
Exercise 1: Dirac and Klein-Gordon equation(*4 points*)
Let ψ be a Dirac spinor solving the Dirac equation

$$[\mathrm{i}\gamma^\mu \partial_\mu - m]\,\psi(x) = 0. \tag{1}$$

Show that it is also a solution of the Klein-Gordon equation

$$\left[\partial_\mu \partial^\mu + m^2\right]\psi(x) = 0. \tag{2}$$

Exercise 2: Bilinear covariants I(*3+3=6 points*)

i) First show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{3}$$

and use your results to show that

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0. \tag{4}$$

ii) In the lecture you found the following Lorentz transformation of a Dirac spinor ψ

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x), \qquad S(\Lambda) = e^{-\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}, \tag{5}$$

with $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$.
Furthermore you have shown that $\bar\psi\psi$ is a Lorentz scalar, therefore the inverse transformation is given by

$$\bar\psi(x) \to \bar\psi'(x')S^{-1}(\Lambda), \qquad S^{-1}(\Lambda) = e^{\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}. \tag{6}$$

Show that $S^{-1}(\Lambda)$ can also be written as

$$S^{-1}(\Lambda) = (\gamma^0 S(\Lambda)\gamma^0)^\dagger \tag{7}$$

(instead of $S^{-1}(\Lambda) = S^\dagger(\Lambda)$ as one would expect naively).

Hint: Use the result of i)

Exercise 3: Bilinear covariants II(*3+3=6 points*)
Recall the infinitesimal representation of the Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu \tag{8}$$

and the Lorentz transformation of a spinor

$$S(\Lambda) = 1 - \frac{\mathrm{i}}{4}\sigma_{\mu\nu}\delta\omega^{\mu\nu} + o(\delta\omega^2), \qquad \sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]. \tag{9}$$

i) Using these transformations, show that the equation

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu, \tag{10}$$

familiar from the lecture, is satisfied up to first order in $\delta\omega$.

ii) In the lecture you showed that $\bar{\psi}\psi$ transforms as a Lorentz scalar and $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz vector. Furthermore you introduced an additional gamma matrix $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ with the transformation property

$$S^{-1}(\Lambda)\gamma^5 S(\Lambda) = \det(\Lambda)\gamma^5. \tag{11}$$

Prove the behaviour under Lorentz transformation of the following bilinear covariants

$$\bar{\psi}\gamma^5\psi \to \det(\Lambda)\bar{\psi}\gamma^5\psi \qquad \text{pseudoscalar} \tag{12}$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \to \det(\Lambda)\Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma^5\psi \qquad \text{axial vector} \tag{13}$$

$$\bar{\psi}\sigma^{\mu\nu}\psi \to \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\bar{\psi}\sigma^{\alpha\beta}\psi \qquad \text{antisymmetric tensor} \tag{14}$$

Hint: Equation (10) will be very useful.

Exercise 4: Continuity equation in relativistic quantum mechanics II(*4 points*)

Let the Dirac spinor ψ be a solution of the Dirac equation

$$\left[\mathrm{i}\gamma^\mu\partial_\mu - m\right]\psi(x) = 0. \tag{15}$$

Show that the current

$$j^\mu(x) = \bar{\psi}\gamma^\mu\psi \tag{16}$$

satisfies a continuity equation

$$\partial_\mu j^\mu(x) = 0. \tag{17}$$

Discuss wether $j^0 = \rho$ is positive definite in the case of the Dirac equation.