Exercise sheet XI, XII

July 1 [solution: July 7, 21]

Problem 1 [*Chiral symmetry of QCD*] Consider the fermionic part of the QCD Lagrangian in Euclidean spacetime:

$$\mathcal{L} = \sum_{f=1}^{N_f} \overline{\psi}^{(f)} \left(\gamma_\mu D_\mu + m^{(f)}\right) \psi^{(f)}, \tag{1}$$

where (f) is the flavour index and N_f the number of flavours.

(i) Check that the above Lagrangian in the massless case is invariant under the group $SU(N_f)_V \times SU(N_f)_A \times U(1)_V \times U(1)_A$ by checking its invariance with respect to the following transformations, respectively:

$$\psi^{(f)} \to \psi^{(f)\prime} = e^{i\omega^a T^a} \psi^{(f)}, \tag{2}$$

$$\psi^{(f)} \to \psi^{(f)\prime} = e^{i\omega^a T^a \gamma^5} \psi^{(f)}, \tag{3}$$

$$\psi^{(f)} \to \psi^{(f)\prime} = e^{i\omega} \psi^{(f)}, \tag{4}$$

$$\psi^{(f)} \to \psi^{(f)\prime} = e^{i\omega\gamma^5} \psi^{(f)}, \tag{5}$$

where T^a are the generators of the $SU(N_f)$ group and ω or ω^a are numbers.

(ii) What happens if masses are non-zero, but degenerate? Which symmetries are explictly broken?

(iii) What happens if the masses are non-degenerate? Which symmetry survives?

In the exercises, we will discuss the consequences of the breaking of these symmetries by the non-degenerate quark masses in the real world. We will also discuss two further breakings that give rise to important effects: the breaking of the $U(1)_A$ symmetry by the axial anomaly (which gives rise to the unexpectedly large mass of the η' meson) and spontaneous breaking of the $SU(N_f)_A$ symmetry (which implies the existence of pseudo-Goldstone bosons – the pions, the kaons and the η meson).

Problem 2 [*Fermion doubling on the lattice*] Let us consider fermions in 1+1 dimensions. The continuum Hamiltonian density is $\overline{\psi}(x)(-\gamma_1 d/dx + m)\psi(x)$, where $\overline{\psi}(x) = \psi^\dagger(x)\gamma_0$ and the gamma matrices are: $\gamma_0 = \mathrm{diag}(1, -1)$, $\gamma_1 = ((0, 1), (-1, 0))$. Taking a symmetric discretization of the derivative: $d\psi(x)/dx = (\psi_{x+1} - \psi_{x-1})/2a$, where a is the lattice spacing, the Hamiltonian can be written as:

$$H = \frac{-i}{2a} \sum_{x=0}^{N-1} \overline{\psi}_x \gamma_1 (\psi_{x+1} - \psi_{x-1}) + m \sum_{x=0}^{N-1} \overline{\psi}_x \psi_x, \tag{6}$$

where N is the number of lattice sites and periodic boundary conditions were used: $\psi_N = \psi_0$. Diagonalize this Hamiltonian in momentum space, by using the inverse Fourier transform:

$$\psi_x = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i k x}{N}} \psi_k. \tag{7}$$

What are the energy eigenvalues? You should find an expression of the form $E = \pm\sqrt{m^2 + p^2}$. Plot the obtained p as a function of k. The zeros of p correspond to fermion species. You will see there are two zeros in the first Brillouin zone $k \in [0, N)$. Hence, our Hamiltonian describes 2 fermion species instead of 1 that we intended. This is a general problem in the formulation of fermions on the lattice, called *fermion doubling*. With d discretized dimensions, there are 2^d fermion flavours, i.e. $2^d - 1$ unwanted doublers. It was proven by Nielsen and Ninomiya in 1981 that one can not get rid of the doublers without sacrificing one of the following features: Hermiticity of the Hamiltonian, locality or chiral symmetry.

For example, one can trace back the origin of the doublers to the symmetric form of the derivative. Consider now a one-sided derivative, e.g. $d\psi(x)/dx = (\psi_{x+1} - \psi_x)/a$. Repeat the whole exercise and show that there are no doublers, but the Hamiltonian becomes non-Hermitian.

Problem 3 [*Discretization of the QCD gauge action*] Consider the simplest Euclidean lattice QCD gauge action, the so-called *Wilson plaquette action*:

$$S_G[U] = \frac{2}{g^2} \sum_n \sum_{\mu<\nu} \mathrm{Re}\ \mathrm{Tr}\left(1 - U_{\mu\nu}(n)\right), \tag{8}$$

where $U_{\mu\nu}(n)$ is the plaquette:

$$U_{\mu\nu}(n) = U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_\mu^\dagger(n+\hat{\nu})\, U_\nu^\dagger(n), \tag{9}$$

$$U_\mu(n) = e^{iaA_\mu(n)}. \tag{10}$$

Show that the action can be rewritten as:

$$S_G[U] = \frac{a^4}{2g^2} \sum_n \sum_{\mu,\nu} \mathrm{Tr}\, F_{\mu\nu}^2(n) + \mathcal{O}(a^2). \tag{11}$$

