Introduction to QCD at zero and finite temperature and density

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- Lecture I: Quantum Field Theory
- Lecture II: QCD in the continuum, weak coupling
- Lecture III: QCD on the lattice
- Lecture IV: QCD at finite temperature and density

References and Sources

- O.Philipsen, “Introduction to Quantum Field Theory”
- M.Walzl and G.Münster, “Introduction to Lattice Gauge theory”
- I have copied smaller and larger bits and pieces from publicly available lecture slides of:
  D. Gütersloh, F. Krauss, J.Heitger, U.Husemann, H.-J.Pirner and H.-C. Schultz-Coulon, H.Wittig

Lecture I: Quantum field theory

Motivation: QCD is a Quantum Field Theory!

QFT is necessary to merge Quantum Mechanics and Special Relativity

- Particles in classical and quantum mechanics, path integral
- Classical and quantum field theory
- Perturbation theory

Units for these lectures

- Natural units: $\hbar = 1, c = 1$
- Small circle units: $\pi = 1$
- Supra-natural units: $-1 = 1, i = 1$
- Einstein sum convention: $a_i b_i \equiv \sum_i a_i b_i$

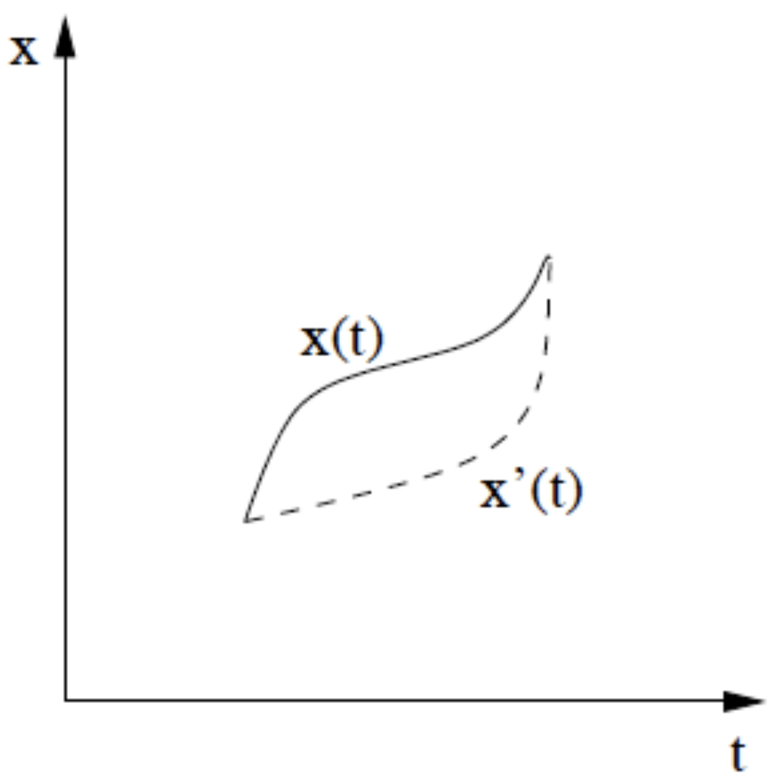
Reminder: Lagrange formalism in classical mechanics

Lagrange function of particle: $$L(x,\dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$

Classical action: $$S = \int_{t_0}^{t_1} dt\, L(x,\dot{x})$$

Hamilton's principle: action stationary, $\delta S = 0$, under small variations of particle trajectory

$x'(t) = x(t) + \delta x(t) \qquad \delta x/x \ll 1$



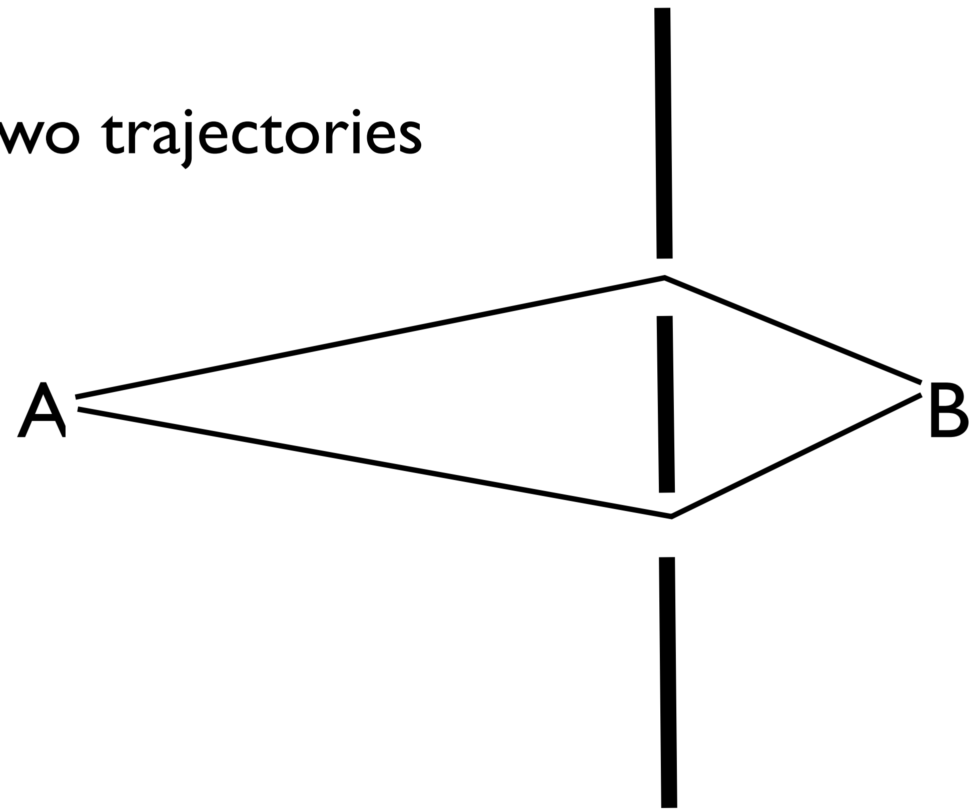
$$\begin{aligned} S + \delta S &= \int_{t_1}^{t_2} L(x+\delta x, \dot{x}+\delta\dot{x})\, dt \\ &= S + \frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \underbrace{\left\{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right\}}_{=0} \delta x\, dt \; + \ldots \end{aligned}$$

