Bachelor Thesis

Computing path integrals using Lefschetz thimbles

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Abstract

Finding minima of functions S is of great physical importance. In particular, in quantum field theory calculating vacuum expectation values in spontaneously broken systems or finding classical equations of motions. For example if $S : \mathbb{R} \to \mathbb{R}$ has a unique minimum x_{min}, then Laplace's method (if applicable) gives

$$x_{\text{min}} = \lim_{\lambda \to \infty} \frac{\int_{\mathbb{R}} \mathrm{d}x \; x e^{-\lambda S}}{\int_{\mathbb{R}} \mathrm{d}x \; e^{-\lambda S}}.$$

We review a technique applicable to more general S by decomposing the integrals into ones over so called Lefschetz thimbles. This basically reduces to the previous situation with a unique minimum. As applications to the theory we choose the Airy function and a function motivated by the Gross-Neveu model.

Zusammenfassung

Das Finden von Minima von Funktionen S hat große physikalische Bedeutung. Insbesondere, wenn man Vakuum Erwartungswerte in der Quantenfeldtheorie für spontan gebrochene Systeme oder klassische Bewegungsgleichungen berechnet. Zum Beispiel, wenn $S : \mathbb{R} \to \mathbb{R}$ genau ein Minimum bei x_{min} besitzt, dann liefert Laplace's Methode (wenn anwendbar)

$$x_{\text{min}} = \lim_{\lambda \to \infty} \frac{\int_{\mathbb{R}} \mathrm{d}x \; x e^{-\lambda S}}{\int_{\mathbb{R}} \mathrm{d}x \; e^{-\lambda S}}.$$

Wir werden eine Methode lernen, die auf allgemeine S anwendbar ist. Dabei werden die Integrale in Summen einfacherer Integrale über sogenannte Lefschetz-Fingerhüte zerlegt. Damit reduziert man die Situation auf den Fall mit eindeutigem Minimum. Als Anwendung wählen wir die Airy-Funktion und eine Funktion, die vom Gross-Neveu-Modell motiviert ist.

Acknowledgements

I want to thank my supervisor Prof. Dr. Marc Wagner for giving me this wonderful topic, for his support, detailed explanations and patience. I am glad to be one of his students. My special thanks also go to my brother Adrian Zorbach for teaching me in mathematics and helping me with the English language. Especially his explanations about homology and our discussions on Witten's article were key to my understanding.

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die Arbeit selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel verfasst habe. Alle Stellen der Arbeit, die wörtlich oder sinngemäß aus Veröffentlichungen oder aus anderen fremden Texten entnommen wurden, sind von mir als solche kenntlich gemacht worden. Ferner erkläre ich, dass die Arbeit nicht - auch nicht auszugsweise - für eine andere Prüfung verwendet wurde.

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Niklas Zorbach

Contents

1 Introduction

One of the four fundamental forces in physics is the *strong interaction*. This force is mathematically described by quantum chromodynamics (QCD), a Yang-Mills theory based on the symmetry group SU(3). Similar to quantum electrodynamics (QED) there exists a gauge boson (the *gluon*) describing the interaction between quarks by carrying out color charge for the strong interaction. This force holds hadrons (e.g. protons) stable and is the strongest of the four fundamental forces. In contrast to QED, the symmetry group SU(3) is nonabelian. This leads to difficult analytical calculations of QCD. Therefore, we rely on numerical calculations.

Furthermore, it is of interest to study the strong interaction for *extreme conditions*: *high temperature* or *high density*. Those conditions appear in nature: shortly after the *"Big Bang"* the strong interaction was exposed to high temperatures. And in neutron stars the strong interaction experiences high density. The project *Collaborative Research Center TransRegio 211* (CRC-TR 211) studies strongly interacting matter under those conditions. For this purpose it is of interest to analyze the *phase structures* and *phase transitions* of strongly interacting matter in the temperature density plane. When temperature and density are low, quarks and gluons are confined (inside hadrons) and the chiral symmetry of the QCD is spontaneously broken. However, if temperature and density are high, then quarks and gluons are deconfined and chiral symmetry is unbroken.

Moreover, one observes inhomogeneous phases occurring at low temperature but high densities. Here the order parameter becomes space dependent and the system is restored at a discrete set of points in space. The A03 project (part of the CRC-TR 211 project) analyses those inhomogeneous phases and develops numerical methods. Starting developing methods, the A03 project investigates the phase transition for a QCD like theory: the Gross-Neveu model [1]. This model is analytically well known in $1 + 1$ dimensions (one space and one time dimension) for the large-N limit with chemical potential and also for finite temperature [2]. One has discovered three phases as depicted in figure 1: the unbroken, the homogeneously broken and the inhomogeneous phase. Also this model is suitable for developing numerical methods. However, the A03 project wants to investigate these phases in higher dimensions. For $2 + 1$ dimensions one has currently no proof that ensures the effective action to be purely real. Therefore, we assume a *complex effective action*. The computation of partition functions with complex effective actions is a problem known as the *sign problem*. Thus we need new mathematical methods to handle those actions. Note that complex actions also appear in other physical theories: QCD with chemical potential and Chern-Simons gauge theory [3]. To handle those, Picard-Lefschetz theory (based on Morse theory [4, 5]) provides a good framework. The calculation of Lefschetz thimbles can be difficult and has been performed using numerical methods such as the axis scan method [6].

Advantageous prerequisites for the reader are complex analysis and homology (see for example [7] or the appendix A).

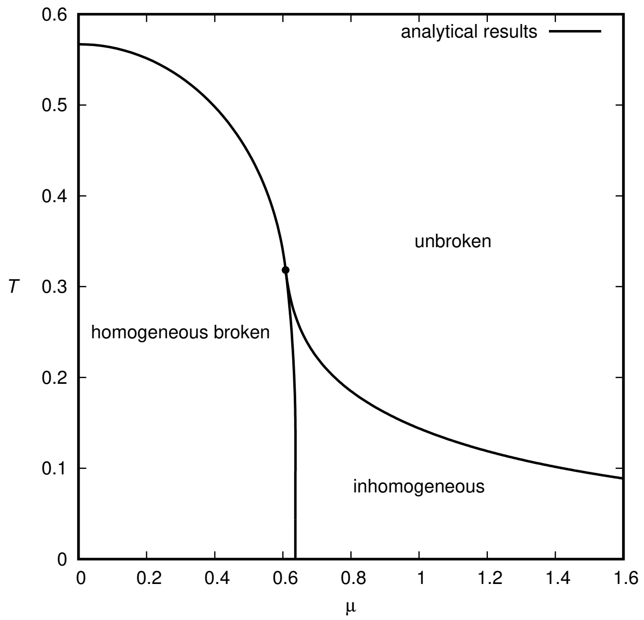


Figure 1: This figure shows the three phases detected in the Gross-Neveu model in $1+1$ dimensions. This data comes from [2].

In this thesis we are interested in contour integrals of the form

$$\int_C \mathrm{d}z \, \exp\{\lambda f(z)\}, \tag{1}$$

where C is a curve in the complex plane $X = \mathbb{C}$, λ is a complex parameter and f is a holomorphic function. Section 2.1 and 2.2 will handle the case for fixed λ. There we will learn how to express (1) as a $\mathbb{Z}$-linear combination of (simpler) contour integrals

$$\int_C \omega = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \omega, \tag{2}$$

where $n_\sigma \in \mathbb{Z}$, $\omega = \mathrm{d}z \, \exp\{\lambda f(z)\}$ and the sum runs over all saddle points σ, to which we associate paths $\mathcal{J}_\sigma$ given by steepest descent called *Lefschetz thimbles* [3, 8]. In section 2.3 we explain how varying λ around the origin can affect the Lefschetz thimbles. In section 2.4 we will learn how to apply *Laplace's method* (or *saddle point method*) [9, 10] to get a simpler expression for (1) when $|\lambda| \to \infty$. Section 3 illustrates the theory at hand of two examples: The Airy function [3] and the Gross-Neveu model in zero dimensions [1, 11, 8]. Both are illustrated by several plots created with gnu-plot and the reader is welcome to read this section alongside the abstract theory for a better understanding. Finally, in section 4 future directions of research are motivated.

2 Lefschetz thimbles

For the Morse theory, this section is based on [4, 5]. The reference for homology is [7] and for the theory about Lefschetz thimbles the article [3] was used.

2.1 Preparation

Recall that we study integrals of the form

$$\int_C \mathrm{d}z \ \exp(\lambda f(z)), \tag{3}$$

where C is a curve in $X = \mathbb{C}$, λ a complex parameter and f a holomorphic function. For convergence reasons we actually want $\lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Let us first illustrate the idea of what we want to achieve in this section. We will define a height function h on X (taking values in $\mathbb{R}$). The integration over regions with height $< T$ ($-T$ large) will be neglectable. In fact, we will collapse $\{x \in X \mid h(x) < T\}$ (see figure 2) to a point, when $-T$ is large enough and this then results in interesting homology groups (more on this later).

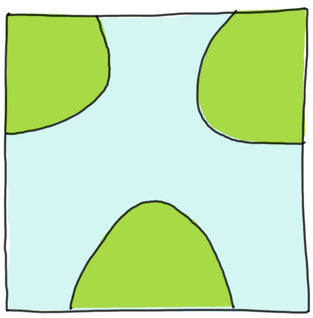


Figure 2: This picture shows the complex plane, the green regions are given by $\{x \in X \mid h(x) < T\}$.

Let us now focus on the other parts of integration, i.e. those where the height is $\geq T$. If the imaginary part of the exponent $\lambda f(z)$ is a constant c along the contour, we may pull $\exp(ic)$ out of the integral and have hence reduced to a real integral. This then can be handled with Laplace's method. In the case of nonconstant imaginary part, we try to deform the contour (without changing the value of the integral) such that the following property is satisfied:

(**LC**) The imaginary part of the exponent is locally constant along the deformed contour where the height is $\geq T$.

