Exercise Sheet VIII

June 10 [Solution June 16]

Problem 1. *The Faddeev-Popov way*

In this exercise we will solve a known integral a Faddev-Popov-inspired way. The integral in question is:

$$I = \int \int \mathrm{d}x\, \mathrm{d}y\, e^{-(x^2+y^2)} \tag{1}$$

The argument of the exponential is invariant under a 2D rotation, which is the symmetry group SO(2). In other words, the exponential is invariant under a transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^g \\ y^g \end{pmatrix} \quad \text{for} \quad g \in \mathrm{SO}(2) \tag{2}$$

Follow the Faddeev-Popov method and insert a gauge-fixing term into the integral

$$1 = \Delta_{FP} \int \mathrm{d}g\, \delta\Big(F(x^g, y^g)\Big) \tag{3}$$

Where $F(x^g, y^g)$ is the gauge fixing term which is so that there exists at least one $g \in \mathrm{SO}(2)$ so that $F(x^g, x^g) = 0$.

- Parametrise the integration measure and simplify the normalisation factor Δ_{FP} as much as possible.
- Choose $F(x^g, y^g) = y^g$. In words, what is this gauge fixing? Explicitly evaluate Δ_{FP} for this gauge fixing function.
- Use this to solve the integral

$$I = \int \mathrm{d}x\, \mathrm{d}y \Delta_{FP} \int \mathrm{d}g\, \delta\Big(F(x^g, y^g)\Big) e^{-(x^2+y^2)} \tag{4}$$

  where you utilise the steps used in the lecture to fix the gauge. Verify with this that $I = \pi$.

Problem 2. *The eight-fold way*

In previous courses you have studied the SU(2) symmetry group and addition of angular momentum. In this exercise we will study the SU(3) symmetry group of QCD, and carry out similar additions to produce the meson- and baryon spectrum.

Similar to how you can read of the angular momentum eigenstates of SU(2) from the diagonal elements of σ_3, one can find the eigenstates of SU(3) by its diagonal matrices. There are two such matrices in SU(3)[1], which means that the eigenstate will be characterised by a two-component vector rather than a single number. We will call the value of this the weight vector, or simply the weight. In the fundamental representation of SU(3), the diagonal matrices are the Gell-Mann matrices λ_3 and λ_8.

- Write down the three eigenstates of SU(3), and plot them as three points in a simple 2D graph. We will refer to this representation as 3.

There is also a second, just as fundamental, complex conjugate conjugate representation of SU(3). The representation is related to the normal representation through

$$\begin{aligned} &\lambda_1^* = -\lambda_1, \;\; \lambda_2^* = -\lambda_2, \;\; \lambda_3^* = -\lambda_3, \;\; \lambda_4^* = \lambda_4 \\ &\lambda_5^* = -\lambda_5, \;\; \lambda_6^* = \lambda_6, \quad\;\; \lambda_7^* = -\lambda_7, \;\; \lambda_8^* = \lambda_8 \end{aligned}$$

and if one wants to relate the weights, they have an additional minus sign.

- Write down the conjugate weights and draw them in the same diagram (remember the minus sign). We will refer to this representation as $\bar{3}$.

Doing addition of angular momentum is now as straight forward as adding vectors together. For instance a composite $3 \otimes \bar{3}$ will be all possible additions of the points in 3 and the points in $\bar{3}$. One can carry this out in a systematic way with raising and lowering operators to create the actual eigenstates just as with SU(2), but let us ignore that for now.

- Carry out the addition $3 \otimes \bar{3}$, and draw the result.
- Do the same for the addition $3 \otimes 3 \otimes 3$.

We will now assign flavour values to the different eigenstates. We know that λ_3 is an element in the SU(2) sub-algebra of SU(3), and that u and d form an isospin doublet.

- Assign a quark flavour to each of the eigenstates, and anti-quark eigenvalues to the complex conjugate eigenstates. What does this say about the two calculations you carried out in the previous task?

Finally, the first weight is related to isospin while the second is $\sqrt{3}/2$ times hypercharge ($Y = B + S$, B = baryon number and S = strangeness).

- Use this information together with the Particle Data Group booklet to classify the mesons and baryons in your diagram. Draw lines of constant strangeness.
- (*Bonus*) Can you find all the SU(2) subalgebras in the meson and baryon diagrams?

[1]This is called the *Cartan subalgebra* of SU(3)