Exercise Sheet IX

June 17 [Solution June 23]

Problem 1. *The static femion propagator*

In this exercise we will calculate the fermion propagator for static quarks, meaning the infinite mass limit. The quark propagator was calculated on a previous exercise sheet and is:

$$S_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i\big(\not{p}+m\big)}{p^2-m^2+i\epsilon} e^{-ip(x-y)} \tag{1}$$

- Carry out the integral over p_0. Cauchy's residue theorem should come in handy.
- Then send the fermion mass to infinity and you should get:

$$S_F(x-y) \underset{m\to\infty}{=} \delta\big(\vec{x}-\vec{y}\big) e^{-im|x_0-y_0|} \left(\frac{1+\gamma_0}{2}\theta(x_0-y_0) + \frac{1-\gamma_0}{2}\theta(y_0-x_0) \right) \tag{2}$$

Problem 2. *The gluonic potential*

In this exercise we will analyse the potential between two static quarks under gluon exchange. Before we can study this we need to set up an appropriate quantum state:

$$|\Phi(t)\rangle = \bar{Q}\big(t, -\frac{\vec{r}}{2}\big)\, \gamma^5\, \Lambda\, Q\big(t, \frac{\vec{r}}{2}\big) |\Omega\rangle \tag{3}$$

Where $Q(t,\vec{x})$ and $\bar{Q}(t,\vec{x})$ are the appropriate quark creation operators, γ^5 makes the state a pseudoscalar and Λ is a colour matrix. The potential between the two particles is defined by the relation:

$$\langle\Phi(t)|\Phi(0)\rangle = \sum_n \Big|\langle\Phi|n\rangle\Big|^2 e^{-i(E_n-E_\Omega)t} \underset{t\to\infty(1-i\epsilon)}{=} \Big|\langle\Phi|0\rangle\Big|^2 e^{-V_{Q\bar{Q}}(\vec{r})\epsilon t} \tag{4}$$

Here the sum over the states is the sum over all states that contain a static quark-anti-quark pair at the given coordinates. The potential is thus defined by:

$$\lim_{t\to\infty(1-i\epsilon)} \langle\Phi(t)|\Phi(0)\rangle \sim A e^{-V_{Q\bar{Q}}(\vec{r})\epsilon t} \tag{5}$$

- Calculate the value of $\big|\langle\Phi(t)|\Phi(0)\rangle\big|^2$ to $\mathcal{O}(g^0)$. Keep the colour matrix Λ undefined for now.

  Hint: Use the standard method of canonical quantization together with Wick contractions. The possible contractions should be restricted by the fact that you have static quark propagators. Writing out the spin- and colour indices explicitly might help as well.

There should be two additional contributions to $\mathcal{O}(g^2)$ when calculating $V_{Q\bar{Q}}$. One of them is absorbed in the definition of the quark propagator, which we will ignore.

- Draw and calculate the additional graph to $\mathcal{O}(g^2)$, you can use the gluon propagator in Feynman gauge.
- Then use the relation in eq (5) to find an expression for the potential $V_{Q\bar{Q}}$.

Just as we did on sheet VII, we will study the effect of having states in different colour groups.

- Calculate $V_{Q\bar{Q}}$ for a colour singlet state ($\Lambda = \frac{1}{\sqrt{6}} I$).
- Repeat the calculation for a colour octet state ($\Lambda \in \{\frac{1}{2}\lambda^a \,|\, a \in \{1, .., 8\}\}$)

The potential you end up with goes like $\sim \frac{1}{r}$, which is just like the electrodynamical and gravitational potential. The strong force is however supposed to be "stronger" than these and you might have heard of the gluon string potential before. What assumption have we made in this calculation

Problem 3. *Feynman rules in axial gauge*

In this exercise we will look at QCD in a different gauge, namely the *axial gauge*, which is defined by the equation:

$$n^\mu A^a_\mu = 0 \text{ , for some fixed } n_\mu : n_\mu n^\mu = 1 \tag{6}$$

- Write down the effective QCD Lagrangian including the gauge-fixing term and the ghost term.
- Derive the propagator (in momentum space)
- Sketch the Feynman rukes for quark-gluon interactions.

Hint: when inverting the gluon propagator you can use the symmetry Ansatz:

$$\Big(-k^2 g_{\mu\nu} + k_\mu k_\nu - \frac{1}{\alpha} n_\mu n_\nu \Big)^{-1} = A g^{\mu\nu} + B k^\mu k^\nu + C\big(k^\nu n^\mu + k^\mu n^\nu\big) \tag{7}$$