Exercise sheet X

`July 3 [correction: July 10]`

Problem 1 [*Lattice discretisation of the harmonic oscillator*]

Consider the Euclidean harmonic oscillator:

$$\begin{aligned} Z[j] &= \frac{1}{Z}\int [\mathscr{D}q] \exp\left\{\frac{1}{2}\int \mathrm{d}\tau\; q(\tau)\Big(m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - m\omega^2\Big)q(\tau) + \int \mathrm{d}\tau\; j(\tau)q(\tau)\right\} = \\ &= \exp\left\{\frac{1}{2}\int \mathrm{d}\tau_1 \int \mathrm{d}\tau_2\; j(\tau_1)\Big[\frac{e^{-\omega|\tau_1-\tau_2|}}{2m\omega}\Big]j(\tau_2)\right\} \;; \end{aligned}$$

assume, in particular, a periodic and finite time extent T.

(i) Discretise now the temporal direction, by dividing T into N intervals of length Δt. Write the corresponding action S_{ij} and generating functional $Z(\mathbf{j})$, where now S_{ij} is a matrix acting on N-component vectors $\mathbf{q} = (q_1, \ldots, q_n)$, the discretised form of the trajectories $q(\tau)$. The source term $\mathbf{j}$ will also be a N-component vector.

(ii) Evaluate the inverse of S_{ij} with a discrete Fourier transform:

$$(S^{-1}_{k\ell}) = G_{k-\ell} = \sum_{n=1}^{N} \exp\Big(\frac{-2\pi i n(k-\ell)}{N}\Big)\widetilde{G}_n \;.$$

(*Hint*: you should get a closed form only in Fourier space, while in coordinate space an explicit summation will remain, over a finite number of terms).

(iii) Express the lattice two-point function

$$\langle 0|T\Big(q(j\;\Delta t)q(k\;\Delta t)\Big)|0\rangle = C\Big(\underbrace{(k-j)\Delta t}_{t}\Big)$$

in terms of $S^{-1}_{k\ell}$. Compute numerically the resulting sum with your favourite method (Python, `C++`, Fortran, ...): to do this, choose $mT = 8$, $\omega/m = 1$ and take $N = 8, 16, 32$. Compare, by means of a plot, the discretised results with the $T \to \infty$ continuum equation you have seen in the Lecture. Are there differences?

(iv) Determine the slope of $-\log\Big(C(t)\Big)$ for "intermediate" t. Interpret what you find.

(v) Look at your results for $t \lesssim T$: can you imagine a technique to make the comparison to the continuum more accurate?