$$m\ddot{x} = F = -\frac{\partial V}{\partial x} \quad \text{(Newton's law)}$$

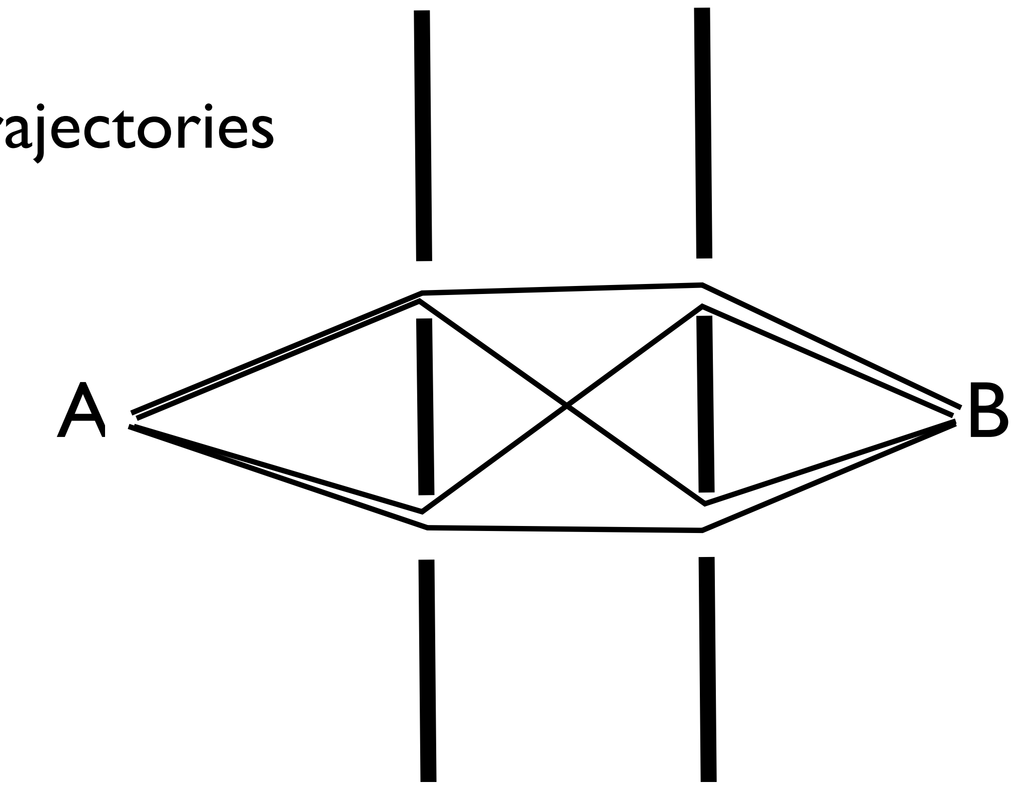
Particle trajectories in classical and quantum physics

Consider classical trajectories through double slits contributing to QM process

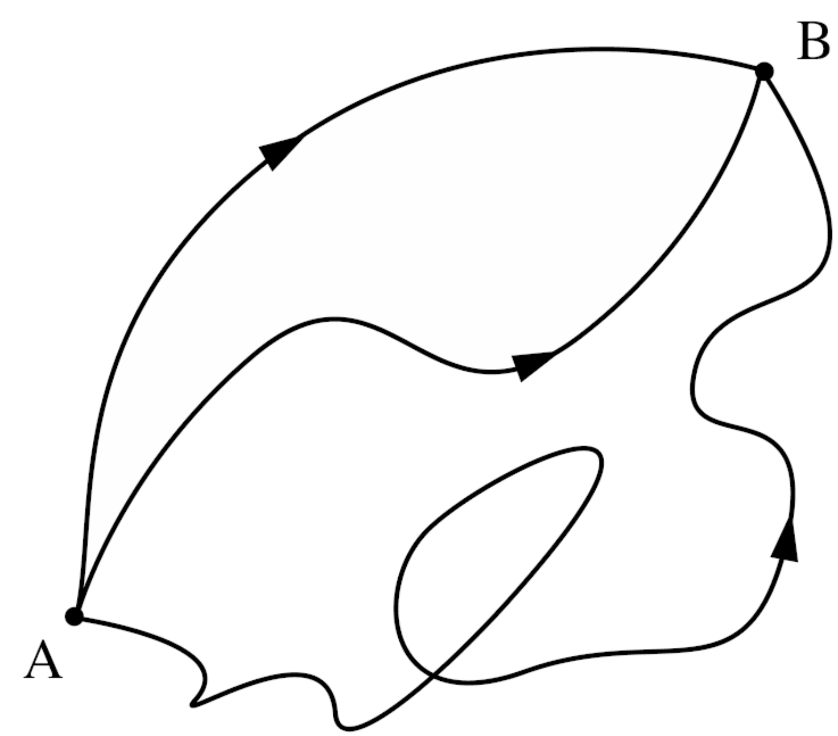
two trajectories



four trajectories



Limit of infinitely many blends with infinitely many slits:
all classically allowed trajectories between A and B contribute
➡ sum them up!



The path integral in 1d-QM

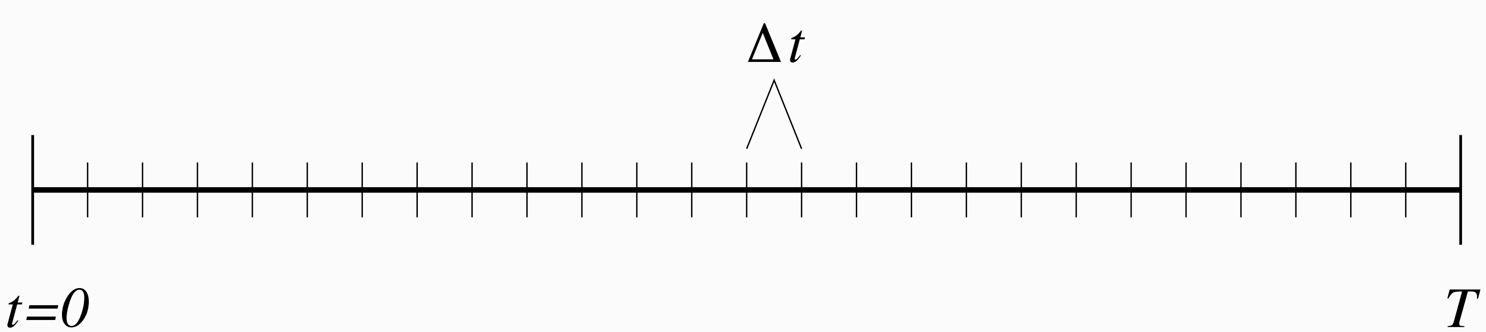
Particle in potential: $$H = \frac{p^2}{2m} + V(x) \equiv H_0 + V.$$

Transition amplitude: $$\langle x', t' | x, t \rangle = \langle x' | e^{-iH(t'-t)} | x \rangle$$