We can then again reduce to real integrals, because integration along regions with non-constant imaginary part are neglectable. For example let our original contour pass a critical point. If the critical point is a saddle point, then there are two paths starting from that critical point in the directions of steepest descent (with respect to the height). Along those paths (**LC**) is fulfilled, because of the Cauchy-Riemann equations. We then combine these two paths into one. Note that we can freely continue our path at its ends if they lie in regions with height $< T$, i.e. there we allow changes in imaginary part. The idea then is to build up our original contour from patching together such constructed paths each satisfying (**LC**). Obviously, the question arises whether such a deformation is always possible. For our situation the answer is *yes*[1]. We will actually have a *unique* combination of such paths, called *Lefschetz thimbles decomposition*[2] (see 21). Let us try to understand this (using Morse theory and homology): we define the *height function* $h = \mathrm{Re}(\lambda f)$ and we assume that h is a *Morse function*, which means that all critical points[3] of h are non degenerate[4]. Also, we assume finitely many critical points. We observe:

1. Since f is holomorphic, we can deform C to C' along a region A without changing the path integral. This follows form Cauchy's integral theorem which says

$$\int_C \omega - \int_{C'} \omega = \int_{\partial A} \omega = 0. \tag{4}$$

2. If $C \subset Z_{\leq T} := \{z \in X \mid h(z) \leq T\}$ for $T \ll 0$[5] and there exists $\lambda_0 = \alpha\lambda$ with $\alpha \in (0,1)$ such that $\int_C \mathrm{d}z \ \left|e^{\lambda_0 f}\right| < \infty$, then

$$\begin{aligned}
\left|\int_C \mathrm{d}z\, e^{\lambda f(z)}\right| &\leq \int_C \mathrm{d}z \ \left|e^{\lambda f(z)}\right| \\
&= \int_C \mathrm{d}z \ \left|e^{\lambda_0 f(z) + \lambda f(z) - \lambda_0 f(z)}\right| \\
&= \int_C \mathrm{d}z \ \left|e^{\lambda_0 f(z)}\right| e^{(1-\alpha)h(z)} \\
&\leq e^{(1-\alpha)T} \int_C \mathrm{d}z \ \left|e^{\lambda_0 f}\right| \\
&= \mathrm{const} \cdot e^{(1-\alpha)T} \approx 0 \text{ for } T \ll 0.
\end{aligned}$$

In other words, for $T \ll 0$ the term $e^{(1-\alpha)T}$ kills any contribution of the contour integral over regions $Z_{\leq T}$. Hence, we better look at the quotient space $X/Z_{\leq T}$, which is obtained from X by gluing all points in $Z_{\leq T}$ to one single point. This gluing is illustrated in figure 3.

[1] One assumption on our contour is that its ends lie in regions with height $< T$.

[2] ... if no *Stokes rays* occur. More on this later.

[3] We call a point $p \in \mathbb{R}^2$ of a function $h : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto h(x, y)$ *critical* if $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$.

[4] A critical point $p \in \mathbb{R}^2$ of a function $h : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto h(x, y)$ is called non degenerate if the Hessian matrix is invertible at p.

[5] We say $\phi(T)$ holds for $T \ll 0$ if $\exists N < 0 \ \forall n \leq N : \ \phi(n)$.

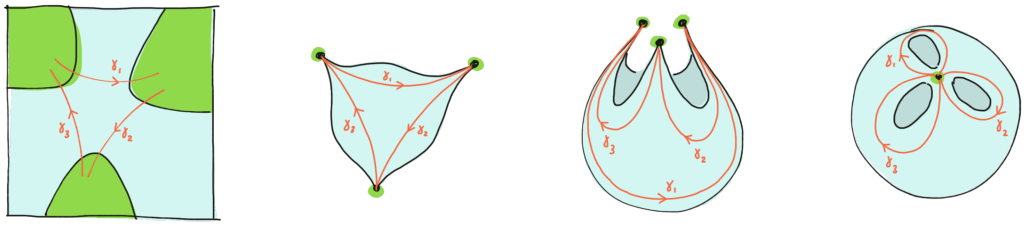


Figure 3: This series of images illustrates the quotient space X/Z where $X = \mathbb{C}$ and Z is the green region. It has first homology of rank two, because $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 = 1$ is contractible.

Although $X = \mathbb{C}$ has trivial higher homology groups $H_i(X;\mathbb{Z}) = 0$ for $i > 0$, $X/Z_{\leq T}$ might have nontrivial ones. Moreover, we have[6]

$$H_i(X/Z_{\leq T};\mathbb{Z}) = H_i(X, Z_{\leq T};\mathbb{Z}) \tag{5}$$

for $i > 0$, with the ith relative homology on the right hand side. Note, for $i = 0$, the left hand side is $\mathbb{Z}$ (i.e. the free abelian group over $\mathbb{Z}$ of rank equal the number of connected components of $X/Z_{\leq T}$), whereas the group on the right is 0, because every point can be deformed into $Z_{\leq T}$.

From now on, we want to assume that the endpoints of C lie in $Z_{\leq T}$ for $T \ll 0$. This ensures that C defines a loop in $X/Z_{\leq T}$. And if $H_1(X/Z_{\leq T};\mathbb{Z}) \neq 0$, then it can happen that C is not contractible. This is measured by $[C]$, the class in $H_1(X/Z_{\leq T};\mathbb{Z})$ represented by C. Indeed, we are only interested in the class $[C]$ of C because for any other $C' \in [C]$, we just observed

$$\int_{C'} \omega \approx \int_C \omega, \tag{6}$$

where the approximation becomes equal in the limit $T \to -\infty$. If it is clear from context we simplify the notation by writing C instead of $[C]$.

The fundamental result for our application comes from Morse theory and says that $H_1(X, Z_{\leq T};\mathbb{Z})$ is a free abelian group of rank $r := \#\sum$, where[7]

$$\sum = \{\sigma \mid \sigma \text{ critical point of } h\}. \tag{7}$$

I want to explain a few ingredients for proving this. First, we proof the assertion concerning the rank. For this we use the *Morse inequality*

$$\#\Sigma \geq \operatorname{rank}(H_1), \tag{8}$$

[6] If clear from context we will also simply write H_i for $H_i(X, Z_{\leq T};\mathbb{Z})$.

[7] The standard Morse theory determines the ordinary homology of compact manifolds. Also $X/Z_{\leq T}$ can be non compact (our case) Morse theory still describes the relative homology groups $H_i(X, Z_{\leq T};\mathbb{Z})$ [3].

which becomes an equality if and only if h is perfect[8]. This is true in our situation. Indeed, h is harmonic as it the real part of a holomorphic function. Hence, every critical point $\sigma \in \Sigma$ must be a saddle point of *Morse index*[9] 1 as shown in figure 4.

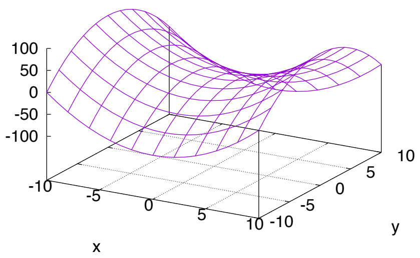


Figure 4: This figure illustrates a non degenerate saddle point of Morse index 1. Every critical point of a Morse function is a non degenerate saddle point.

It follows that the difference of Morse indices for any two critical points is zero. This is a criterium for h to be perfect [5]. This proves the claim about the rank. It remains to check that $H_1(X, Z_{\leq T}; \mathbb{Z})$ is *free*[10]. Next, we will study elements in $H_1(X, Z_{\leq T}; \mathbb{Z})$. For T sufficiently small $Z_{\leq T}$ decomposes into $\#\Sigma + 1$ disjoint connected components $\mathcal{V}_{T,i}$ with $i \in \{1, \dots, \#\Sigma + 1\}$. Let C be a curve in $\mathbb{C}$ with a chosen orientation. We say that C starts in $\mathcal{V}_m$ and ends in $\mathcal{V}_n$ if there is a parameterization $C(t)$ respecting the orientation such that for all T there exists t_0 such that for all $t < t_0$ $C(t) \in \mathcal{V}_{T,m}$ and similar $C(t) \in \mathcal{V}_{T,n}$ for some t_0 and any $t > t_0$[11]. We also write $C(-\infty) \in \mathcal{V}_m$ and $C(+\infty) \in \mathcal{V}_n$. Let $C_{\mathcal{V}_m \to \mathcal{V}_n}$ denote any curve starting in $\mathcal{V}_m$ and ending in $\mathcal{V}_n$. If there are no holes in X then any two choices for $C_{\mathcal{V}_m \to \mathcal{V}_n}$ represent the same element $[C_{\mathcal{V}_m \to \mathcal{V}_n}]$ in $H_1(X/Z_{\leq T}; \mathbb{Z})$. Also, the $[C_{\mathcal{V}_m \to \mathcal{V}_n}]$ for fixed m and $n \neq m$ define a basis for $H_1(X/Z_{\leq T}; \mathbb{Z})$. Therefore, we have a *unique* $\mathbb{Z}$-linear combination

$$[C] = \sum_{n \in \{1,\dots,\#\Sigma+1\} \setminus \{m\}} c_n [C_{\mathcal{V}_m \to \mathcal{V}_n}], \tag{9}$$

where c_n are integer coefficients. Thus the integral (3) can be written as

$$\int_C \mathrm{d}z \; \exp(\lambda f(z)) = \sum_{n \in \{1,\dots,\#\Sigma+1\} \setminus \{m\}} c_n \int_{C_{\mathcal{V}_m \to \mathcal{V}_n}} \mathrm{d}z \; \exp(\lambda f(z)). \tag{10}$$

This discussion can be generalized for functions with singularities. As we will see in the zero-dimensional Gross-Neveu model in section 3.2.

[8] This is the definition of perfect.
[9] The Morse index of h at σ is defined as the number of negative eigenvalues of the Hessian of h at σ.
[10] An abelian group is called free if it has a basis. A basis is a linear independent set of generators. A proof for this can be found in [5]
[11] Note that there is no definition for $\mathcal{V}_m$ since $\bigcap_T \mathcal{V}_{T,m} = \emptyset$.

