
Exercise 1: Yang-Mills theory (*4+4 points*)

In the lecture you derived the covariant derivative of $SU(N)$ gauge theory

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu^a T^a, \tag{1}$$

with $SU(N)$ gauge field A_μ^a and the generators T^a of the $SU(N)$ group, with $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.

i) Use the covariant derivative to calculate the field strength tensor $F_{\mu\nu}^a$ of $SU(N)$ gauge theory via

$$F_{\mu\nu} = \frac{\mathrm{i}}{g}[D_\mu, D_\nu]. \tag{2}$$

ii) The Lagrangian of Yang-Mills theory is constructed from the field strength tensor $F^{\mu\nu}$ in the following, gauge invariant way

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{3}$$

Use your result from i) to express the Lagrangian in terms of $A^{\mu a}$ and identify the self interaction terms.

Exercise 2: Vektor and axial flavour symmetry of QCD (*4+4 points*)

Consider the quark sector of the QCD Lagrangian with N_f flavours $\psi_f = (\psi_1, \psi_2, ..., \psi_{N_f})$

$$\mathcal{L} = \sum_{f=1}^{N_f} \bar{\psi}_f(\mathrm{i}\gamma^\mu D_\mu - m_f)\psi_f.$$

i) Show that in case of degenerate quark masses, $m_f = m$ with $f \in \{1, ..., N_f\}$, the theory is invariant under the following global flavour transformation,

$$\psi \to \psi' = U\psi = e^{-\mathrm{i}\theta^a T^a}\psi,$$

with $U \in SU(N_f)$, constant parameters θ^a and the generators T^a of the $SU(N_f)$ group.

ii) Consider the following axial flavour transformation of Dirac spinors

$$\psi \to \psi' = U\psi = e^{-\mathrm{i}\omega^a T^a \gamma_5}\psi.$$

Determine first the corresponding transformation of $\bar{\psi}$, and then show that the Lagrangian is invariant only if $m_f = 0$.

Exercise 3: Quark-antiquark scattering (*4+4+4 points*)

Quarks carry electric charge as well as color charge and hence interact both via the electromagnetic as well as the strong interaction.

i) Draw all tree level Feynman diagrams for quark-antiquark scattering of an up and an antidown quark

$$u + \bar{d} \to u + \bar{d} \tag{4}$$

both for QED and QCD.

ii) Write down the corresponding scattering amplitudes $\mathrm{i}\mathcal{M}_{QED}$ and $\mathrm{i}\mathcal{M}_{QCD}$. Use the QCD Feynman rules in momentum space familiar from exercise 1 for the latter one.

iii) The spin sums of QCD in the case of massless quarks take the following form

$$\sum_s u_s^i(\mathbf{p})\bar{u}_s^j(\mathbf{p}) = \not{p}\delta^{ij}, \quad \sum_s v_s^i(\mathbf{p})\bar{v}_s^j(\mathbf{p}) = \not{p}\delta^{ij} \tag{5}$$

with $i, j \in \{1, ...3\}$ the group indices of the fundamental representation of the gauge group $SU(3)$. Use these sums to average over initial spins and colors and sum over final spins and colors to relate the scattering amplitude of QCD to the scattering amplitude of QED

$$|\mathcal{M}_{QCD}|^2 = R|\mathcal{M}_{QED}|^2. \tag{6}$$

Determine the factor R as a function of strong coupling g_s and the QED quark charges q_u and q_d.
Hint: You do not have to carry out any traces of gamma matrices after performing the spin sums. It is sufficient to look at the Kronecker deltas and group generators of the $SU(3)$ *gauge group to evaluate R.*