Inserting complete sets of coordinate eigenstates

$$1 = \int dx_1 \, |x_1\rangle\langle x_1| \qquad\qquad T = (t' - t)$$

$$\langle x', t' | x, t \rangle = \int dx_1 \dots dx_{n-1} \; \langle x' | e^{-iH\Delta t} | x_{n-1} \rangle \langle x_{n-1} | e^{-iH\Delta t} | x_{n-2} \rangle \dots \langle x_1 | e^{-iH\Delta t} | x \rangle$$



discretised!

using $\quad e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\cdots}$

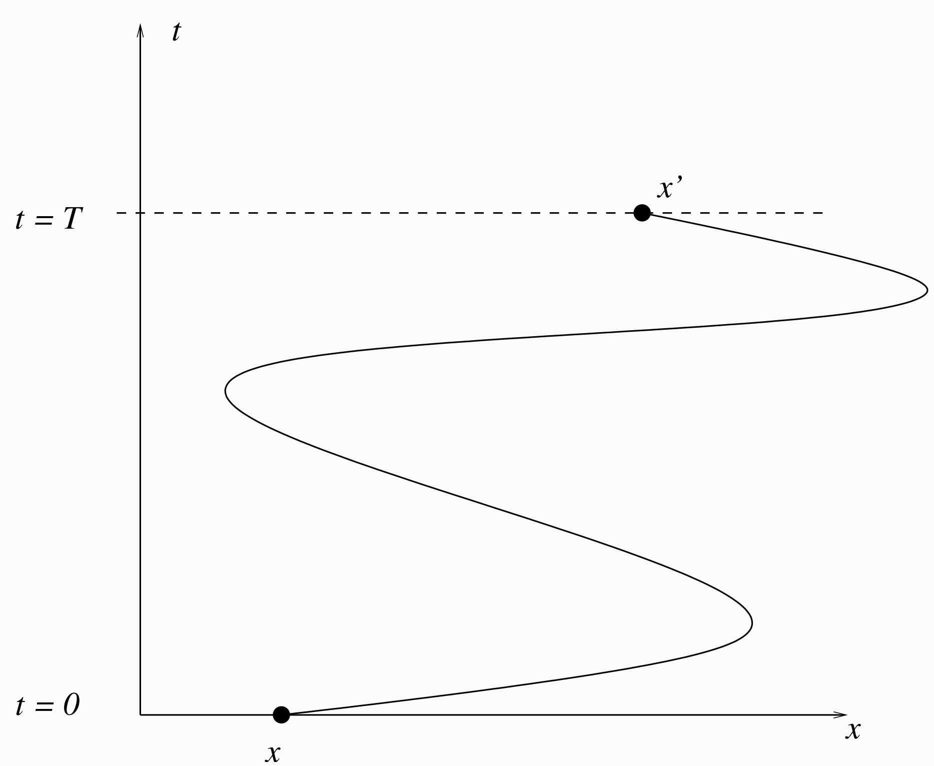
$$\langle x_{k+1}|e^{-iH\Delta t}|x_k\rangle \approx \langle x_{k+1}|e^{-iH_0\Delta t}e^{-iV\Delta t}|x_k\rangle = \langle x_{k+1}|e^{-iH_0\Delta t}|x_k\rangle\, e^{-iV(x_k)\Delta t}$$

Evaluating:

$$\langle x'|e^{-iHT}|x\rangle = \int \frac{dx_1\ldots dx_{n-1}}{\left(\frac{2\pi i\Delta t}{m}\right)^{n/2}} \exp\, i \underbrace{\sum_{k=0}^{n-1}\Delta t\left\{\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right\}}_{\Downarrow}$$

Continuous time: $\quad \Delta t \to 0, T = const.$

$$\int_0^T \mathrm{d}t \left[\frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - V(x)\right] = \int_0^T \mathrm{d}t\, L(x,\dot{x}) \equiv S$$



classical action for path x(t) with

$$x_k = x(k\Delta t)$$

interpretation as integration of the system over all possible paths

$$\left(\frac{m}{2\pi i\Delta t}\right)^{n/2} \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \longrightarrow \text{constant} \times \prod_t \mathrm{d}x(t) \equiv \mathcal{D}x$$

$\Rightarrow$ path integral representation of the QM amplitude:

$$\langle x'|\mathrm{e}^{-iHT}|x\rangle = \int \mathcal{D}x\, \mathrm{e}^{iS}$$

Restoring $\hbar$:

$$\langle x'|e^{\frac{i}{\hbar}HT}|x\rangle = \int \mathcal{D}x\; e^{\frac{i}{\hbar}S}$$

sum over all paths $x(t)$

Note: no operators necessary as in canonical quantisation!

Interpretation: in classical mechanics, there is only one particle trajectory, the solution of the eqs. of motion with initial conditions

in quantum mechanics all possible trajectories contribute, weighted by the classical action

Can the path integral be calculated?

$$\langle x' | e^{\frac{i}{\hbar}HT} | x \rangle = \int \mathcal{D}x \; e^{\frac{i}{\hbar}S}$$

- In general mathematically ill defined:
- Integral measure infinite dimensional
- Oscillatory integrand, convergence??
- Cure to first:
  back to intermediate step with discrete time (lattice!), integrate, take continuum limit
- Cure to second:
  analytically continue to Euclidean time

Even then: only Gauss integrals can be evaluated in closed form, only quadratic potentials, harmonic oscillator!

$$I = \int d^n x \;\; \exp\left(-\frac{1}{2} x_i A_{ij} x_j\right) = (2\pi)^{n/2} (\det A)^{-1/2}$$

Euclidean path integral

Consider Green function: $G(t_1, t_2) = \langle 0|X(t_1)X(t_2)|0\rangle, \qquad t_1 > t_2$

Choose imaginary, or Euclidean, times

$$t = -i\tau \qquad x_\mu = (x_1, x_2, x_3, x_4 = \tau), \quad x_\mu x_\mu = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