2.2 Lefschetz thimbles

A basis for $H_1(X/Z_{\leq T};\mathbb{Z})$ better suited for computations (using Laplace's method) is given by the *Lefschetz thimbles*. Namely, let σ be a critical point and define the *stable Lefschetz thimbles*

$$\begin{aligned}\mathcal{J}_\sigma = &\text{ points reached from } \sigma \text{ by } \textit{steepest descent}\\ &\text{ with respect to the standard metric on } X.\end{aligned}$$

If $\mathcal{J}_\sigma$ contains exactly one critical point, it is called *good* and otherwise *bad* or a *Stokes ray*[12]. Let us now discuss why good Lefschetz thimbles define elements $[\mathcal{J}_\sigma]$ in the relative homology. And if no Stokes rays occur those actually define the basis we were looking for. We can ensure this (i.e. no Stokes rays occur) by slightly varying $\lambda \in \mathbb{C}$. The $\mathcal{J}_\sigma$ have two key properties:

- $\mathrm{Im}(\lambda f)$ is constant on $\mathcal{J}_\sigma$.
- $h = \mathrm{Re}(\lambda f)$ decreases on $\mathcal{J}_\sigma$ in directions away from σ.

The second property ensures that h actually tends to $-\infty$ at the ends of $\mathcal{J}_\sigma$ and the integration over $\mathcal{J}_\sigma$ converges, if $\mathcal{J}_\sigma$ is good. It follows that $[\mathcal{J}_\sigma]$ is a well-defined element in H_1. To prove those key properties about $\mathcal{J}_\sigma$ it is helpful to have the following characterization:

$$\mathcal{J}_\sigma = \left\{ u(0) \in X \;\middle|\; \lim_{t\to-\infty} u(t) = \sigma \right\}, \tag{11}$$

where $u(t) = u^1(t) + iu^2(t) \in X$ is a solution to the downward flow equations (with respect to the metric g)

$$\frac{du^i}{dt} = -g^{ij}\frac{\partial h}{\partial u^j}. \tag{12}$$

This means $u(t)$ is a path starting at σ (for $t = -\infty$) and going in direction of steepest descent, reaching some point $u(0)$. It is crucial to start at σ only in the limit $t \to -\infty$, since σ is a critical point and solutions to the downward flow equations satisfying $u(t_0) = \sigma$ for finite t_0 must be constant. This is, because

$$\frac{dh}{dt} = \sum_i \frac{\partial h}{\partial u^i}\frac{du^i}{dt} = -\sum_i \left(\frac{\partial h}{\partial u^i}\right)^2. \tag{13}$$

So h is decreasing along downward flows showing the second key property for $\mathcal{J}_\sigma$. In particular, if $u(t)$ crosses a critical point at finite t_0, then $u(t)$ becomes constant since otherwise $h(u(t))$ would increase.

[12] In this thesis a Stokes ray is a special Lefschetz thimble. This may differ from the literature, where a $\mathcal{J}_\sigma$ is either a Lefschetz thimble or a Stokes ray depending on the number of critical points it contains.

Finally, the constancy of $\mathrm{Im}(\lambda f)$ on $\mathcal{J}_\sigma$ follows from the Cauchy-Riemann equations and the holomorphicity of f: let $\mathcal{I} = \lambda f$ and $H = \mathrm{Im}(\mathcal{I})$, then by using a Kähler metric $\mathrm{d}s^2 = |\mathrm{d}z|^2$ one can rewrite the downward flow equation to

$$\frac{dz}{dt} = -\frac{\partial \bar{\mathcal{I}}}{\partial \bar{z}}, \tag{14}$$

where we used $\frac{\partial \mathcal{I}(z)}{\partial \bar{z}} = 0$ (from holomorphicity of $\mathcal{I}$) and one observes

$$\frac{dH}{dt} = \frac{\partial H}{\partial z}\frac{dz}{dt} + \frac{\partial H}{\partial \bar{z}}\frac{d\bar{z}}{dt} = \frac{1}{2i}\left(\frac{\partial \mathcal{I}}{\partial z}\frac{dz}{dt} - \frac{\partial \bar{\mathcal{I}}}{\partial \bar{z}}\frac{d\bar{z}}{dt}\right) = 0. \tag{15}$$

Hence, H is a conserved quantity along the Lefschetz thimbles and therefore the imaginary part stays constant[13].

Let us assume that no Stokes rays appear. For example, any two critical points have different imaginary values of $\mathcal{I}$. Then the number of $[\mathcal{J}_\sigma]$, for $\sigma \in \Sigma$ is exactly the rank of $H_1(X/Z_{\leq T}; \mathbb{Z})$ and hence we would like to show that they actually generate H_1. For this, we need to define the *intersection number*. First, we define the *unstable Lefschetz thimbles* $\mathcal{K}_\sigma$ by

$$\begin{aligned}\mathcal{K}_\sigma = {} & \text{points reached from } \sigma \text{ by } \textit{steepest ascent} \\ & \text{with respect to the standard metric on } X.\end{aligned}$$

Similar to previous observations we have

$$\mathcal{K}_\sigma = \left\{ u(0) \in X \;\middle|\; \lim_{t\to\infty} u(t) = \sigma \right\}, \tag{16}$$

where as before $u(t)$ is a solution of the downward flow equations (12). Also we define

$$Z_{\geq T} = \{x \in X \mid h(x) \geq T\}. \tag{17}$$

Then, once we choose an orientation for $\mathcal{K}_\sigma$ we obtain elements $[\mathcal{K}_\sigma]$ for $H_1(X, Z_{\geq T}; \mathbb{Z})$. Again, we have a notion of *good* and *bad* (unstable) Lefschetz thimbles and we assume that no bad thimbles occur to actually have well-defined homology elements. Since this time, h is increasing on $\mathcal{K}_\sigma$ away from σ but decreasing on $\mathcal{J}_\sigma$ away from σ, it follows that $\mathcal{K}_\sigma$ and $\mathcal{J}_\sigma$ only intersect at σ. Moreover, for $\sigma \neq \tau$ $\mathcal{J}_\sigma$ and $\mathcal{K}_\tau$ do not intersect because distinct critical points are not connected by gradient flows (we assumed no Stokes rays). Therefore, the *intersection pairing* is

$$\langle \mathcal{J}_\sigma, \mathcal{K}_\tau \rangle = \delta_{\sigma\tau}, \tag{18}$$

after possibly reorienting some thimbles. Now we are able to check that the $\mathcal{J}_\sigma$ are linearly independent and hence define a basis for $H_1(X/Z_{\leq T}; \mathbb{Z})$. For this let $\sum_\sigma a_\sigma [\mathcal{J}_\sigma] = 0$. Applying $\langle \cdot, \mathcal{K}_\tau \rangle$ we get

$$0 = \sum_\sigma a_\sigma \langle \mathcal{J}_\sigma, \mathcal{K}_\tau \rangle = \sum_\sigma a_\sigma \delta_{\sigma,\tau} = a_\tau.$$

[13] One can interpret H as a Hamiltonian from classical mechanics and the downward flow equations are nothing but Hamilton's equation of motion $\frac{dz}{dt} = \{H, z\}_P$, where $\{f, g\}_P = -2i\left(\partial f \bar{\partial} g - \bar{\partial} f \partial g\right)$ denotes the Poisson bracket with $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$.

This is true for all $\tau \in \Sigma$ and so the linear combination must be trivial.
As before, the basis of $\mathcal{J}_\sigma$'s allows us to write

$$[C] = \sum_\sigma n_\sigma [\mathcal{J}_\sigma], \tag{19}$$

with *unique* integer coefficients n_σ, given by

$$n_\sigma = \langle C, \mathcal{K}_\sigma \rangle \tag{20}$$

and the integral (3) can be written as

$$\int_C \mathrm{d}z\, e^{\lambda f(z)} = \sum_{\sigma\in\Sigma} \langle C, \mathcal{K}_\sigma \rangle \int_{\mathcal{J}_\sigma} \mathrm{d}z\, e^{\lambda f(z)}. \tag{21}$$

2.3 Stokes rays

In the last section we have learned that for integration contours C defining elements $[C] \in H_1(X, Z_{\leq T}; \mathbb{Z})$ there is a Lefschetz thimbles decomposition. For this however we assumed that no Stokes rays occur. In this section we will investigate Stokes rays in more detail. We will analyze the behavior of the decomposition (21) by varying λ such that one crosses a Stokes ray.

Assume given λ such that there appears a Stokes ray. By definition that ray connects two distinct critical points. Let those be σ_+ and σ_-. The first key property implies that $\mathrm{Im}\, \mathcal{I}(\sigma_+) = \mathrm{Im}\, \mathcal{I}(\sigma_-)$. Hence, λ lies in the set of *Stokes lines*

$$S := \{\lambda \in \mathbb{C}^\times \mid \mathrm{Im}\, \mathcal{I}(\sigma_+) = \mathrm{Im}\, \mathcal{I}(\sigma_-)\}. \tag{22}$$

Also, the second key property ensures $h(\sigma_+) = \mathrm{Re}\, \mathcal{I}(\sigma_+) \neq \mathrm{Re}\, \mathcal{I}(\sigma_-) = h(\sigma_-)$. Thus, λ does not lie in the set of *anti Stokes lines*

$$A := \{\lambda \in \mathbb{C}^\times \mid \mathrm{Re}\, \mathcal{I}(\sigma_+) = \mathrm{Re}\, \mathcal{I}(\sigma_-)\}. \tag{23}$$

Let us recall the problem that a Stokes ray (may) fail to define an element of H_1. However, S is closed and therefor no ϵ-ball centered at λ is contained in S. This allows us to slightly vary λ such that $\lambda \notin S$ in which case all thimbles are good. Or in other words we can write $\lambda = \lim_{n\to\infty} \lambda_n$ with $\lambda_n \notin S$[14]. Now that we have good thimbles $J_{\sigma_\pm}$ for all n we also want to understand the dependency of the choice of presentation $\lambda = \lim_{n\to\infty} \lambda_n$. Indeed, approximating λ from different directions shows the phenomenon of *Stokes jumps*

$$\begin{aligned} [\mathcal{J}_{\sigma_+}] &\to [\mathcal{J}_{\sigma_+}] \pm [\mathcal{J}_{\sigma_-}] \\ [\mathcal{J}_{\sigma_-}] &\to [\mathcal{J}_{\sigma_-}], \end{aligned} \tag{24}$$

[14]Later we want to study the asymptotic behavior of the Airy function. For this we need additional information about the λ_n. First, since A is closed and $\lambda \notin A$ we can assume all $\lambda_n \notin A$. Then we choose a subsequence of the λ_n such that $\mathrm{Re}\, \mathcal{I}(\sigma_+) > \mathrm{Re}\, \mathcal{I}(\sigma_-)$ for all n (or $<$ instead of $>$).

as is schematically depicted in figure 5.

