Exercise sheet VIII

June 15 [correction: June 22]

Symmetry transformations $\widehat{R}$ are closely related to conserved quantities T: for instance, translational symmetry $\leftrightarrow$ momentum conservation.

Conversely, the operator $\widehat{T}$ associated to the conserved quantity generates the symmetry transformation via

$$\widehat{R}(\alpha) = e^{i\alpha\widehat{T}} \quad .$$

Problem 1 [*Quantum mechanics "warm up"*]

Let $\widehat{P}$ be the momentum operator. Show that

$$\widehat{S}(\alpha) = \exp(i\alpha\widehat{P}) \quad ,$$

acting on a wave function $\psi(x)$, yields the same wave function translated by a length α, i.e. $\psi(x+\alpha)$.

Problem 2 [*Quantum numbers of QCD states*]

Consider in the following the QCD state

$$|\Phi\rangle = \widehat{\bar{u}}(\mathbf{x})\gamma_5\widehat{d}(\mathbf{x})|\Omega\rangle \quad .$$

(i) Spatial rotations w.r.t the j-axis are realised by the angular momentum/spin operator $\widehat{J}_j$: $\widehat{R}_j(\alpha) = \exp(i\alpha\widehat{J}_j)$. Their action on quark fields is given by

$$\widehat{R}_j(\alpha)\Big(\widehat{\psi}\Big) = \exp\left(\alpha\epsilon_{jk\ell}\frac{\gamma^k\gamma^\ell}{4}\right)\widehat{\psi} \quad .$$

Verify that the state $|\Phi\rangle$ possesses definite quantum numbers J_z and J^2 and evaluate them.

(ii) Isospin-rotations w.r.t the j-axis are given by $\widehat{\widehat{R}}_j(\alpha) = \exp(i\alpha\widehat{I}_j)$, where $\widehat{I}_j$ is the isospin operator. Their action on a isospin-doublet of quarks is

$$\widehat{\widehat{R}}_j(\alpha)\left[\begin{pmatrix}\widehat{u}\\ \widehat{d}\end{pmatrix}\right] = \exp\left(i\alpha\frac{\sigma_j}{2}\right)\begin{pmatrix}\widehat{u}\\ \widehat{d}\end{pmatrix} \quad ;$$

check that $|\Phi\rangle$ has definite quantum numbers, and calculate them, for the operators $\widehat{I}_z$ and $\widehat{I}^2 = \widehat{I}_x^2 + \widehat{I}_y^2 + \widehat{I}_z^2$.

♣ *Exercise continues on next page!* ♣

(iii) Parity is a *discrete* symmetry, with eigenvalues ± 1. Its action on a quark field is given by $\widehat{P}(\widehat{\psi}) = \gamma^0 \widehat{\psi}$. What is the parity (if defined) of $|\Phi\rangle$?

(iv) Browse the PD-Booklet and infer which hadron(s) may have a nonzero overlap with the trial state $|\Phi\rangle$.

(v) Argue, for $t \to +\infty$, the following behaviour for the correlation function in the Euclidean formulation of QCD:

$$\langle \Phi(t) | \Phi(0) \rangle \quad \sim \quad e^{-mt} \quad .$$

Which hadronic mass corresponds to m?

(vi) Repeat the above steps to identify the state

$$|\Phi'\rangle = \widehat{\overline{u}}(\mathbf{x}) \gamma_j \widehat{d}(\mathbf{x}) |\Omega\rangle \quad .$$