**Exercise 1: QCD running coupling** *(4+2 points)*

In the lecture you discussed that $\alpha_s = \frac{g^2}{4\pi}$, with $g$ the strong coupling of QCD, is a function of energy. This running coupling is given by the following differential equation,

$$Q^2 \frac{d\alpha_s}{dQ^2} = -\beta_0 \alpha_s^2(Q^2) + \mathcal{O}(\alpha_s^3), \tag{1}$$

where $Q^2$ is the squared energy scale of the system and $\beta_0$ is given as

$$\beta_0 = \frac{11N_c - 2N_f}{12\pi}, \tag{2}$$

with $N_c$ colors and $N_f$ flavors.

i) Solve the differential equation up to first order, ignoring all terms $\mathcal{O}(\alpha_s^3)$ and higher.

ii) The value for $\alpha_s$ at the energy scale of the mass of the Z-boson, $Q = M_Z = 91.1$ GeV is

$$\alpha_s(Q^2 = M_Z^2) = 0.12. \tag{3}$$

Use $Q = 10$ GeV with $N_f = 5$ flavors and calculate the value of $\alpha_s$.

**Exercise 2: Fermion mass generation** *(4+4+4 points)*

Consider the following theory for real scalar fields and fermions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \bar{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - g\phi\bar{\psi}\psi, \tag{4}$$

with the following potential

$$V(\phi) = \frac{\lambda}{4!}\left(\phi^2 - v^2\right)^2. \tag{5}$$

i) Sketch the potential $V(\phi)$ of the scalar field. What is the vacuum expectation value $\langle\phi\rangle = \langle\Omega|\,\phi\,|\Omega\rangle$?

ii) Reparametrise the scalar field by its fluctuations around the expectation value $\phi = \langle\phi\rangle + \chi$ and express $\mathcal{L}$ in terms of the field variables $\psi$ and $\chi$.

iii) Identify the particle masses $m_\psi$ and $m_\xi$ and the basic vertices for all interaction terms in this theory, without specifying the Feynman rules. Is there a Goldstone boson? Discuss without calculations what changes in case of a complex scalar field.