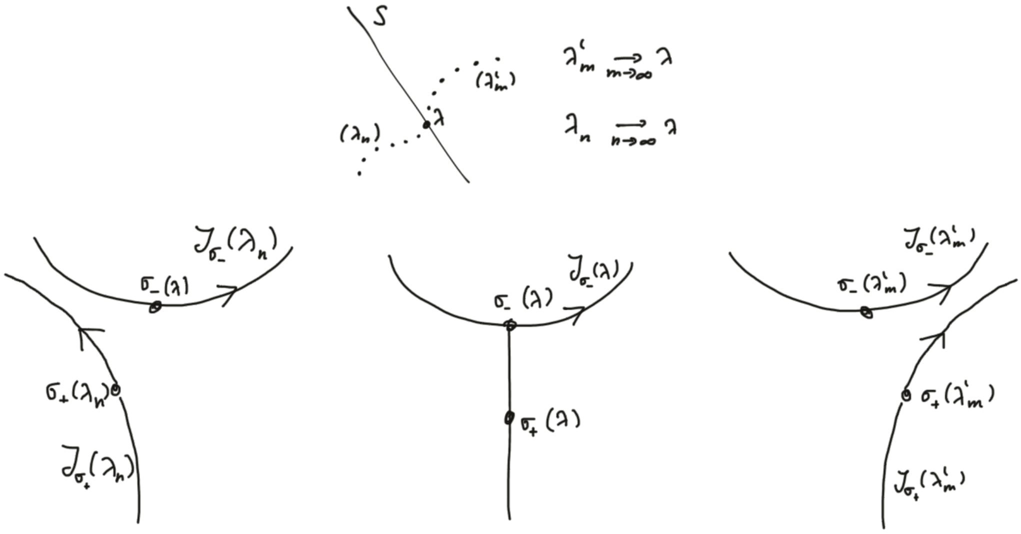


Figure 5: Approximating $\lambda \in S$ from different directions can show Stokes jumps.

2.4 Laplace's method

In this section we complete our analysis of $H_1(X/Z_{\leq T};\mathbb{Z})$ by studying the asymptotic behavior when $|\lambda| \to \infty$. This will also show why a basis given by Lefschetz thimbles can be more useful for computations (for large λ).

We begin with the *Laplace's method* (a proof can be found in [9]), which says that[15]

$$\int_a^b \mathrm{d}x\, g(x)e^{\lambda f(x)} \sim g(x_0)e^{\lambda f(x_0)}\sqrt{\frac{2\pi}{\lambda(-f''(x_0))}} \text{ as } \lambda \to \infty, \tag{25}$$

where $\lambda \in \mathbb{R}^+$, $g \in C[a,b]$ a positive function ($g(x) > 0\ \forall x \in [a,b]$), $f \in C^2[a,b]$ with a global maximum x_0 in the open interval (a,b). We can use Laplace's method to compute integrals over Lefschetz thimbles. We assume $\lambda \in \mathbb{R}$[16] and parameterize J_σ by $\gamma : \mathbb{R} \to J_\sigma$ such that $\gamma(0) = \sigma$. Using that $\mathrm{Im}(\lambda f)$ is constant on J_σ we can write

$$f = \mathrm{Re}(f) + i\underbrace{\mathrm{Im}(f)}_{\text{const}} = \underbrace{\mathrm{Re}(f) - \mathrm{Re}(f(\sigma))}_{=h} + \underbrace{\mathrm{Re}(f(\sigma)) + i\,\mathrm{Im}(f(\sigma))}_{=f(\sigma)} = h + f(\sigma), \tag{26}$$

where the *real* function h satisfies

$$\ddot{h}(0) = \ddot{f}(0) = f'(\sigma)\ddot{\gamma}(0) + f''(\sigma)\dot{\gamma}^2(0) = f''(\sigma)\dot{\gamma}^2(0). \tag{27}$$

[15] For two functions f and g we write $f \sim g$ if and only if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$.
[16] If $\lambda \in \mathbb{C}$ one can redefine f to $\tilde{f} = e^{i\arg(\lambda)}f$ and use $\tilde{f}$ instead.

This allows us to write

$$\int_{\mathcal{J}_\sigma} \mathrm{d}z\; e^{\lambda f} = \int_{\mathbb{R}} \mathrm{d}t\; \dot\gamma e^{\lambda(h+f(\sigma))} = e^{\lambda f(\sigma)} \int_{\mathbb{R}} \mathrm{d}t\; \dot\gamma e^{\lambda h}.$$

Although $\dot\gamma$ need not satisfy the conditions for applying Laplace's method (directly) there exists an extension of Laplace's method called *method of steepest descent*, which reduces to the Laplace's method by deforming the integration contour. One can learn more about this method in [9, 10]. The upshot is that we still have the same type of asymptotics, namely

$$e^{\lambda f(\sigma)} \int_{\mathbb{R}} \mathrm{d}t\; \dot\gamma e^{\lambda h} \sim e^{\lambda f(\sigma)} \sqrt{\frac{2\pi}{-\lambda \ddot h(0)}}\dot\gamma(0) e^{\lambda \overbrace{h(0)}^{=0}} = \sqrt{\frac{2\pi}{-\lambda \ddot h(0)}}\dot\gamma(0) e^{\lambda f(\sigma)}.$$

Finally, being careful with complex roots[17] we see

$$\sqrt{\frac{2\pi}{-\lambda \ddot h(0)}}\dot\gamma(0) e^{\lambda f(\sigma)} = \pm e^{\lambda f(\sigma)} \sqrt{\frac{2\pi}{-\lambda f''(\sigma)}}$$

with $+$ if and only if $\arg(\dot\gamma(0)) \in (-\frac{\pi}{2}, +\frac{\pi}{2}]$. Combining this calculation yields

$$\int_{\mathcal{J}_\sigma} \mathrm{d}z\; e^{\lambda f} = \begin{cases} +e^{\lambda f(\sigma)} \sqrt{\frac{2\pi}{-\lambda f''(\sigma)}} & \text{if } \arg[\dot\gamma(0)] \in (-\frac{\pi}{2}, +\frac{\pi}{2}] \\ -e^{\lambda f(\sigma)} \sqrt{\frac{2\pi}{-\lambda f''(\sigma)}} & \text{otherwise} \end{cases}.$$

Actually, the sign is no surprise, because both sides must depend on the orientation of the integration cycle.

[17] We use $\sqrt{z} = \sqrt{r} e^{i\phi/2}$ if $z = re^{i\phi}$ and $\phi \in (-\pi, +\pi]$.

3 Applications

In this section we want to use the theory of Lefschetz thimbles to understand the asymptotic behavior of the Airy function and after that study a simplified Gross-Neveu model with respect to spontaneous symmetry breaking.

Also a remark about the plots: 13, 14 and 17. The plots always show the complex plane $\mathbb{C}$. The coloring is given by the respective height function h. In particular, green regions correspond to $Z_{\leq T}$. The stable and unstable thimbles are depicted as black and dashed red lines. Finally, dots represent saddle points, unless they are red. Those stand for singularities.

3.1 Airy function

The Airy function is a solution of the Airy differential equation

$$\frac{\partial^2 f}{\partial z^2} = zf. \tag{28}$$

First some remarks about this equation:

- It's space of solutions has dimension two.
- The set of poles of the coefficients (1 and z) only have a (simple) pole at $z = \infty$ in $\hat{\mathbb{C}}$[18]. It follows from this that the monodromy of (28) is trivial and hence, every solution to (28) can be analytically continued to the whole complex plane. In particular the Airy function will be a function on $\mathbb{C}$.

To solve (28) one formally applies a Fourier transformation to obtain a simpler equation

$$\frac{\partial g}{\partial x} = ix^2 g \tag{29}$$

with a space of solutions spanned by

$$g = e^{i\frac{x^3}{3}}. \tag{30}$$

It may seem strange that the dimensions of the solution spaces of (28) and (29) are different, but we will see how to construct solutions to (28) from those of (29) by integrating over curves C in the complex plane representing elements $[C] \in H_1$ for a certain homology group H_1 of rank *two*:

$$f(z) = \int_C \mathrm{d}x\ g(x)e^{izx} = \int_C \mathrm{d}x\ e^{i\left(\frac{x^3}{3}+zx\right)} \tag{31}$$

[18] $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

Indeed, $[C] \in H_1$ will ensure that this inverse Fourier transform actually is well-defined and in turn makes $f(z)$ a solution to (28):

$$\begin{aligned}\frac{\partial^2 f}{\partial z^2}(z) &= \int_C \mathrm{d}x\ (ix)^2 e^{izx} g(x) \\ &= i \int_C \mathrm{d}x\ e^{izx} \frac{\partial g}{\partial x}(x) \\ &= i \left(\left[e^{izx} g(x) \right]_{\partial C} - \int_C \mathrm{d}x\ (iz) e^{izx} g(x) \right) \\ &= i \left[e^{izx} g(x) \right]_{\partial C} + z f(z).\end{aligned}$$

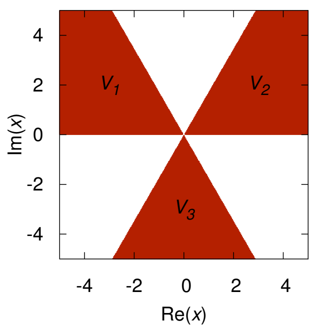
In particular, note that $[C] \in H_1$ has to imply vanishing of the boundary term. To define H_1, let $h(x, z) = \mathrm{Re}(\mathcal{I}(x, z))$, where $\mathcal{I}(x, z) = i \left(\frac{x^3}{3} + zx \right)$. Then the integral (31) converges and also the boundary term $\left[e^{izx} g(x) \right]_{\partial C}$ vanishes if C is closed (a loop) or its "ends"[19] lie in regions

$$Z_{\leq T}(z) = \{ x \in \mathbb{C} \mid h(x, z) \leq T \} \tag{32}$$

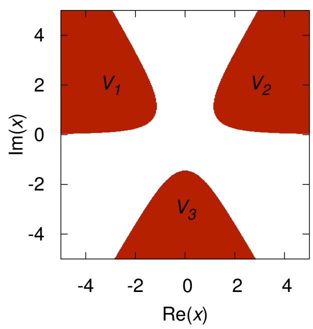
for every T. This is satisfied if $[C] \in H_1(X, Z_{\leq T}(z); \mathbb{Z})$ for all T. Intuitively only the leading term of $\mathcal{I}$ should matter. Indeed, define

$$Z'_{\leq T'} = \left\{ x \in \mathbb{C} \,\middle|\, \mathrm{Re} \left(\frac{ix^3}{3} \right) \leq T' \right\} \tag{33}$$

and observe that for every T and z there exists T' such that $Z'_{\leq T'} \subset Z_{\leq T}(z)$ and conversely for every T' and z there exists a T with $Z_{\leq T}(z) \subset Z'_{\leq T'}$. Illustrated in figure 6 are two plots of $Z'_{\leq T'}$ for $T' = 0$ and $T' = -1$.



