Topological objects in gauge theories – Problem Set 3

May 14, 2010 (will be discussed on May 28, 2010)

Problem 5

Quite often it is convenient to consider Euclidean versions of relativistic field theories, which mainly arise by replacing the Minkowski metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ by the Euclidean metric $g_{\mu\nu} = \text{diag}(+1, +1, +1, +1)$. Since there is no difference between covariant and contravariant components anymore, one usually writes lower indices only.

(a) Consider the Euclidean version of Maxwell theory, i.e.

$$S_E = \int d^D x \, \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

What is the minimum value of S_E and which field configurations yield that value? For which number of spacetime dimensions D is it possible to exclude the existence of further solutions of the equations of motion with finite Euclidean action?

(b) Consider the Euclidean version of SU(2) Yang-Mills theory, i.e.

$$S_E = \int d^D x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \quad , \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu$$

(the additional upper index a of the gauge field A^a_μ is a color index, where $a = 1, 2, 3$, i.e. the gauge field has three times as many components as in Maxwell theory). What is the minimum value of S_E and which field configurations yield that value? For which number of spacetime dimensions D is it possible to exclude the existence of further solutions of the equations of motion with finite Euclidean action?

Problem 6

The Abelian Higgs model,

$$\mathcal{L} \;=\; (D^\mu\phi)^* D_\mu\phi - V(|\phi|) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad , \quad V(|\phi|) \;=\; \frac{\lambda}{4}\Big(|\phi|^2 - a^2\Big)^2 ,$$

is a model closely related to superconductivity.

One of the major phenomena of superconductivity is the ejection of any sufficiently weak magnetic field from the interior of a superconductor (Meissner effect).

(a) Derive the equations of motion for ϕ and A_μ.

(b) Show that for $\phi = a$ and $\mathbf{E} = 0$ the spatial components of the equation of motion for A_μ reduce to the London equation[1]

$$\Big(\mathrm{rot}\,\mathbf{B}\Big)_j \;=\; 2e^2a^2A^j .$$

(c) Solve the London equation. Interpret the obtained solution with respect to the Meissner effect. Are the remaining equations of motion of the Abelian Higgs model also fulfilled?

[1] Developed in 1935 by F. and H. London by means of phenomenological considerations.