.

ii) We consider a charged particle with mass m and charge q moving in an external electromagentic field. In the lecture you discussed a relativistic description of such a system, given by the following Dirac equation

$$\left[\mathrm{i}\gamma^\mu \left(\partial_\mu + \mathrm{i}qA_\mu\right) - m\right]\psi(x) = 0. \tag{4}$$

Determine the corresponding Dirac equation for the charge conjugate spinor

$$C : \quad \psi \to \psi^C = \mathrm{i}\gamma^2\gamma^0\bar{\psi}^T = \mathrm{i}\gamma^2\psi^*, \tag{5}$$

as you have done it in the lecture in case of the free Dirac equation. Comment on the name „charge conjugation“.

Exercise 2: Solution of the Dirac equation

In the lecture you showed that the general solution of the Dirac equation

$$\left[\mathrm{i}\gamma^\mu\partial_\mu - m\right]\psi(x) = 0, \tag{6}$$

is given by the following Dirac spinor

$$\psi(x) = \int \sum_{r=1,2} \left(b_r(\mathbf{k})u_r(\mathbf{k})e^{-\mathrm{i}kx} + d_r^*(\mathbf{k})v_r(\mathbf{k})e^{\mathrm{i}kx}\right) \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}, \tag{7}$$

with $b_r(\mathbf{k}), d_r(\mathbf{k}) \in \mathbb{C}$ complex numbers. The $u_r(\mathbf{k}), v_r(\mathbf{k})$ are referred to as basis spinors.

i) Show that the basis spinors are orthogonal

$$\bar{u}_r(\mathbf{k})u_s(\mathbf{k}) = -\bar{v}_r(\mathbf{k})v_s(\mathbf{k}) = 2m\delta_{rs}, \tag{8}$$

$$\bar{u}_r(\mathbf{k})v_s(\mathbf{k}) = \bar{v}_r(\mathbf{k})u_s(\mathbf{k}) = 0. \tag{9}$$

ii) Show the completeness relations

$$\sum_{s=1,2} u_{s,\alpha}(\mathbf{k})\bar{u}_{s,\beta}(\mathbf{k}) = (\not{k} + m)_{\alpha\beta}, \tag{10}$$

$$\sum_{s=1,2} v_{s,\alpha}(\mathbf{k})\bar{v}_{s,\beta}(\mathbf{k}) = (\not{k} - m)_{\alpha\beta}. \tag{11}$$

iii) We define the following two operators

$$\Lambda^{+}_{\alpha\beta}(\mathbf{k}) := \frac{1}{2m}\sum_{s=1,2} u_{s,\alpha}(\mathbf{k})\bar{u}_{s,\beta}(\mathbf{k}), \qquad \Lambda^{-}_{\alpha\beta}(\mathbf{k}) := -\frac{1}{2m}\sum_{s=1,2} v_{s,\alpha}(\mathbf{k})\bar{v}_{s,\beta}(\mathbf{k}). \tag{12}$$

Show that they are orthogonal projection operators, i.e., $(\Lambda^{\pm})^2 = \Lambda^{\pm}$, $\Lambda^{+}\Lambda^{-} = 0$. Let them act on the general solution of the Dirac equation $\psi(x)$. What do they project out?