(a) $Z'_{\leq 0}$ is colored in red.



(b) $Z'_{\leq -1}$ is colored in red.

Figure 6

[19] If $z(t)$ is a parameterization of C, then $z(t) \in Z_{\leq T}(z)$ for $t \ll 0$ and $t \gg 0$.

For T' sufficiently small (in fact $T' < 0$) $Z'_{\leq T'}$ decomposes into three disjoint connected components $\mathcal{V}_{T',1}, \mathcal{V}_{T',2}$ and $\mathcal{V}_{T',3}$. We therefor define $H_1 := H_1(X, Z'_{\leq T'}; \mathbb{Z})$ for some sufficiently small T'. In particular, H_1 is independent of z. The theory just learned shows that H_1 is free abelian of rank two and generated by curves $C_{\mathcal{V}_1 \to \mathcal{V}_2}$, $C_{\mathcal{V}_2 \to \mathcal{V}_3}$ connecting the regions $\mathcal{V}_i$. We can now define generators for the space of solutions of (28):

$$\mathrm{Ai}(z) = \int_{C_{\mathcal{V}_1 \to \mathcal{V}_2}} \omega(z), \tag{34}$$

$$\mathrm{Bi}(z) = \int_{C_{\mathcal{V}_2 \to \mathcal{V}_3}} \omega(z) - \int_{C_{\mathcal{V}_3 \to \mathcal{V}_1}} \omega(z), \tag{35}$$

where $\omega(z) = \frac{\mathrm{d}x}{2\pi}\, e^{\mathcal{I}(x,z)}$. $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are called *Airy functions*.

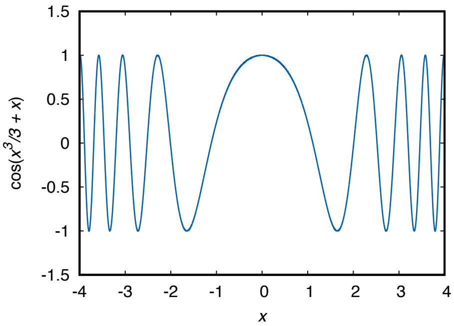


Figure 7: $\mathrm{Re}\left(e^{\mathcal{I}(x,z=1)}\right)$ shows oscillations on the real axis.

Note that the standard definition for $\mathrm{Ai}(z)$ seems to be $\frac{1}{2\pi}\int_{C_\mathbb{R}} \mathrm{d}x\, e^{\mathcal{I}(x,z)}$, where $C_\mathbb{R}$ denotes the curve given by the real axis $\mathbb{R}$, oriented from left to right. However $C_\mathbb{R}$ defines no element $[C_\mathbb{R}] \notin H_1$. In fact integrating over $C_\mathbb{R}$ shows oscillations (see figure 7), which can be avoided by shifting the integration contour $C_\mathbb{R}$ by $i\epsilon$ for any $\epsilon > 0$[20]. Indeed,

$$\lim_{x\to\pm\infty} \left| e^{\mathcal{I}(x+i\epsilon,z)} \right| = \lim_{x\to\pm\infty} \left| e^{-x^2\epsilon + \frac{\epsilon^3}{3} - \mathrm{Im}(z)x - \mathrm{Re}(z)\epsilon} \right| = \lim_{x\to\pm\infty} \left| e^{-x^2\epsilon} \right| = 0$$

and we have $[C_{\mathbb{R}+i\epsilon}] = [C_{\mathcal{V}_1 \to \mathcal{V}_2}]$ ($C_{\mathbb{R}+i\epsilon}$ being orientated from left to right).

3.1.1 Asymptotic analysis

Next we want to compute the asymptotic behavior of $\mathrm{Ai}(re^{i\phi})$ for $r \to \infty$. For ϕ with $z = re^{i\phi}$ not lying on Stokes line this is done in the following steps:

[20] Actually, one can apply the residue theorem to show $\int_{C_\mathbb{R}} \omega = \int_{C_{\mathbb{R}+i\epsilon}} \omega$.

1. Decompose the Airy function as
$$\mathrm{Ai}(z) = \int_{C_{\mathbb{R}+i\epsilon}} \omega(z) = \sum_{\sigma} n_\sigma \int_{\mathcal{J}_\sigma} \omega(z), \tag{36}$$
where $n_\sigma = \langle C_{\mathbb{R}+i\epsilon}, \mathcal{K}_\sigma \rangle$ and the sum runs over the critical points σ_+ and σ_- of $\mathcal{I}(x,z) = i\left(\frac{x^3}{3} + zx\right)$:
$$\frac{\partial \mathcal{I}}{\partial x}(\sigma) = i(x(\sigma)^2 + z) = 0 \Leftrightarrow x(\sigma) = \pm i\sqrt{z}. \tag{37}$$
For this decomposition we need the critical points to be non degenerate and the thimbles to be good. This is fulfilled, if $z \neq 0$ and
$$\mathrm{Im}(\mathcal{I}(\sigma_+, z)) \neq \mathrm{Im}(\mathcal{I}(\sigma_-, z)). \tag{38}$$
So let σ_+ and σ_- have x coordinates $x(\sigma_+) = i\sqrt{z}$ and $x(\sigma_-) = -i\sqrt{z}$. Note that reorienting the thimbles $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ to $\mathcal{J}'_\sigma = -\mathcal{J}_\sigma$ and $\mathcal{K}'_\sigma = -\mathcal{K}_\sigma$ has no effect on
$$\langle C_{\mathbb{R}+i\epsilon}, \mathcal{K}_\sigma \rangle \int_{\mathcal{J}_\sigma} \omega(z) = -\langle C_{\mathbb{R}+i\epsilon}, \mathcal{K}'_\sigma \rangle \int_{-\mathcal{J}'_\sigma} \omega(z) = \langle C_{\mathbb{R}+i\epsilon}, \mathcal{K}'_\sigma \rangle \int_{\mathcal{J}'_\sigma} \omega(z). \tag{39}$$

2. Find the coefficients $n_\pm = \langle C_{\mathbb{R}+i\epsilon}, \mathcal{K}_{\sigma_\pm} \rangle$ as *locally constant*[21] functions on $\mathbb{C}^\times \setminus S$, where S denotes the Stokes lines. We compute those numbers by looking at the plots in the appendix B. For example the plot 8 (or 17e) for $z = e^{i0.1\pi}$ shows $n_+ = 1$ and $n_- = 0$.

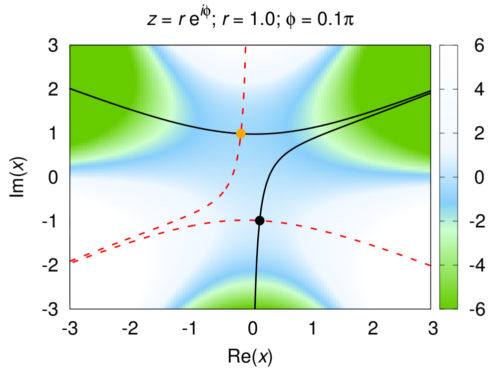


Figure 8: This plot shows the stable (black) and unstable thimbles (dashed red) for $z = e^{i0.1\pi}$.

[21] First note that for T' small enough and any $T \leq T'$:
$$H_1(z,T) := H_1\left(X, \left\{\mathrm{Re}\left(i\left(\frac{x^3}{3} + zx\right)\right) \leq T\right\}; \mathbb{Z}\right) \cong H_1\left(X, \left\{\mathrm{Re}\left(i\left(\frac{x^3}{3}\right)\right) \leq T'\right\}; \mathbb{Z}\right) \overset{\text{Def.}}{=} H_1,$$
the latter obviously being independent of z and T. The above isomorphism maps $[\mathcal{J}_\sigma(z)] \in H_1(z,T)$ to the class in H_1 also represented by the curve $\mathcal{J}_\sigma(z)$. A similar argument works for the $\mathcal{K}_\sigma(z)$. In particular, we can define the intersection pairing for H_1 and since the function $X \setminus S \to H_1$, $z \mapsto [\mathcal{J}_\sigma(z)]$ is locally constant, the same is true for the intersection pairing.

The result is summarized in figure 9.

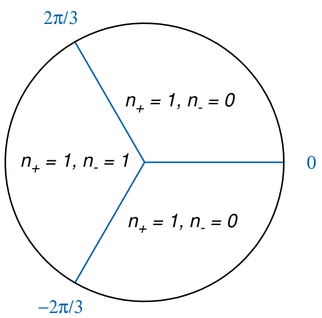


Figure 9: In this picture the z-plane of is shown. Stokes lines are depicted in blue. On those Lefschetz thimbles need not be good. This can result in jumps of the intersection numbers $n_\pm = \langle C_{\mathbb{R}+i\epsilon}, \mathcal{K}_{\sigma_\pm}\rangle$.

