

Exercise 1: Infinitesimal Lorentz transformation (3+3 points)

An infinitesimal Lorentz transformation and its inverse can be written as

$$x'^{\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta})x_{\beta}, \quad x^{\alpha} = (g^{\alpha\beta} + \epsilon'^{\alpha\beta})x'_{\beta}, \quad (1)$$

with the Minkowski metric $(g^{\alpha\beta}) = \text{diag}(1, -1, -1, -1)$ and $\epsilon^{\alpha\beta}$ and $\epsilon'^{\alpha\beta}$ are infinitesimal.

- i) Show that $\epsilon'^{\alpha\beta} = -\epsilon^{\alpha\beta}$ by making use of the definition of an inverse transformation.
- ii) Show from the preservation of the norm that the infinitesimal shift $\epsilon^{\alpha\beta}$ is antisymmetric $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$.

Exercise 2: Collider experiments at the LEP (4 points)

In the LEP storage ring at CERN, head-on collisions between (equally accelerated) electrons and positrons were produced, such that the total energy in the center of mass was equal to that of the Z boson ($m_z = 91 \text{ GeV}$). What is the velocity of each particle before the collision? If an electron is accelerated toward a positron at rest, what velocity does it need in order to reach the same center-of-mass total energy?

Hint: The total momentum of two particles a, b in the center of mass frame is $\mathbf{p}_a + \mathbf{p}_b = 0$.

Exercise 3: Continuity equation in relativistic quantum mechanics (3+2 points)

In the lecture you derived the Klein-Gordon equation of relativistic quantum mechanics

$$-\frac{\partial^2}{\partial t^2}\phi(x) = [-\Delta + m^2]\phi(x). \quad (2)$$

- i) Show that the current

$$j^{\mu}(x) = i(\phi^*(\partial^{\mu}\phi) - (\partial^{\mu}\phi^*)\phi) \quad (3)$$

satisfies a continuity equation

$$\partial^{\mu}j_{\mu} = 0. \quad (4)$$

- ii) Since the Klein-Gordon equation is a relativistic wave equation, it is solved by plane waves. Calculate the zero-component of the current $j^0 = \rho$ for a plane wave solution of the form $\phi \sim \exp(ipx)$ and interpret your result.

Exercise 4: Lorentz invariant integration measure (5 points)

Given the relativistic invariance of d^4k show that the integration measure

$$\frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \quad (5)$$

is Lorentz invariant, provided that $E(\mathbf{k}) = k_0 = \sqrt{m^2 + \mathbf{k}^2}$. Use this to argue that $2E(\mathbf{k})\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ is a Lorentz invariant distribution.

Hint: Start from the Lorentz-invariant expression $\frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k_0)$ and use $\delta(x^2 - x_0^2) = \frac{1}{2|x_0|}(\delta(x + x_0) + \delta(x - x_0))$.