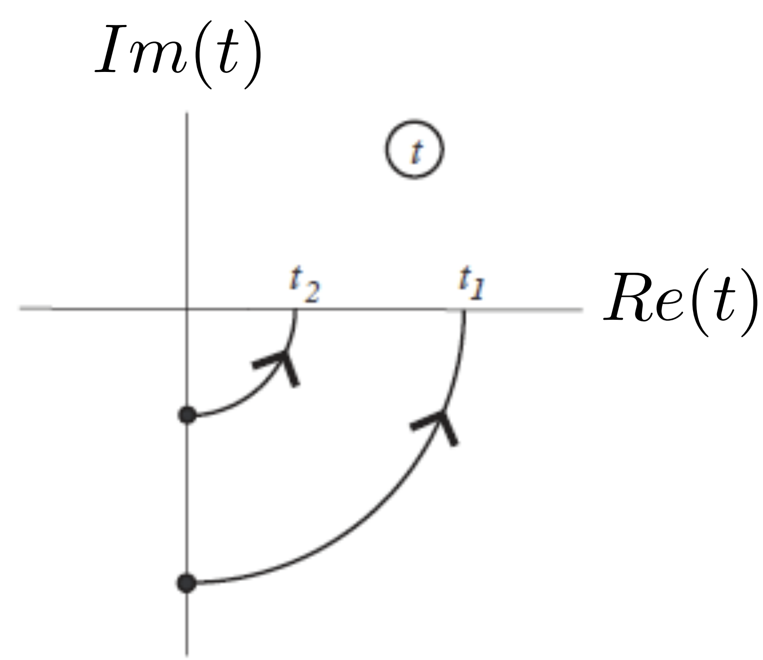
Going through similar steps one can show

$$G_E(\tau_1, \tau_2) = \frac{1}{Z}\int \mathcal{D}x \; x(\tau_1)x(\tau_2)\, e^{-S_E},$$

$$Z = \int \mathcal{D}x \; e^{-S_E}$$

$$S_E = \int d\tau \left\{ \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x(\tau)) \right\}.$$

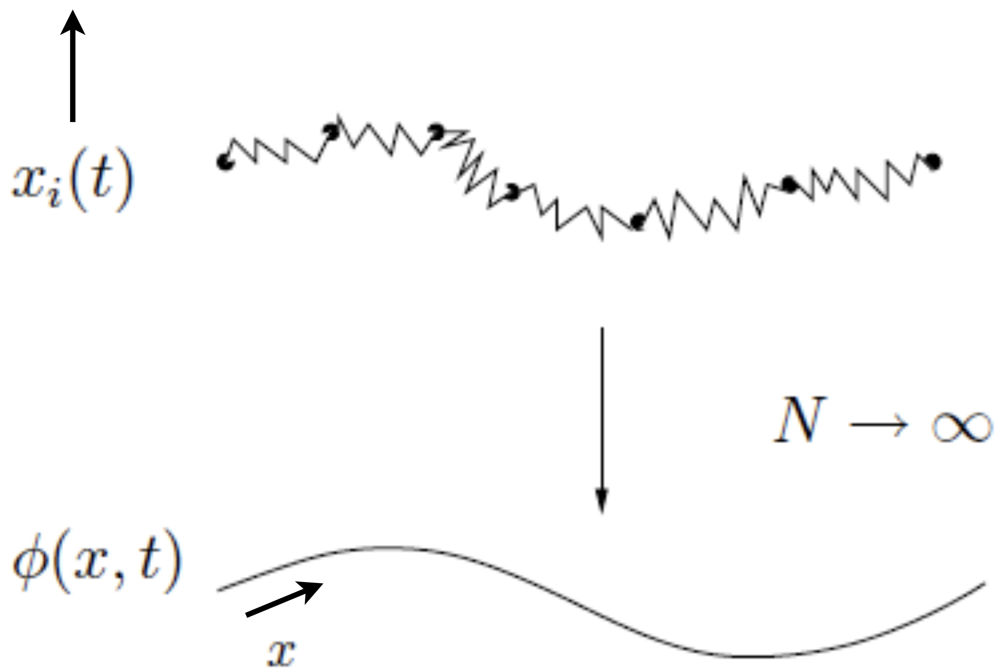
To recover Minkowski: Wick rotation



From N-point mechanics to relativistic field theory

N-particle system:

Field theory:



$$
\begin{aligned}
x(t) &\longrightarrow \phi(x,t) \\
\dot{x}(t) &\longrightarrow \dot{\phi}(x,t) \\
i &\longrightarrow x \\
L(x,\dot{x}) &\longrightarrow \mathcal{L}[\phi,\dot{\phi}]
\end{aligned}
$$

Relativistic field theory:

Action and Lagrange density must be invariant (=scalar) under Lorentz transformations

$$
S = \int d^4x\, \mathcal{L}[\phi, \partial^\mu \phi] \quad L[\phi, \dot{\phi}] = \int d^3x\, \mathcal{L}[\phi, \partial^\mu \phi]
$$

Relativistic fields and actions

Lorentz transformation: $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$ $\quad \mu = 0, \ldots 3$

Field types defined by transformation behaviour:

$\phi'(x') = \phi(x)$ scalar field

$V'^{\mu}(x') = \Lambda^{\mu}_{\nu} V^{\nu}(x)$ vector field

$V'^{\mu\nu}(x') = \Lambda^{\mu}_{\gamma}\Lambda^{\nu}_{\delta} V^{\gamma\delta}(x)$ tensor field

$\psi'_{\alpha}(x') = S(\Lambda)_{\alpha\beta}\psi_{\beta}(x), \quad \alpha = 1, \ldots 4$ spinor field

2 spin states, particle and anti-particle

$S = \int d^4x \, \mathcal{L}$

Hamilton's principle:

$$\begin{aligned} \delta S &= \int d^4x \left\{ \frac{\delta\mathcal{L}}{\delta\phi} - \partial_{\mu}\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\phi)} \right\} \delta\phi \\ &= 0 \end{aligned}$$

Euler Lagrange eqns for relativistic free fields

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \phi \partial_\mu \phi - \tfrac{1}{2} m^2 \phi^2$$

$$(\Box + m^2)\phi(x) = 0$$

Klein Gordon equation

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$$

$$A^\mu(x) = (\varphi(x), \vec{j}(x))$$

$$\partial_\mu F^{\mu\nu}(x) = 0, \partial_\mu A_\mu(x) = 0$$

$$\Box A^\nu(x) = 0$$

Maxwell equations for plane waves

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

$$(i\gamma_\mu \partial^\mu - m)\psi(x) = 0$$

$$\bar{\psi}(x)(i\gamma^\mu \partial_\mu - m) = 0$$

Dirac equation

These are classical field equations! Generalises to interacting theories

Solutions for free fields:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E(p)} \left[a(p)\, e^{-ip\cdot x} + a^*(p)\, e^{ip\cdot x}\right]$$

$$\downarrow$$

$$(\partial_0^2 - \partial_i^2 - m^2)\phi(x) = 0 \qquad \longrightarrow \qquad (E^2 - \vec{p}^2 - m^2)\phi(x) = 0$$

$$E^2 = \vec{p}^2 + m^2$$

Wave functions of particles correspond to field excitations with specific energy and momentum, i.e. the Fourier coefficients

The sum of all excitations in Fourier space makes up the field

Free gauge fields (photons, gluons)

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3 2E(p)} \left[\epsilon^\mu(\lambda,p)\, e^{-ip\cdot x} + \epsilon^{*\mu}(\lambda,p)\, e^{ip\cdot x}\right]$$

$$\Box A^\nu(x) = 0 \qquad \longrightarrow \qquad E^2 = \vec{p}^2, \quad p^2 = 0$$

mass zero!