3. Use Laplace's method to compute the asymptotic behavior of (36) with the coordinate transformation $y = \frac{x}{\sqrt{r}}$ and $\lambda = \sqrt{r}^3$:

$$\begin{aligned}\int_{\mathcal{J}_{\sigma_\pm}} \mathrm{d}x\; e^{i\left(\frac{x^3}{3}+zx\right)} &= \sqrt{r}\int_{\mathcal{J}_{\sigma_\pm}} \mathrm{d}y\; e^{i\sqrt{r}^3\left(\frac{y^3}{3}+e^{i\phi}y\right)} \\ &= \lambda^{\frac{1}{3}}\int_{\mathcal{J}_{\sigma_\pm}} \mathrm{d}y\; e^{i\lambda\left(\frac{y^3}{3}+e^{i\phi}y\right)} \\ &\sim \pm\lambda^{\frac{1}{3}}\sqrt{\frac{\pi}{-\lambda i y(\sigma_\pm)}}\, e^{i\lambda\left(\frac{y(\sigma_\pm)^3}{3}+e^{i\phi}y(\sigma_\pm)\right)},\end{aligned}$$

with the sign $\pm$ depending on the orientations of $\mathcal{J}_{\sigma_\pm}$.

4. Compare the asymptotic contributions of both critical points (see figure 10):

$$\int_{\mathcal{J}_{\sigma_+}} \omega(z) + \int_{\mathcal{J}_{\sigma_-}} \omega(z) \sim \begin{cases} \int_{\mathcal{J}_{\sigma_+}} \omega(z) & \text{if } h(\sigma_+, z) > h(\sigma_-, z) \\ \int_{\mathcal{J}_{\sigma_-}} \omega(z) & \text{if } h(\sigma_+, z) < h(\sigma_-, z) \\ \int_{\mathcal{J}_{\sigma_+}} \omega(z) + \int_{\mathcal{J}_{\sigma_-}} \omega(z) & \text{otherwise} \end{cases}.$$

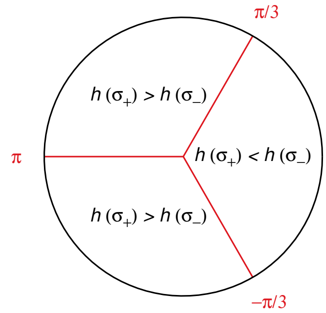


Figure 10: In this picture the z-plane of is shown. The three regions are separated by anti Stokes lines (red) and determine the dominating thimble(s). Note however that the contribution for $\mathrm{Ai}(z)$ of a dominating thimble might be zero if the corresponding intersection number vanishes. So although $h(\sigma_+) < h(\sigma_-)$ between $-\pi/3$ and $\pi/3$, J_{σ_-} does not contribute, because there $n_- = 0$.

Next we extend the analysis to the case that z is lying on a Stokes line. Then one of the thimbles might be a Stokes ray and does not define an element of the homology. However, $h(\sigma_+, z) \neq h(\sigma_-, z)$ (since $\mathrm{Im}(\mathcal{I}(\sigma_+, z)) = \mathrm{Im}(\mathcal{I}(\sigma_-, z))$) and therefore there exists (ϕ_n) with $\phi = \lim_{n\to\infty} \phi_n$ such that the $re^{i\phi_n}$ do not lie on a Stokes lines and $h(\sigma_+, z) > h(\sigma_-, z)$ for all n (or $<$ instead of $>$). Hence above steps work for ϕ_n's and the dominating summand in $\int_{\mathcal{J}_{\sigma_+}} \omega(z) + \int_{\mathcal{J}_{\sigma_-}} \omega(z)$ from step 4. is the same for all n. This "continuity" of step 4. away from $h(\sigma_+, z) = h(\sigma_-, z)$ then yields

$$\mathrm{Ai}(re^{i\phi}) \sim \begin{cases} \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}\sqrt{z}^3} & \phi \in (-\pi, \pi) \\ \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \cos\left(\frac{2}{3}\sqrt{z}^3 - \frac{\pi}{4}\right) & \phi = \pi \end{cases} . \tag{40}$$

This sort of asymptotic behavior is called a *Stokes phenomenon.*

3.2 The Gross-Neveu model

As we have seen in section 3.1 one can determine the asymptotic behavior of certain integrals by applying the Lefschetz thimble approach. The next application is physically motivated by the *Gross-Neveu model* [1]. This model is a relativistic quantum field theory for Dirac fermion fields $\bar{\psi} = (\bar{\psi}_1, \ldots, \bar{\psi}_N)^{\mathrm{T}}$ and $\psi = (\psi_1, \ldots, \psi_N)^{\mathrm{T}}$ with N flavors and a single quark color. It is characterized by a quadratic four-point interaction term $\frac{g^2}{2}\left(\bar{\psi}\psi\right)^2$ (coupled by the coupling constant g) and has QCD-like features in the *large-N limit*, including *asymptotic freedom*, *spontaneous symmetry breaking* and *renormalization* (in two dimensions) [12]. Therefore, this model is suitable as a *toy model* for QCD. The Gross-Neveu model is defined by the Lagrangian density in Euclidean space

$$\mathcal{L}_{\psi,m} = \bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - \frac{g^2}{2}\left(\bar{\psi}\psi\right)^2, \tag{41}$$

where $\gamma_0 = \sigma_1$, $\gamma_1 = \sigma_3$ (σ_i are the Pauli matrices) and m is a mass term. For $m = 0$ the Lagrangian is invariant under the $\mathbb{Z}_2$ symmetry (*chiral symmetry*). This is realized by the transformation $\psi \to \gamma_5\psi$ and $\bar{\psi} \to -\bar{\psi}\gamma_5$, where $\gamma_5 = \sigma_2$. In the massive case $m \neq 0$ this chiral symmetry is explicitly broken. Therefore to observe spontaneous symmetry breaking we set $m = 0$ and write $\mathcal{L}_\psi = \mathcal{L}_{\psi,m=0}$. One calls the chiral symmetry spontaneously broken if the *vacuum state* $|0\rangle$[22] satisfies no chiral symmetry. In particular, one observes for the *vacuum expectation values*[23] under the $\mathbb{Z}_2$ transformation

$$\langle 0|\, g\bar{\psi}\psi\, |0\rangle \to -\langle 0|\, g\bar{\psi}\psi\, |0\rangle\,. \tag{42}$$

Therefore, $\langle 0|\, g\bar{\psi}\psi\, |0\rangle = 0$ in unbroken systems and thus $\langle 0|\, g\bar{\psi}\psi\, |0\rangle$ is suitable as an *order parameter*. In order to determine $\langle 0|\, g\bar{\psi}\psi\, |0\rangle$ it would be wonderful to have a function f such that the minima of f correspond to vacuum expectation values $\langle 0|\, g\bar{\psi}\psi\, |0\rangle$. This function is called an *effective action* [13].

First, we define a second Lagrangian density

$$\mathcal{L}_\sigma = \bar{\psi}(\gamma_\mu \partial_\mu)\psi + \frac{1}{2g^2}\sigma^2 + \bar{\psi}\psi\sigma, \tag{43}$$

where σ is an auxiliary bosonic field. Both Lagrangians $\mathcal{L}_\psi$ and $\mathcal{L}_\sigma$ have the same fermion Green's functions and same effective potentials ($\langle 0|\, g\bar{\psi}\psi\, |0\rangle = \langle 0|\, \sigma\, |0\rangle$) [1]. Thus, $\mathcal{L}_\sigma$ and $\mathcal{L}_\psi$ describe the same four fermion theory.

To construct the effective potential we start with the functional

$$Z[J] = \int D\bar{\psi} D\psi D\sigma\ \exp\left\{\int \mathrm{d}^2x\ \left[-\mathcal{L}_\sigma[\sigma, \bar{\psi}, \psi] + J\sigma\right]\right\}, \tag{44}$$

where J is called the *external source*. Furthermore, one can integrate over the fermion fields by using

$$\int D\bar{\psi} D\psi\ \exp\left\{-\int \mathrm{d}^2x \mathrm{d}^2y\ \left[\bar{\psi}(x) D(x,y) \psi(y)\right]\right\} = \left[\det(D(x,y))\right]^N. \tag{45}$$

[22] Here $|0\rangle$ is the "right" vacuum state (ground state). This can be a symmetric (for unbroken) or asymmetric (for broken) vacuum state.

[23] One calls $\langle 0|\, g\bar{\psi}\psi\, |0\rangle$ also *fermionic condensate*.

Here, we define $D(x,y) = (\gamma_\mu \partial_\mu^x + \sigma)\delta(x-y)$. Finally, we have[24]

$$Z[J] = \int D\sigma \ \exp\left\{ \int \mathrm{d}^2x \ \left[-\tilde{\mathcal{L}}_\sigma[\sigma] + J\sigma \right] \right\}, \tag{46}$$

where $\lambda = Ng^2$ and

$$\tilde{\mathcal{L}}_\sigma = N \left[\frac{1}{2\lambda}\sigma^2 - \ln(\det(D(x,y))) \right]. \tag{47}$$

The vacuum expectation value is

$$\sigma_v = \langle 0 | \, \sigma \, | 0 \rangle_{J=0} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J}\bigg|_{J=0} = \frac{\delta W[J]}{\delta J}\bigg|_{J=0}, \tag{48}$$

where $W[J] = \ln(Z[J])$ is the *generating functional of connected correlation functions*. Both functions $Z[J]$ and $W[J]$ depend on J and therefore they are *no* candidates for f. To construct the effective action we have to do a Legendre transformation ($f^*(x) = p(x)x - f(p(x))$) with respect to the external source field J

$$\Gamma[\sigma_{cl}] = \int \mathrm{d}^2x \ \sigma_{cl} J[\sigma_{cl}] - W[J[\sigma_{cl}]]. \tag{49}$$

This function depends on the *classical field* $\sigma_{cl} = \langle 0 | \, \sigma \, | 0 \rangle_J$ and is the effective action we are looking for. From the Legendre transformation we conclude

$$\frac{\delta \Gamma}{\delta \sigma}[\sigma_{cl}] = -J. \tag{50}$$

In particular for the vacuum expectation value σ_v we have $\frac{\delta\Gamma}{\delta\sigma}[\sigma_v] = 0$. In the $1/N$-expansion we have

$$\Gamma[\sigma_{cl}] = \int \mathrm{d}^2x \ \tilde{\mathcal{L}}_\sigma[\sigma_{cl}] + \mathcal{O}(1/N). \tag{51}$$

In the large-N limit only the leading correction of $\Gamma[\sigma_{cl}]$ (the 1-loop contribution) becomes necessary. Higher L-loops will be suppressed by a factor N^{1-L} [14] and we conclude

$$\Gamma[\sigma_{cl}] = \int \mathrm{d}^2x \ \tilde{\mathcal{L}}_\sigma[\sigma_{cl}]. \tag{52}$$

Discovering the spontaneous symmetry breaking in the large-N limit reduces to finding the global minima of Γ or equivalent of $\int \mathrm{d}^2x \ \tilde{\mathcal{L}}_\sigma$.

