Exercise Sheet II

April 28.

Task 1. *Partition function of the Harmonic Ocillator*

In the first task you shall calculate the partition function $\mathcal{Z}$ of the Harmonic Oscillator for both the classical and the quantum mechanical system. The partition function is defined as:

$$\mathcal{Z} = \operatorname{tr} e^{-tH} = \sum \langle \psi | e^{-tH} | \psi \rangle \tag{1}$$

- Calculate $\mathcal{Z}$ for a classical harmonic oscillator system using the equation:

$$\mathcal{Z} = \frac{1}{2\pi} \int \mathrm{d}x \, \mathrm{d}p \, e^{-tH} \tag{2}$$

- Then calculate $\mathcal{Z}$ for a quantum mechanical harmonic oscillator using the eigenstates of the Hamilton operator.

- Repeat this calculation using the position eigenstates. You can start with the transition amplitude calculated in the lecture, but you need to rotate it to Euclidean time first:

$$\langle x, \tau | x, 0 \rangle = \sqrt{\frac{m\omega}{2\pi i \sin(\omega\tau)}} \exp\left\{ i \frac{m\omega}{\sin(\omega\tau)} x^2 \big(\cos(\omega\tau) - 1 \big) \right\} \tag{3}$$

  Also compare this result with the previous one.

- Finally calculate the free energy $F = -\frac{1}{t} \log \mathcal{Z}$ in all three cases, and compare them for $t \ll 1$. Can you explain the apparent behaviour?

Task 2. *Functional derivatives*

Start with the four dimensional electromagnetic action:

$$S[A_\mu] = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \right), \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor, and j_μ is an external current.

- Evaluate the functional derivative:

$$\frac{\delta S[A_\mu]}{\delta A_\mu(x)} \tag{5}$$

- Show that setting this derivative to zero gives you Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{6}$$