But the fields must also satisfy a gauge condition, here radiation gauge:

$$A_0(x) = 0, \partial_i A_i(x) = 0 \qquad \longrightarrow \qquad \vec{p}\cdot\vec{\epsilon}(\lambda,p) = 0$$

transversality of the wave!

Out of 4 real field components only two are physical!

The Feynman propagator

$$G_F(x-y) \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

Feynman propagator: 2-point function without interactions

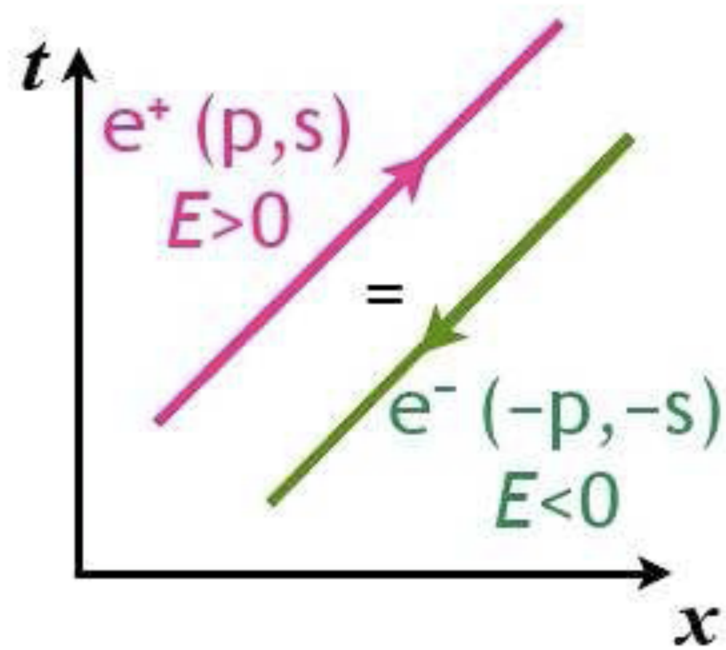
Pole of the Feynman propagator: particle mass of free particle

$$= \int \frac{d^3p}{(2\pi)^3\, 2E(\mathbf{p})}\left(e^{-ip\cdot(x-y)}\theta(t_x-t_y) + e^{ip\cdot(x-y)}\theta(t_y-t_x)\right)$$

= forward and backwards travelling plane waves

For fermions:

$$\langle 0|T\{\bar\psi(x)\psi(y)\}|0\rangle$$



backward wave=anti-particle!
Feynman, Stueckelberg

The path integral in quantum field theory

Scalar field, time evolution: $\phi(\mathbf{x},t) = \mathrm{e}^{iHt}\,\phi(\mathbf{x},t=0)\,\mathrm{e}^{-iHt}$

All physical information encoded in Green functions

$$\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \qquad t_1 > t_2 > \cdots > t_n$$

$$\text{e.g. } n=2: \quad \langle 0|\phi(x)\phi(y)|0\rangle \qquad \text{propagators}$$

Formal analogies:

$$
\begin{aligned}
x_i(t) \quad &\longleftrightarrow \quad \phi(\mathbf{x}, t) \\
i \quad &\longleftrightarrow \quad \mathbf{x} \\
\prod_t \prod_{i=1}^{3} \mathrm{d}x_i(t) \quad &\longleftrightarrow \quad \prod_t \prod_{\mathbf{x}} \mathrm{d}\phi(\mathbf{x}, t) \equiv \mathcal{D}\phi \\
S = \int \mathrm{d}t\, L \quad &\longleftrightarrow \quad S = \int \mathrm{d}t\, \mathrm{d}^3x\, \mathcal{L}
\end{aligned}
$$

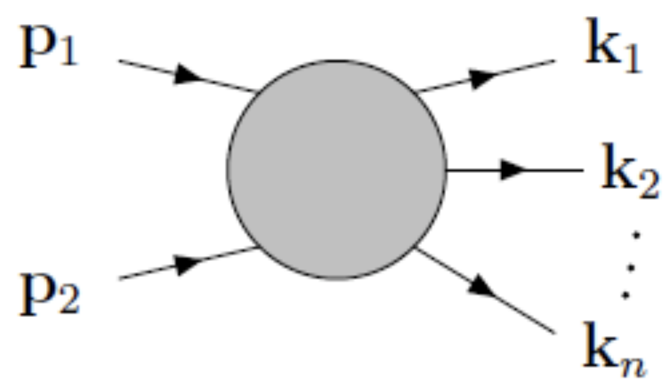
Example:
$$\mathcal{L}(x) = \frac{1}{2}\,(\partial_\mu\phi)\,(\partial^\mu\phi) - \frac{m_0^2}{2}\,\phi^2(x) - \frac{g_0}{4!}\,\phi^4(x)$$

Green functions:
$$\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\; \phi(x_1)\phi(x_2)\cdots\phi(x_n)\, \mathrm{e}^{iS}$$

Integral over all field configurations!
$$\mathcal{Z} = \int \mathcal{D}\phi\, \mathrm{e}^{iS}$$

The Lehman-Symanzik-Zimmerman formula

Scattering matrix element:



$$S_{\mathrm{fi}} = \left\langle \mathrm{k}_1, \mathrm{k}_2, \ldots, \mathrm{k}_n; \mathrm{out} \middle| \mathrm{p}_1, \mathrm{p}_2; \mathrm{in} \right\rangle$$

LSZ:

$$\begin{aligned} S_{\mathrm{fi}} &= (i)^{n+2} \int d^4x_1 \int d^4x_2 \int d^4y_1 \cdots \int d^4y_n \, \mathrm{e}^{(-ip_1\cdot x_1 - ip_2\cdot x_2 + ik_1\cdot y_1 + \cdots + k_n\cdot y_n)} \\ &\times \left(\Box_{x_1} + m^2\right)\left(\Box_{x_2} + m^2\right)\left(\Box_{y_1} + m^2\right)\cdots\left(\Box_{y_n} + m^2\right) \\ &\times \left\langle 0; \mathrm{out} \middle| T\{\phi(y_1)\cdots\phi(y_n)\phi(x_1)\phi(x_2)\} \middle| 0; \mathrm{in} \right\rangle, \end{aligned}$$