[24] To lift the determinant in the exponent, $\det(D)$ has to be positive. Indeed, this is the case [12].

3.2.1 A simpler Gross-Neveu model

For this section we study the Gross-Neveu model in *zero dimensions* following [8, 11]. This simplifies the discussion by

- $\Omega \to C_{\mathbb{R}}$, where Ω is the σ-field configuration space,
- $\sigma(x) \to \sigma$,
- $D\sigma \to \mathrm{d}\sigma$,
- and $\Gamma[\sigma] \to \Gamma_G(\sigma) = N\left[\frac{1}{G}\sigma^2 - \ln\left(p^2 + \sigma^2\right)\right]$, $G = 2\lambda$.

To apply our methods we view σ and G as complex coordinates. The goal of this section is to study the minima of $\Gamma_G|_{\mathbb{R}}$ using Lefschetz thimbles[25]. Since minima of $\Gamma|_{\mathbb{R}}$ correspond to maxima of $e^{-\Gamma_G|_{\mathbb{R}}}$ it is of interest to study

$$Z(G) = \int_{C_{\mathbb{R}}} \mathrm{d}\sigma\, e^{-\Gamma_G(\sigma)}. \tag{53}$$

From the definition of Γ_G it is clear that Z does not converge if $\mathrm{Re}(G) < 0$. Also $\mathrm{Re}(G) = 0$ is problematic, because of oscillations. So we restrict to G with $\mathrm{Re}(G) > 0$. We follow the discussion about $\mathrm{Ai}(z)$ with a similar analysis.

0. We first compute $H_1(G,T) := H_1(\mathbb{C}, \{\mathrm{Re}(-\Gamma_G(\sigma)) \leq T\}; \mathbb{Z})$. As before, for $T \ll 0$ one has $H_1(G,T) \approx H_1(G,T')$ for all $T' \leq T$. For such T we define $H_1 := H_1(G,T)$ and $Z_{\leq T} := \{\sigma \in \mathbb{C} \mid \mathrm{Re}(-\Gamma_G(\sigma)) \leq T\}$. This time $Z_{\leq T}$ decomposes into four connected components, two of which coming from the two singularities (of $-\Gamma_G(\sigma)$)

$$p^2 + \sigma^2 = 0 \Leftrightarrow \sigma = \pm ip. \tag{54}$$

In particular H_1 is (free) abelian of rank three and one set of generators is depicted in figure 11.

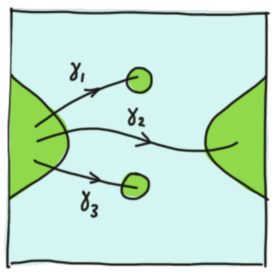


Figure 11: A choice of three generators for H_1 is shown.

[25] When talking about minima of Γ_G we always mean minima of $\Gamma_G|_{\mathbb{R}}$.

$-\Gamma_G(\sigma)$ also has saddle points:

$$\frac{\partial \Gamma_G}{\partial \sigma}(\sigma_i) = 2N\sigma(\sigma_i)\left[\frac{1}{G} - \frac{1}{p^2+\sigma^2(\sigma_i)}\right] = 0 \Leftrightarrow \sigma(\sigma_i) \in \Sigma, \tag{55}$$

where $\Sigma = \{0, \pm\sqrt{G-p^2}\}$. In particular, if $G = p^2$, then the critical points are degenerate. Therefor we assume $G \neq p^2$. Let us denote $\mathcal{J}_{\sigma_0}$ and $\mathcal{K}_{\sigma_0}$ the thimbles for $\sigma = 0$ and let $\mathcal{J}_{\sigma_1}$, $\mathcal{K}_{\sigma_1}$, $\mathcal{J}_{\sigma_2}$ and $\mathcal{K}_{\sigma_2}$ be the thimbles corresponding to the other two critical points.

1. The $[\mathcal{J}_{\sigma_i}]$ generate H_1 and therefore we have a unique decomposition

$$Z_G = \sum_{\sigma_i \in \{\sigma_0,\sigma_1,\sigma_2\}} \langle C_\mathbb{R}, \mathcal{K}_{\sigma_i}\rangle \int_{\mathcal{J}_{\sigma_i}} \mathrm{d}\sigma\, e^{-\Gamma_G(\sigma)}. \tag{56}$$

2. We compute the intersection numbers summarized in figure 12. This figure also shows that the real axis is contained in the Stokes lines $\mathbb{R} \subset S$. So, even if we are interested in some $G \in \mathbb{R}$ we might better study $G + i\epsilon (\notin S)$ for small $\epsilon \in \mathbb{R}$.

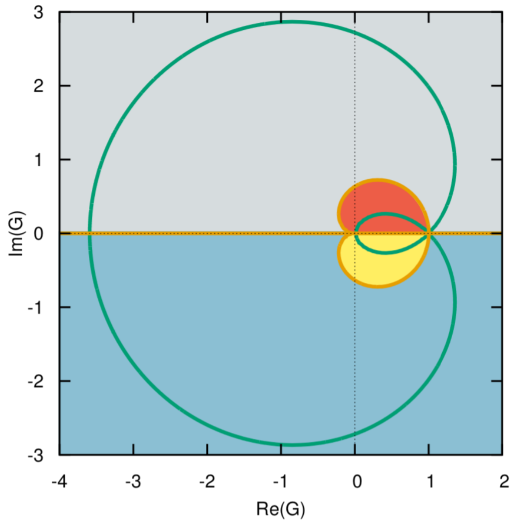
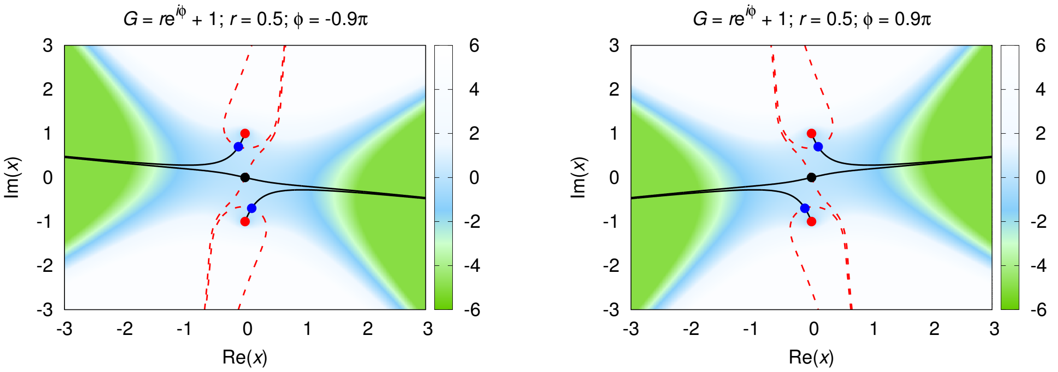


Figure 12: This figure shows the G-plane and the intersection numbers $n_i = \langle C_\mathbb{R}, \mathcal{K}_{\sigma_i}\rangle$ for $p = 1$. In the red and yellow regions $n_1 = n_2 = 0$ and $n_0 = 1$ and in the gray and blue regions $n_0 = n_1 = n_2 = 1$. In addition, the Stokes lines are depicted in gold and the anti Stokes lines in green.

We skip a discussion of the asymptotic behavior (i.e. steps 3 and 4) and instead use the intersection number to give a (partial) answer to whether $\sigma = 0$ is a global minimum of $-\Gamma_G|_\mathbb{R}$. Namely, the key properties of Lefschetz thimbles imply that the maximum of $\mathrm{Re}(-\Gamma_G)$ over $\mathcal{J}_{\sigma_i}$ is located at the corresponding saddle point. In particular, if

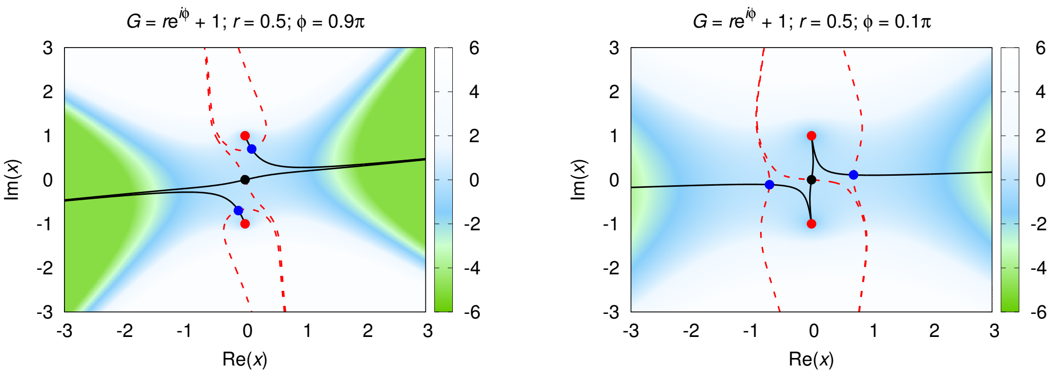
$n_1 = n_2 = 0$, then $\mathcal{J}_{\sigma_0}$ is the only thimble contributing to $Z = \int_{\mathcal{J}_{\sigma_0}} \mathrm{d}\sigma\, e^{-\Gamma_G(\sigma)}$ and we have a situation as shown in figure 13.



