Exercise sheet XI

January 14 [solution: January 23, 30]

Problem 1 [*Normal order and chronological order*] Show the relation between the normal and the chronological ordering. Start with the decomposition of the scalar field into:

$$\phi(x) = \phi^+(x) + \phi^-(x), \tag{1}$$

where ϕ^+ contains only annihilation operators and ϕ^- only creation operators. Then, consider the cases $x^0 < y^0$ and $x^0 > y^0$ and write the chronologically ordered product $T\{\phi(x)\phi(y)\}$ in normal order (all creation operators to the left of annihilation operators), apart from one commutator term: $[\phi^+(x), \phi^-(y)]$ or $[\phi^+(y), \phi^-(x)]$. Finally, define the *contraction* of two fields:

$$\overbrace{\phi(x)\phi}(y) = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{if } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{if } y^0 > x^0 \end{cases} \tag{2}$$

(which is just the Feynman propagator $i\Delta_F(x-y)$) and show that the final result can be written as:

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \overbrace{\phi(x)\phi}(y)\}. \tag{3}$$

The generalization to an arbitrary number of fields is:

$$T\{\phi(x_1)\ldots\phi(x_n)\} = N\{\phi(x_1)\ldots\phi(x_n) + \text{all possible contractions}\}. \tag{4}$$

Note that $\langle 0|N\{\text{anything with uncontracted operators}\}|0\rangle = 0$ and hence:

$$\langle 0|T\{\phi(x_1)\ldots\phi(x_n)\}|0\rangle = \text{all possible full contractions (no operator left uncontracted).} \tag{5}$$

The last equation is known as the *Wick's theorem.*

Problem 2 [*Wick's theorem*] Using the Wick's theorem, calculate $\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle$ and $\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\}|0\rangle$, where $\phi_n \equiv \phi(x_n)$. Introduce the graphical notation, in which the contraction $\overbrace{\phi(x_i)\phi}(x_j)$ is represented by a straight line connecting points x_i and x_j. This gives the *Feynman diagrams* of the free scalar field theory.

Problem 3 [*Interacting scalar field theory*] Consider one of the simplest interacting scalar field theories, the ϕ^4 theory with the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{6}$$

where m is the mass parameter and λ is the coupling constant. We want to compute the expectation values of certain time-ordered products of operators in the *interacting theory* *vacuum*, denoted by $|\Omega\rangle$, in contrast to the *free theory vacuum*, $|0\rangle$. For the case of 2 fields, one has:

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi(x)\phi(y)\exp\left(-i\int_{-T}^{T} dt H_{\text{int}}(t)\right)\right\}|0\rangle}{\langle 0|T\left\{\exp\left(-i\int_{-T}^{T} dt H_{\text{int}}(t)\right)\right\}|0\rangle}, \tag{7}$$

where $H_{\text{int}} = -\int d^3x \mathcal{L}_{\text{int}} = \int d^3x \frac{\lambda}{4!}\phi(x)^4$ is the interaction Hamiltonian.

a) Compute the first 2 terms (free theory (*tree-level*) term and the 1st order term) in the numerator of Eq. (7) and draw the corresponding Feynman diagrams.

b) Compute the third term (2nd order in λ) in the numerator of Eq. (7) and draw the corresponding Feynman diagrams.

c) The inclusion of the denominator of Eq. (7) actually simplifies the computation. It can be shown that:

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \text{sum of all } \textit{connected} \text{ diagrams with } n \text{ external points}, \tag{8}$$

which means that the *disconnected* diagrams cancel between the numerator and the denominator. Use this rule to compute the full expression $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$ up to 2nd order in the coupling.

Note: disconnected diagrams are those that contain pieces not connected to any external point. Diagrams where each of e.g. 4 external points is connected to only one other external point (but there are no pieces disconnected from external points) are still connected.

d) Draw all diagrams contributing to $\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle$ up to 3rd order in the coupling.

e) Draw at least 10 examples of *distinct* diagrams contributing to $\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle$ at N-th order, where you choose n and N yourself, but $n \geq 4$ and $N \geq 4$.