All physics encoded in Green functions = vacuum expectation values of field products

$$G_{n+2}(y_1, y_2, \ldots, y_n, x_1, x_2) = \left\langle 0 \middle| T\{\phi(y_1)\cdots\phi(y_n)\phi(x_1)\phi(x_2)\} \middle| 0 \right\rangle$$

Euclidean QFT

Schwinger functions: $$G_{\mathrm{E}}\left((\mathbf{x}_1,\tau_1),\ldots,(\mathbf{x}_n,\tau_n)\right) = G\left((\mathbf{x}_1,-it_1),\ldots,(\mathbf{x}_n,-it_n)\right)$$

=analytic cont. of Minkowski Green functions

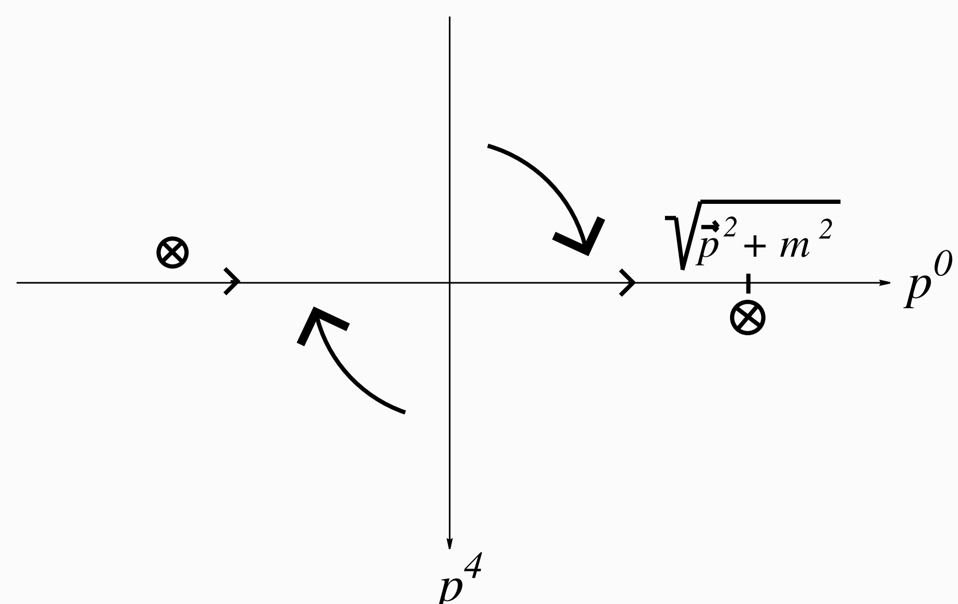
- no oscillatory integrand
- strongly fluctuating fields exp. suppressed

$$G_{\mathrm{E}}(x_1,\ldots,x_n) = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, \mathrm{e}^{-S_{\mathrm{E}}}$$

$$\mathcal{Z} = \int \mathcal{D}\phi\, \mathrm{e}^{-S_{\mathrm{E}}}$$

Example: Feynman propagator

$$i\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{e}^{-ip*x}}{p^2-m_0^2+i\varepsilon} = \lim_{\varphi\to\pi/2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{e}^{ipx}}{p^2+m_0^2}\Bigg|_{x=(\mathbf{x},t\mathrm{e}^{i\varphi})}$$



Spectrum of stable particles calculable from Euclidean theory

Can we calculate path integrals in field theory?

As in QM: discretise, evaluate, then take continuum limit

Again only Gaussian integrals (quadratic in the fields = free fields) in closed form:

Bosons:
$$\int D\phi(x)\ \exp\left[-\int d^4x\,\frac{1}{2}\phi(x)\Delta^{-1}\phi(x)\right] = N\det(\Delta^{-1})^{-1/2}$$

Fermions: anti-commuting Grassman variables (for Pauli principle) $\{\psi(x),\psi(y)\}=0$

$$\int D\bar{\psi}(x)D\psi(x)\ \exp\left[\int d^4x\,\bar{\psi}(x)\Delta^{-1}\psi(x)\right] = N\det(\Delta^{-1})$$

For interacting fields: approximate methods (perturbation theory) or numerical (lattice)

Euclidean field theory = Classical statistical system

Euclidean Field Theory	Classical Statistical Mechanics
Action	Hamiltonian
unit of action h	units of energy $\beta = 1/kT$
Feynman weight for amplitudes $e^{-\mathcal{S}/h} = e^{-\int \mathcal{L}dt/h}$	Boltzmann factor $e^{-\beta H}$
Vacuum to vacuum amplitude $\int \mathcal{D}\phi e^{-\mathcal{S}/h}$	Partition function $\sum_{conf.} e^{-\beta H}$
Vacuum energy	Free Energy
Vacuum expectation value $\langle 0 \vert \mathcal{O} \vert 0 \rangle$	Canonical ensemble average $\langle \mathcal{O} \rangle$
Time ordered products	Ordinary products
Green's functions $\langle 0 \vert T[\mathcal{O}_1 \ldots \mathcal{O}_n] \vert 0 \rangle$	Correlation functions $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$
Mass M	correlation length $\xi = 1/M$
Mass-gap	exponential decrease of correlation functions
Mass-less excitations	spin waves
Regularization: cutoff Λ	lattice spacing a
Renormalization: $\Lambda \to \infty$	continuum limit $a \to 0$
Changes in the vacuum	phase transitions

➡ common methods: mean field, block-spin trafos = RG, high T expansions...

Perturbation theory

$$\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\,\mathrm{e}^{iS}$$

$$S = S_0 + S_{\text{int}} = \int d^4x\,\left[\frac{1}{2}(\partial^\mu\phi(x))\partial_\mu\phi(x) - \frac{1}{2}m_0^2\phi^2(x) - \frac{g_0}{4!}\phi^4(x)\right]$$

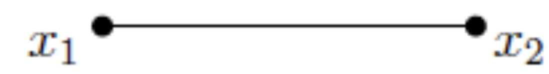
$$S_{\text{int}} = -\int d^4x\,\frac{g_0}{4!}\phi^4(x)$$

Perturbative evaluation of Green function as power series in the coupling constant

$$\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \frac{\sum_{r=0}^{\infty}\left(-\frac{ig_0}{4!}\right)^r\frac{1}{r!}\left\langle 0\middle|\;\left\{\phi\ (x_1)\cdots\phi\ (x_n)\left(\int d^4y\,\phi^4\,(y)\right)^r\right\}\middle|0\right\rangle}{\sum_{r=0}^{\infty}\left(-\frac{ig_0}{4!}\right)^r\frac{1}{r!}\left\langle 0\middle|T\left(\int d^4y\,\phi^4\,(y)\right)^r\middle|0\right\rangle}$$

The Feynman propagator

$$G_F(x-y) \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{e}^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$