You will need to use the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$. Take the continuum limit of this expression and finally evaluate the trace, writing $F_{\mu\nu}(x)$ in terms of generators of the gauge group:

$$F_{\mu\nu}(x) = F^{(c)}_{\mu\nu}(x)\, T_c \tag{12}$$

and using the property $\mathrm{Tr}\,[T_a T_b] = \frac{1}{2}\delta_{ab}$. In the end, you should obtain the continuum gauge action:

$$S_G[U] = \frac{1}{4g^2}\int d^4x\, F^{(c)}_{\mu\nu}(x) F^{(c)}_{\mu\nu}(x), \tag{13}$$

where a sum over colour indices is implied.

Problem 4 [*Wilson fermions and mesonic correlators*] In this problem, we will have another look at the fermion doubling problem and see its simple solution by adding a second derivative term to the Lagrangian, the so-called *Wilson term*. Then, we will use the fermionic propagator to find a semi-analytical expression for mesonic correlation functions in the free theory.

Our starting point is the free 4-dimensional Dirac operator (no gauge fields) in momentum space, corresponding to the *naive* discretization of fermions using a symmetric derivative, as in Problem 2:

$$D(p) = m\mathbb{1} + \frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu \sin(ap_\mu). \tag{14}$$

(i) Calculate the free naive fermion propagator $S(p) = D(p)^{-1}$. Show that it has the form:

$$S(p) = \sum_{\mu=0}^{4} S_\mu(p)\gamma_\mu, \tag{15}$$

where we have defined $\gamma_0 \equiv \mathbb{1}$. Find the explicit forms of $S_\mu(p)$.

(ii) Show that the massless propagator has the right naive continuum limit $S(p) = \frac{-i\gamma_\mu p_\mu}{p^2}$. However, note that the lattice expression has poles not only at $p = (0,0,0,0)$ but also whenever one or more of the momentum components are π/a (and the other are zero). How many fermions does thus this naive propagator describe?

(iii) Now consider adding a second derivative term to the Lagrangian, the Wilson term. In momentum space, the Dirac operator becomes:

$$D(p) = m\mathbb{1} + \frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu \sin(ap_\mu) + \frac{r}{a}\sum_{\mu=1}^{4}(1-\cos(ap_\mu)), \tag{16}$$

where r is called the Wilson parameter and is typically set to $r = 1$. Repeat the computation of the propagator and discuss how the Wilson term affects the doubler modes. How many fermion species do we have in the continuum limit?

(iv) Compute the flavour non-singlet pseudoscalar correlation function, defined as:

$$C_{PP}(x) = -\langle 0|\mathcal{P}^+(x)\mathcal{P}^-(0)|0\rangle, \tag{17}$$

where:

$$\mathcal{P}^+(x) = \bar{d}(x)\gamma_5 u(x), \tag{18}$$

$$\mathcal{P}^-(x) = \bar{u}(x)\gamma_5 d(x). \tag{19}$$

To do this, you need to write e.g. $\bar{d}\gamma_5 u = \bar{d}_\alpha(\gamma_5)_{\alpha\beta}u_\beta$ and perform Wick contractions, e.g. $u_\alpha(x)$ contracted with $\bar{u}_\beta(0)$ is the fermion propagator $S^u_{\alpha\beta}(x)$. Use also the so-called γ_5-Hermiticity property of the Wilson-type fermion propagator: $\gamma_5 S^d(x)\gamma_5 = (S^u)^\dagger(-x)$. You should obtain:

$$C_{PP}(x) = \mathrm{Tr}\left[(S^u)^\dagger(x)S^u(x)\right]. \tag{20}$$

Using the Fourier transformation, show finally that the correlator can be written in terms of the functions $S_\mu(p)$ defined via Eq. (15):

$$C_{PP}(x) = \frac{N_c N_d}{V^2}\sum_p \sum_{p'} \sum_{\mu=0}^{4} (S^u_\mu)^*(p) S^u_\mu(p') e^{i(p'-p)x}, \tag{21}$$

where V is the lattice volume and the trivial factor $N_c N_d$ comes from evaluating the traces ($N_c = 3$ colours, $N_d = 4$ spacetime dimensions). This is a semi-analytical expression that can be evaluated on a computer. We will discuss it during the tutorial. Consider also what changes if you want to evaluate other kinds of correlation functions (e.g. scalar, vector, axial vector).

Bonus. If you are interested, implement Eq. (21) on a computer. This will allow you to compute e.g. the pion mass at tree-level of perturbation theory, which can be accessed from the Euclidean time decay of the correlator $C_{PP}(t) \equiv \sum_{\vec{x}} C_{PP}(\vec{x}, t)$. For more details about such computations, look at the Ph.D. preparation report of Jenifer Lopez which can be found under:

`http://www-zeuthen.desy.de/~kjansen/etmc/Publications/thesis.pdf`

or at the paper arXiv:0802.3637[hep-lat].