Exercise sheet V

`May 12 [solution: May 26 (start on May 19 if we have time)]`

Problem 1 [*Interacting scalar field theory*] In this exercise, we move on to the ϕ^4 theory and see how the path integral formalism works for the interacting case. Consider the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

Write down the resulting generating functional $Z[J]$. You will see that the path integral can no longer be done analytically. There are two ways to proceed to compute the Green's functions.

Method 1. It can be shown (see the Lecture or some book) that the generating functional for an interacting scalar field theory, with an action $S[\phi] = S_0[\phi] + S_I[\phi]$, where $S_0[\phi] = \int d^4x \left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right)$ and $S_I[\phi]$ is general (for ϕ^4 theory: $S_I[\phi] = -\int d^4x \frac{\lambda}{4!}\phi^4$), can be written as:

$$Z[J,\lambda] = Z[0,0] \exp\left(iS_I\left[\frac{\delta}{i\delta J}\right]\right) \exp\left(-\frac{i}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right), \tag{2}$$

where:

$$Z[0,0] = \int D\phi \exp\left(iS[\phi]\right). \tag{3}$$

The n-point Green's functions are then:

$$G_n(x_1,\ldots,x_n) = (-i)^n \frac{1}{Z[0,0]} \frac{\delta^n Z[J,\lambda]}{\delta J(x_1)\ldots\delta J(x_n)}\Big|_{J=0}. \tag{4}$$

They can be computed by a perturbative expansion of the exponential $\exp\left(iS_I\left[\frac{\delta}{i\delta J}\right]\right)$ in powers of the coupling. For the ϕ^4 theory:

$$\exp\left(iS_I\left[\frac{\delta}{i\delta J}\right]\right) = \exp\left(-\frac{i\lambda}{4!}\int d^4w \left(\frac{\delta}{\delta J(w)}\right)^4\right). \tag{5}$$

Using Eq. (4), compute the 2-point function for the ϕ^4 theory, $G_2(x_1,x_2)$, up to $\mathcal{O}(\lambda^2)$ (you can use the result from the Lecture for $\mathcal{O}(\lambda)$). Draw the corresponding diagrams and make sure that the obtained answer is as expected (e.g. compare to what we did in QFT1). Pay attention how the numerical factors in front of each diagram come up from the computation.

Method 2. Alternatively, one can first expand the path integral in powers of J:

$$Z[J,\lambda] = \sum_{s=0}^{\infty} \frac{1}{s!} J(x_1)\dots J(x_s) \int D\phi e^{i\int d^4x\left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)} \phi(x_1)\dots\phi(x_s). \tag{6}$$

Since Eq. (4) still has to be valid, we can write:

$$Z[J,\lambda] = Z[0,0] \sum_{s=0}^{\infty} \frac{i^s}{s!} J(x_1)\dots J(x_s) G_s(x_1,\dots,x_s). \tag{7}$$

Hence, the Green's functions are:

$$G_n(x_1,\dots,x_n) = \frac{1}{Z[0,0]} \int D\phi e^{i\int d^4x\left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)} \phi(x_1)\dots\phi(x_n). \tag{8}$$

Now, the exponential $e^{-\frac{i\lambda}{4!}\int d^4x\phi^4}$ can be expanded in powers of the coupling, giving a pertubative series in λ. Each term of this series is a variant of the Gaussian integral and can be calculated as a sum over Wick contractions. As such, this method becomes very similar to what we did in QFT1 in the canonical quantization formalism, but the way to it is very different now!

Using this approach, calculate again $G_2(x_1,x_2)$ for the ϕ^4 theory, up to $\mathcal{O}(\lambda^2)$ and check that you get the same answer as with Method 1.

Bonus exercises. If you want to gain more practice, consider:

(i) $\mathcal{O}(\lambda^3)$

(ii) and/or higher orders in λ

(iii) and/or 4-point functions

(iv) and/or higher n-point functions,

with one or both methods.