Numerische Methoden der Physik

WiSe 2023-2024 – Prof. Marc Wagner

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Exercise sheet 8

To be handed in on 06.12.2023 and discussed on 08.12.2023 and 11.12.2023.

Exercise 1 [*Condition number and preconditioning*] (2+6=8 pts.)

(i) Construct a 2×2 and a 3×3 symmetric and positive definite matrix with large condition number

$$\text{cond}(A) = \frac{\max_j(\lambda_j)}{\min_j(\lambda_j)} > 10^5,$$

with eigenvalues λ_j.

(ii) Consider the sparse, symmetric and positive definite $N \times N$ matrix

$$A_{ij} = \begin{cases} \frac{1}{i+j-1} & \text{if } i = j \text{ or } (i \bmod 100 = 0 \text{ and } j \bmod 100 = 0) \\ 0 & \text{else} \end{cases} \quad (1)$$

Construct a Matrix $\tilde{A} \approx A$, such that $\text{cond}(\tilde{A}^{-1}A) \approx 1$. Solve the system of linear equations,

$$A\mathbf{x} = \mathbf{b} \qquad \text{with } \mathbf{b} = (1, 1, ..., 1)^T \quad (2)$$

once directly and once with preconditioning. Compare the performance difference with and without preconditioning for $N = 101, 501, 1001$, i.e. count the number of iterations to reach a given precision.

Exercise 2 [*Integration in 1D*] (5+5+2=12 pts.)

Consider the integral

$$I^{(k)} = \int_0^\pi dx \; \frac{sin(kx)}{kx} cos(x),$$

with $i = 0, 1, 2 \ldots$.

(i) Compute $I^{(k)}$ using the iterated trapezoidal rule with step size $h_n = \pi/n$, $n = 2^{10}, 2^{11}, \ldots, 2^{30}$ for $k = 0, 1, 10, 50$. Create a double logarithmic plot of $I^{(k)}_{\text{trpz.}}(h_n)$ and describe what you observe.

(ii) Repeat task (i) using the iterated Simpson rule. How does the result $I^{(k)}_{\text{Simps.}}(h_n)$ compare to $I^{(k)}_{\text{trpz.}}(h_n)$?

(iii) What problem occurs for large k, when numerically computing $I^{(k)}$?