Feynman propagator: 2-point function without interactions, r=0

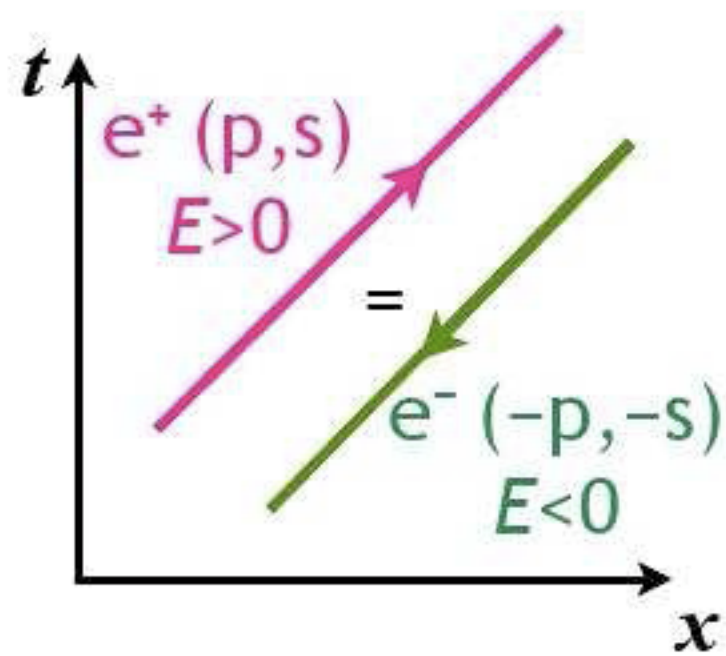
Pole of the Feynman propagator: mass from the Lagrange density

$$= \int \frac{d^3p}{(2\pi)^3\,2E(\mathbf{p})}\left(\mathrm{e}^{-ip\cdot(x-y)}\theta(t_x-t_y)+\mathrm{e}^{ip\cdot(x-y)}\theta(t_y-t_x)\right)$$

= forward and backwards travelling plane waves

For fermions:

$$\langle 0|T\{\bar{\psi}(x)\psi(y)\}|0\rangle$$

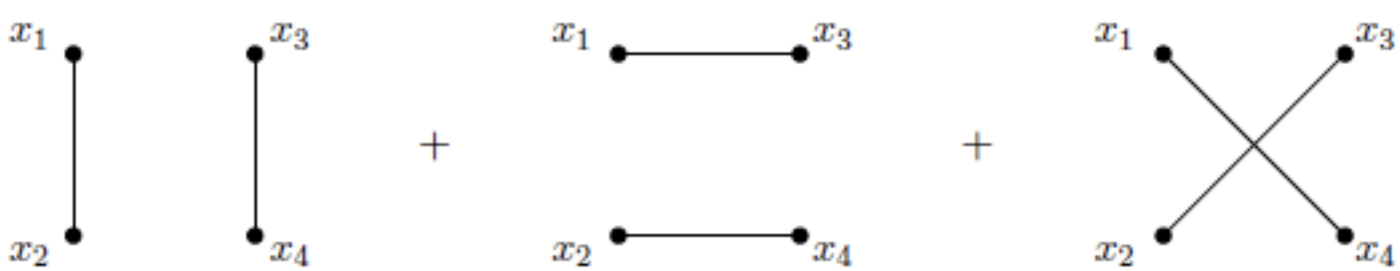


backward wave=anti-particle!
Feynman, Stueckelberg

Example: 4-point function

$$r=0: \quad \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle$$

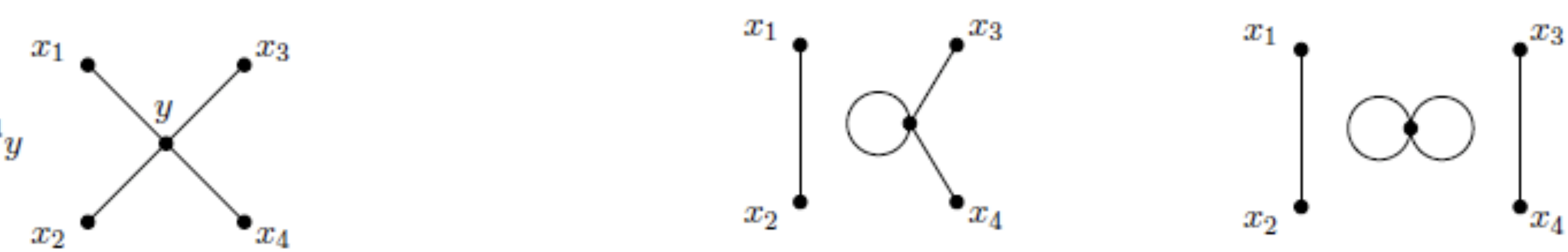
$$= G_F(x_1-x_2)\,G_F(x_3-x_4) + G_F(x_1-x_3)\,G_F(x_2-x_4) + G_F(x_1-x_4)\,G_F(x_2-x_3),$$



$$r=1: \quad -\frac{i\,g_0}{4!}\left\langle 0 \left| T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \int d^4y\,\phi^4(y)\right\}\right| 0\right\rangle$$

$$= -\frac{i\,g_0}{4!}\int d^4y\; 4!\; G_F(x_1-y)G_F(x_2-y)G_F(x_3-y)G_F(x_4-y) \qquad + \;\ldots$$

$-ig_0\int d^4y$



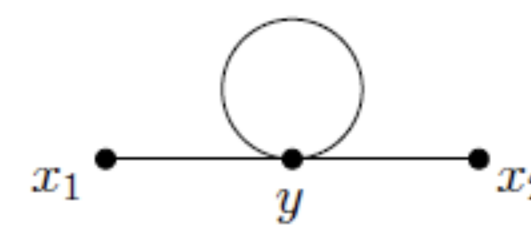
$+\ldots$

Example: 2-point function

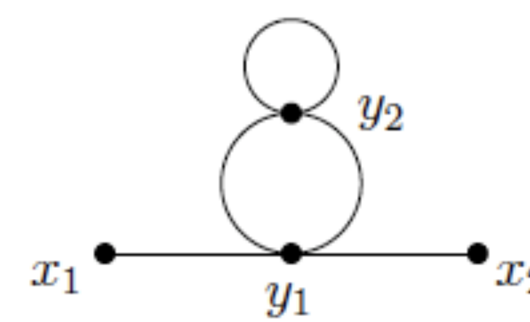
$$G_F(x-y) \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = i\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{e}^{ip\cdot(x-y)}}{p^2-m_0^2+i\epsilon}$$