(a) (b)

Figure 13: Here the stable (black lines) and unstable (dashed red lines) thimbles are shown with respect to the critical points σ_0 (black dot), σ_1, σ_2 (blue dots) for $p = 1$. In the green area the integral (53) converges. Only $\mathcal{J}_{\sigma_0}$ intersects $C_{\mathbb{R}}$.

This suggests that one should also expect 0 to be the global minimum of $\Gamma_G|_{\mathbb{R}}$. If this is true it is interesting to study the G's for which $\mathcal{J}_{\sigma_0}$ undergoes a jump and pose the question whether this corresponds to a transition between broken and unbroken symmetry, i.e. spontaneous symmetry breaking (shown in figure 14).



(a) (b)

Figure 14: Here thimbles are shown as in figure 13. Between 14a and 14b the thimble $\mathcal{J}_{\sigma_0}$ jumps.

4 Outlook

While this thesis restricted to one dimensional Lefschetz thimbles the theory extends to higher dimensions and this is of special interest to the author for computing Feynman path integrals or partition functions for more advanced QCD like theories (in particular inhomogeneous phases). Picard-Lefschetz theory seems to be the correct framework and the author is motivated to learn more about this. Especially the notion of monodromy (as appearing in the article of Witten [3]) has caught the author's attention. Applying such more sophisticated mathematical theories to physically motivated problems was enlightening. In that respect the author wants to better understand the relation (if one exists) between Stokes jumps and phase transitions.

A Homology

This appendix is based on [7]. Let X be a topological space. For every integer $i \in \mathbb{Z}$, one can construct an invariant of X, the so-called ith homology group $H_i(X;\mathbb{Z})$. Intuitively, $H_i(X;\mathbb{Z})$ measures n-dimensional holes in X.

To define homology, it is natural to first define *chain complexes*. A chain complex $(C_\bullet, d_\bullet)$ is a collection of abelian groups C_i and group homomorphisms $d_i : C_i \to C_{i-1}$ connecting them such that $d_{i-1} \circ d_i = 0$. We refer to

1. d_i as *differentials*,
2. elements in C_i as *i-chains*,
3. elements in $\ker(d_i) \subset C_i$ as *closed i-chains* or *i-cycles*,
4. elements in $\mathrm{im}(d_{i+1}) \subset C_i$ as *exact i-chains* or *i-boundaries.*

It is common to omit the index from the d_i. For example, $d_{i-1} \circ d_i = 0$ becomes $d^2 = 0$. This has the crucial consequence: $\mathrm{im}(d) \subset \ker(d)$ (or with indices $\mathrm{im}(d_{i+1}) \subset \ker(d_i)$). In particular, we can define the ith *homology group*

$$H_i(X) = \ker(d_i)/\mathrm{im}(d_{i+1}). \tag{57}$$

So, elements in $H_i(X)$ are represented by i-cycles subject to the relation that boundaries are (artificially made) zero. It is common notation to write $[C] \in H_i$ for the class represented by the i-cycle C.

It remains to construct a chain complex from our topological space. We will use the so called *singular chain complex.* A singular i-chain is a continuous map $\sigma : \Delta^i \to X$ from the standard i-simplex in $\mathbb{R}^{i+1}$ to X. We define C_i to be the free abelian group, generated by singular i-chains. In other words

$$C_i = \mathbb{Z}[\mathrm{Top}(\Delta^i, X)], \tag{58}$$

where $\mathrm{Top}(\Delta^i, X)$ is the set of continuous maps $\Delta^i \to X$ and for a set M, $\mathbb{Z}[M] = \{\sum_{\text{finite}} a_i m_i \mid a_i \in \mathbb{Z}, m_i \in M\}$[26] is the free abelian group of *formal $\mathbb{Z}$-linear combinations of elements in M*. One can think of a singular i-chain as a deformed i-simplex in X (shown in figure 15 below).

[26] $\mathbb{Z}[M]$ is called $\mathbb{Z}$ *adjoined* M.

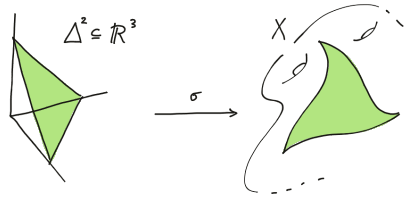


Figure 15

To define the differential, observe that Δ^i comes with the extra structure of *faces*

$$\text{face}_j : \Delta^{i-1} \to \Delta^i, (x_0, ..., x_{i-1}) \mapsto (x_0, ..., 0, ..., x_{i-1}), \tag{59}$$

with the zero at the jth position (counting from $j = 0$). Precomposing a singular i-chain σ with face_j gives a $(i-1)$-chain, the jth *face* σ_j of σ (shown in figure 16 below).

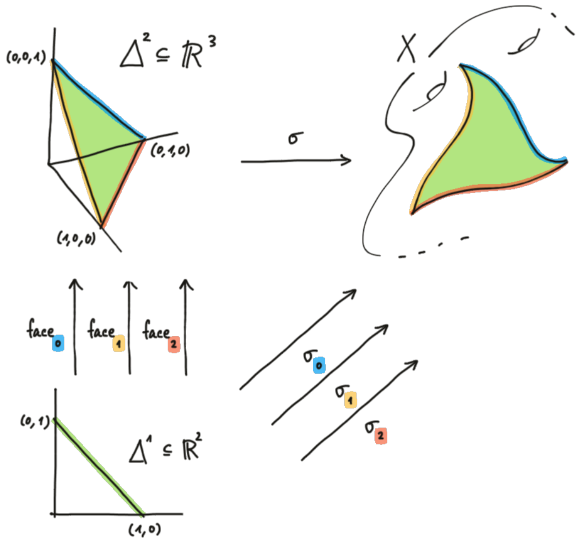


Figure 16

The fundamental idea is to define the differential d as the *alternating sum* of faces:

$$d(\sigma) = \sum_{j=0}^{i} (-1)^j \sigma_j. \tag{60}$$

The alternating sum ensures $d^2 = 0$. Hence, we have indeed defined a chain complex and can apply our previous definition of homology. We write $H_i(X)$ for its ith homology group and call it the ith *singular homology* (group).
By construction, H_i is abelian and although C_i are usually huge (uncountably generated), H_i is usually finitely generated.

B Plots: Airy function

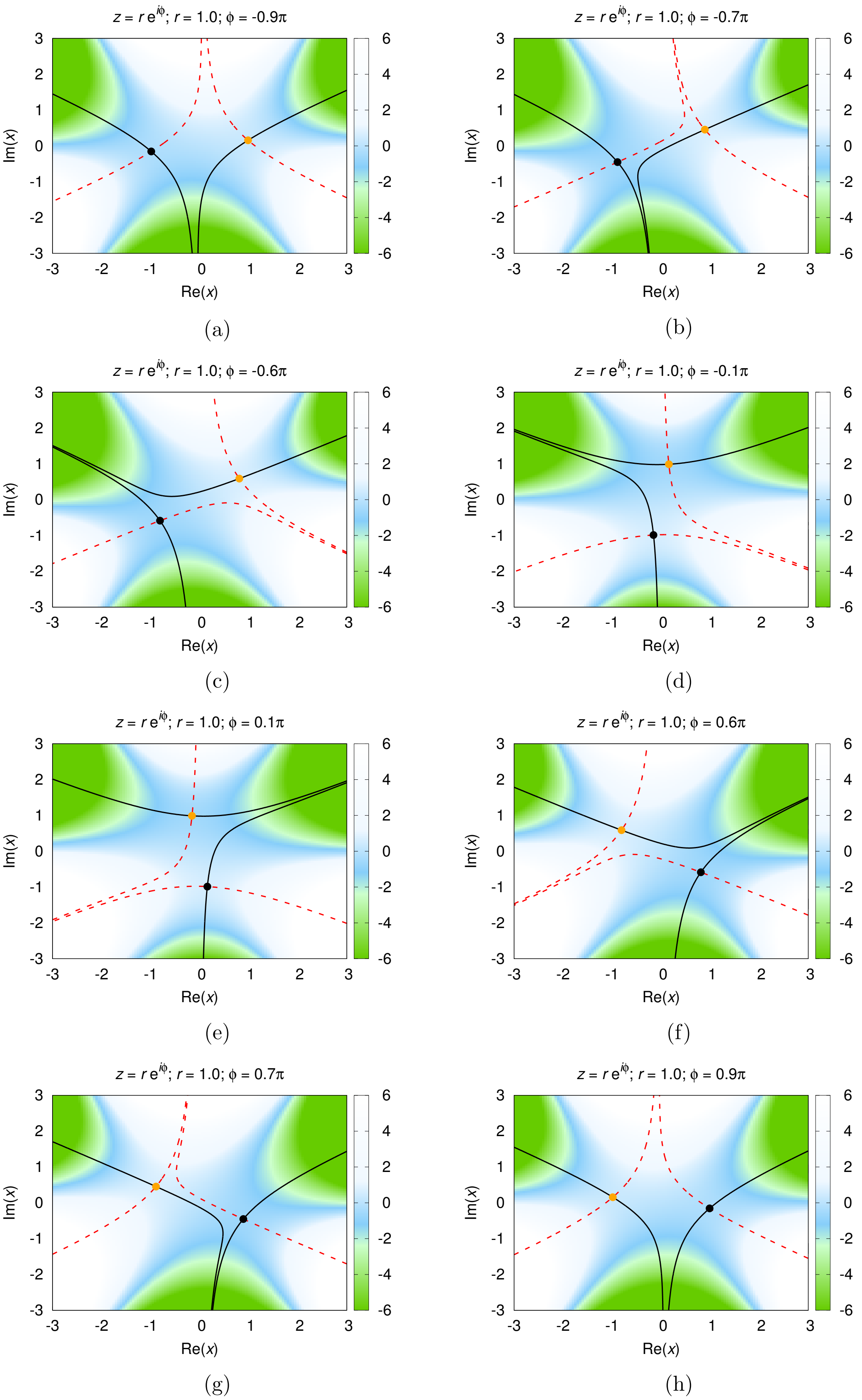


Figure 17: This series of plots shows the stable (black lines) and unstable (dashed red lines) thimbles while varying z on the unit circle around the origin. The coloring is given by the respective height function h. In particular, green regions correspond to $Z_{\leq T}$.

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