Exercise sheet IV

November 5 [solution: November 14]

Problem 1 [*3-vector bilinear*] Show that

$$\bar{\psi}_1 \gamma^0 \gamma^i \gamma^5 \psi_2 \tag{1}$$

transforms as a 3-vector (e.g. x^i) under rotations.

Problem 2 [*Chiral projectors*] The projectors on the left- and right-components of Dirac spinors are given by $P_L = \frac{1-\gamma_5}{2}$ and $P_R = \frac{1+\gamma_5}{2}$, respectively. Use the properties of γ_5 to prove the following projector identities:

$$P_L P_R = P_R P_L = 0\,, \quad P_R P_R = P_R\,, \quad P_L P_L = P_L\,. \tag{2}$$

Verify that, given a spinorial solution to the massless Dirac equation, the spinors $P_{L,R} u_s(p)$ are eigenstates of the helicity operator

$$\mathfrak{h} = \frac{1}{2|\mathbf{p}|} \begin{pmatrix} \sigma \cdot \mathbf{p} & 0 \\ 0 & \sigma \cdot \mathbf{p} \end{pmatrix} \tag{3}$$

with eigenvalues $\pm\frac{1}{2}$. Does this hold also when $m \neq 0$?

Problem 3 [*Weyl equations*] Show that for massless particles, the Dirac equation reduces to 2 Weyl equations:

$$i\bar{\sigma} \cdot \partial \psi_L = 0 \tag{4}$$

for left-handed particles and

$$i\sigma \cdot \partial \psi_R = 0 \tag{5}$$

for right-handed particles, where $\sigma^\mu \equiv (1, \vec{\sigma})$ and $\bar{\sigma}^\mu \equiv (1, -\vec{\sigma})$.

Problem 4 [*Maxwell equations*] Consider the electromagnetic field-strength tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ +E_x & 0 & -B_z & +B_y \\ +E_y & +B_z & 0 & -B_x \\ +E_z & -B_y & +B_x & 0 \end{pmatrix}. \tag{6}$$

It satisfies the equation:

$$\partial_\nu F^{\nu\mu} = j^\mu, \tag{7}$$

where $j^\mu = (\rho, \vec{j})$ is the 4-current, ρ is the charge density and $\vec{j}$ is the electromagnetic current.

a) Show that the following identity holds:

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{8}$$

b) Show that (7)-(8) imply 4 Maxwell equations.