Feynman propagator: 2-point function without interactions, r=0

The full propagator contains all interactions, vacuum bubbles

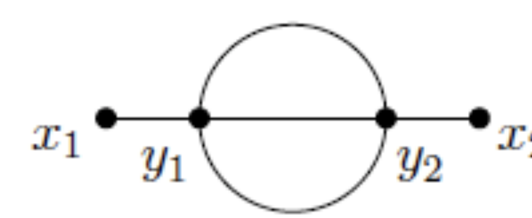


r=1



r=2

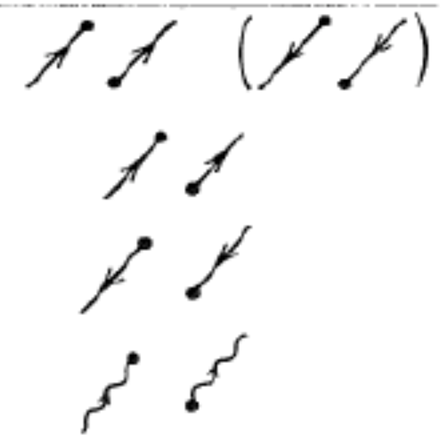
$$\mathcal{C}\left(-\frac{ig_0}{4!}\right)^2\int d^4y_1 d^4y_2\, G_F(x_1-y_1)\,[G_F(y_1-y_2)]^3\, G_F(y_2-x_2)$$



General Feynman rules

Shortcut to perturbative evaluation of QFT's

- **External Lines**



Spin 0 boson (or antiboson)	1
Spin $\frac{1}{2}$ fermion (in, out)	$u, \bar{u}$
antifermion (in, out)	$\bar{v}, v$
Spin 1 photon (in, out)	$\varepsilon_\mu, \varepsilon_\mu^*$

- **Internal Lines — Propagators (need $+i\varepsilon$ prescription)**

Spin 0 boson	$\dfrac{i}{p^2 - m^2}$
Spin $\frac{1}{2}$ fermion	$\dfrac{i(\not{p} + m)}{p^2 - m^2}$
Massive spin 1 boson	$\dfrac{-i\left(g_{\mu\nu} - p_\mu p_\nu / M^2\right)}{p^2 - M^2}$
Massless spin 1 photon (Feynman gauge)	$\dfrac{-ig_{\mu\nu}}{p^2}$

- **Vertex Factors**

Photon — spin 0 (charge $-e$)	$ie(p + p')^\mu$
Photon — spin $\frac{1}{2}$ (charge $-e$)	$ie\gamma^\mu$

Loops: $\int d^4k/(2\pi)^4$ over loop momentum; include -1 if fermion loop and take the trace of associated γ-matrices

Identical Fermions: -1 between diagrams which differ only in $e^- \leftrightarrow e^-$ or initial $e^- \leftrightarrow$ final e^+

Summary of basic features of QFT

Special relativity:

- spins (not there in Schrödinger equation!)
- anti-particles (not there in Schrödinger equation!)
- $E = mc^2$: pair creation and annihilation,
  infinitely many particles, infinitely many degrees of freedom, need field theory!

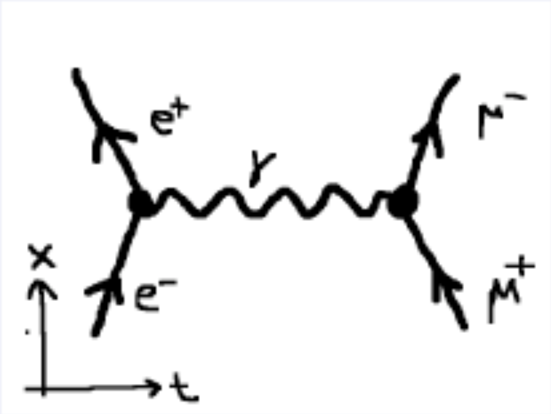
Quantum theory:

Virtual particles

- Recall Heisenberg's uncertainty relation: $\Delta E \Delta t \geq 1$.
- In scattering, this allows to create an unphysical particle with a lifetime $\tau \simeq \Delta t \leq 1/\Delta E$

(See right, the photon is called virtual.)

- No problem, if (local) conservation of energy-momentum guaranteed.



Virtual particles: Quantifying the example

- Consider the reaction $e^+e^- \to \gamma \to \mu^+\mu^-$.

$$E_{e^+e^-} = E_+ + E_- = \sqrt{m^2 + \vec{p}_+^2} + \sqrt{m^2 + \vec{p}_-^2} \geq 0.$$

  Energy-momentum conservation ensures that

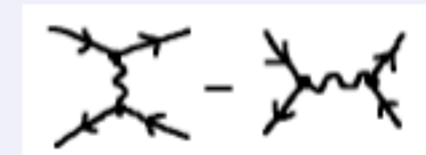
$$E_\gamma = E_{e^+e^-} \quad \text{and} \quad \vec{p}_\gamma = \vec{p}_+ + \vec{p}_-.$$

  However, due to the electron's rest mass it is impossible to satisfy

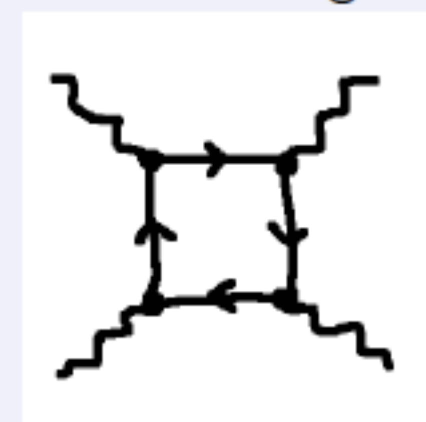
$$E_\gamma^2 - \vec{p}_\gamma^2 = m_\gamma^2 = 0,$$

  the photon is off its mass shell! ($E^2 - p^2 \neq 0! \longrightarrow$ unphysical)
- This implies that the lifetime of the photon is limited: $\Delta t < 1/\Delta E$ in the centre-of-mass frame of the photon ($\vec{p}_\gamma = 0$)
  For a photon with $E_{c.m.} = 200 MeV$, $\tau \leq 1\text{fm}/c \approx 10^{-24}$s.
- The photon cannot be observed at all - they remain intermediate.

Interference of amplitudes in e^-e^+-scattering



A QM effect: Light-by-light scattering



Constructing a specific QFT

- Which particles do we want to describe?
- Spins determine field types
- What are the symmetries of the interaction?
- Internal symmetries determine interactions

QCD:

- Description in terms of quarks (spinors) and gluons (vector fields)
- Baryon number conservation, U(1)
- Flavour quantum numbers
- SU(2) Isospin
- Approximate SU(3) Flavour
- Approximate chiral symmetry SU(2)xSU(2)