

Exercise 1: Adjoint Representation of $SU(N)$ (4+4+4 points)

Consider the Lie group $SU(N)$. The fundamental representation of the group is given by the $N \times N$ matrices $U \in SU(N)$

$$U = e^{-i\theta^a T^a}, \quad a = 1, \dots, N^2 - 1, \quad (1)$$

with generators T^a of the Lie group and antisymmetric structure constants f^{abc} , that satisfy the Lie algebra

$$[T^a, T^b] = i f^{abc} T^c, \quad (2)$$

and are normalized as

$$\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}. \quad (3)$$

A field ϕ^a is said to transform with the adjoint representation of the group $SU(N)$ if the associated matrix valued field, defined by $\phi := \phi^a T^a$ transforms as

$$\phi' = U \phi U^\dagger. \quad (4)$$

- i) Show that the transformation of the components ϕ^a , and therefore the adjoint representation $D^{ab}(U)$ is given as

$$\phi^a \rightarrow \phi^{a'} = D^{ab}(U) \phi^b = 2 \text{tr}(U^\dagger T^a U T^b) \phi^b. \quad (5)$$

- ii) The covariant derivative acting on the matrix valued field $\phi = \phi^a T^a$ is defined as

$$D_\mu \phi(x) = \partial_\mu \phi(x) - i g [A_\mu, \phi(x)]. \quad (6)$$

Show that this indeed transforms as the adjoint representation

$$(D_\mu \phi)'(x) = U(x) (D_\mu \phi(x)) U^\dagger(x), \quad (7)$$

with $U(x) \in SU(N)$ and that

$$(D_\mu \phi)^a = \partial_\mu \phi^a - g f^{abc} A_\mu^c \phi^b. \quad (8)$$

- iii) Prove that for two matrices $U, U' \in SU(N)$ in fundamental representation, one has

$$D^{ab}(U) D^{bc}(U') = D^{ac}(UU'), \quad (9)$$

by making use of the following relation

$$T_{li}^a T_{no}^a = \frac{1}{2} \left(\delta_{lo} \delta_{in} - \frac{1}{N} \delta_{li} \delta_{no} \right). \quad (10)$$

Exercise 2: Isospin (*4+4 points*)

In the lecture you discussed the following interaction term for a nucleon $N = (p, n)$, pion $\phi = T^a \phi^a = \mathbf{T} \cdot (\frac{1}{\sqrt{2}}(\pi^+ + \pi^-), \frac{i}{\sqrt{2}}(\pi^+ - \pi^-), \pi^0)$ interaction

$$\bar{N} \gamma^5 T^a N \phi^a, \quad (11)$$

with T^a the generators of $SU(2)$.

- i) Show that the interaction term is invariant under isospin transformations. Recall that the nucleon transforms with the fundamental representation of $SU(2)$ and the components of the pion field with the adjoint representation.
Hint: Make use of the relation for group generators T^a given in exercise 1 iii).
- ii) Can you think of alternative pion nucleon interaction terms respecting isospin, baryon number, parity and Lorentz invariance?